New Modular Curves

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Modular Background

Let $A_{/\mathbb{Q}}$ be an abelian variety. A is **modular of level** N if

$$\exists\, \nu_{/\mathbb{Q}} : J_1(N) \longrightarrow A\,.$$

In that case, A is **new of level** N if

$$\begin{tikzcd} J_1(N) \arrow[rr, "\nu"] \arrow[dr] & & A\,. \\ & J_1(N)^{\text{new}} \arrow[ur] & \end{tikzcd}$$

Then

$$\nu^*\, H^0(A,\Omega^1) \hookrightarrow S_2(N)^{\text{new}}\frac{dq}{q}.$$

Let $C_{/\mathbb{Q}}$ be a curve. C is **modular of level** N if

$$\exists\, \pi_{/\mathbb{Q}} : X_1(N) \longrightarrow C\,.$$

Then $J(C)$ is modular of level N, since we have

$$\begin{tikzcd} J_1(N) \arrow[r, "\pi_*"] & J(C) \\ X_1(N) \arrow[u] \arrow[r, "\pi"] & C\,. \arrow[u] \end{tikzcd}$$

In that case, C is **new of level** N if $J(C)$ is new of level N.

Finitness Results

Let $g \in \mathbb{Z}_{>0}$, we denote by:

$$\begin{aligned} \mathcal{MC}_g &= \{\text{modular curves of genus } g\}_{/\cong}^{\mathbb{Q}} \\ \mathcal{MC}_g^{\text{new}} &= \{[C] \in \mathcal{MC}_g \,|\, C \text{ is new}\} \end{aligned}$$

$\boxed{g=0}$ $\mathcal{MC}_0 = \mathcal{MC}_0^{\text{new}} = \{X_1(1)\}$

$\boxed{g=1}$ $\mathcal{MC}_1 = \mathcal{MC}_1^{\text{new}} \overset{[2][5]}{=\!=} \{\text{elliptic curves defined over } \mathbb{Q}\}_{/\cong}^{\mathbb{Q}}$

Let $G, g \geq 2$ be positive integers, we denote by

$$\begin{aligned} \mathcal{MC}_g^{\text{new}}(G) &= \{[C] \in \mathcal{MC}_g^{\text{new}} \,|\, C \text{ has gonality } G\} \\ \mathcal{MC}^{\text{new}}(G) &= \bigcup_{g\geq 2} \mathcal{MC}_g^{\text{new}}(G) \end{aligned}$$

Theorem[1]: $\#\mathcal{MC}_g^{\text{new}}<\infty$

g	0	1	2	>2
$\#\mathcal{MC}_g^{\text{new}}$	1	∞	213	$<\infty$
$\#\mathcal{MC}_g$	1	∞	?	?

Theorem[1]: $\#\mathcal{MC}(G)^{\text{new}}<\infty$

G	2	>2
$\#\mathcal{MC}^{\text{new}}(G)$	300?	$<\infty$
$\#\mathcal{MC}(G)$	?	?

Conjecture[1]: $\#\mathcal{MC}_g<\infty$ $\Rightarrow$ **Conjecture**[1]: $\#\mathcal{MC}(G)<\infty$

Hyperelliptic Curves

Let $C_{/\mathbb{Q}}$ be a curve of genus $g \geq 2$, C is *hyperelliptic* if C has an affine model

$$y^2 = F(x),$$

where $F(X) \in \mathbb{Q}[X]$ separable with $\deg F = 2g+1$ or $2g+2$.

New Modular Hyperelliptic Curves[1][3]

C	:	$y^2 = F(x)$
$X_1(13)$	:	$y^2 = x^6 - 2x^5 + x^4 - 2x^3 + 6x^2 - 4x + 1$
$X_1(16)$	:	$y^2 = x^6 + 2x^5 - x^4 - x^2 - 2x + 1$
$X_1(18)$	:	$y^2 = x^6 - 4x^5 + 10x^4 - 10x^3 + 5x^2 - 2x + 1$
$C^A_{21A_{\{0,2\}}}$	:	$y^2 = (x^2 - x + 1)(x^6 + x^5 - 6x^4 - 3x^3 + 14x^2 - 7x + 1)$
$X_0(23)$	:	$y^2 = x^6 - 8x^5 + 2x^4 + 2x^3 - 11x^2 + 10x - 7$
$X_0(26)$	:	$y^2 = x^6 - 8x^5 + 8x^4 - 18x^3 + 8x^2 - 8x + 1$
$C_{28A_{\{0,2\}}}$	:	$y^2 = x^5 - 9x^4 + 13x^3 - 4x^2 - x$
$X_0(29)$	:	$y^2 = x^6 + 2x^5 - 17x^4 - 66x^3 - 83x^2 - 32x - 4$
$\vdots$		$\vdots$
		Total: 300 curves
$\vdots$		$\vdots$
C_{4160II}	:	$y^2 = x^5 - 7x^3 - 4x$
C_{7280BB}	:	$y^2 = x^5 - 5x^4 + 17x^3 - 15x^2 + 6x - 5$
C_{7424A}	:	$y^2 = x^5 + 10x^3 - 4x$
C_{7424B}	:	$y^2 = x^5 - 10x^3 - 4x$
C_{7664A}	:	$y^2 = x^5 - 3x^4 - 7x^3 - 5x^2 - 2x - 1$

Conjecture[1]: $\#\mathcal{MC}^{\text{new}}(2) = 300$

Non-Hyperelliptic Curves

Let $C_{/\mathbb{Q}}$ be a non-hyperelliptic curve of genus $g \geq 3$, and

$$H^0(C, \Omega^1_C) = \langle \omega_1, \dots, \omega_g \rangle_{\mathbb{C}}\,.$$

Then there exists the *canonical embedding* defined by:

$$C \hookrightarrow \mathbb{P}^{g-1} \;:\; z \mapsto [\omega_1(z) : \cdots : \omega_g(z)]$$

where $i(C)$ is a nonsingular projective curve. In fact:

Petri's Theorem $\quad i(C) = \bigcap_{d=2}^{4} \left\{ \mathcal{HS}_d \;\middle|\; i(C) \subset \mathcal{HS}_d\,,\ \begin{array}{l} codim(\mathcal{HS}_d) = 1 \\ deg(\mathcal{HS}_d) = d \end{array} \right\}$

New Modular Non-Hyperelliptic Curves[4]

C	:	$i(C)$	g
$X_1(17)$	:	$\begin{cases} yu - zt + zu + t^2 - 2tu - 2u^2 = 0 \\ yt - z^2 + 5zu - t^2 - 5tu - 2u^2 = 0 \\ x^2 - y^2 + 2yz - z^2 + 2zt - 4zu - 2t^2 + 8tu - 5u^2 = 0 \end{cases}$	5
$X_1(20)$	:	$x^4 - y^4 + 8y^2z^2 - 8yz^3 = 0$	3
$X_1(21)$	:	$\begin{cases} 4xz + 3y^2 - 4yz - z^2 - 3t^2 + 18tu - 18u^2 = 0 \\ 4xy + 2y^2 + 2yz - 3z^2 - 6t^2 + 18tu - 9u^2 = 0 \\ x^2 - y^2 + yz - z^2 = 0 \end{cases}$	5
$X_1(24)$	:	$\{ yz - z^2 - tu - u^2 = 0\,,\ y^2 - 2z^2 - t^2 - 2u^2 = 0\,,\ x^2 - z^2 - t^2 + u^2 = 0 \}$	5
$C_{25A_{\{4\}}}$	:	$\begin{cases} xz - y^2 + yt - 2zt + t^2 = 0 \\ x^2t - xt^2 - y^3 + y^2z - 3yzt + 3yt^2 + z^2t - 2zt^2 + t^3 = 0 \end{cases}$	4
$X_0(43)$	:	$x^4 + 2x^2y^2 + 8x^2yz + 16x^2z^2 - 3y^4 + 8y^3z + 16y^2z^2 + 48yz^3 + 64z^4 = 0$	3
$\vdots$		$\vdots$	$\vdots$

References

[1] M.H. Baker, E. González-Jiménez, J. González, and B. Poonen. *Finiteness results for modular curves of genus at least 2*, Amer. J. Math. 127 (2005), no. 6, 1325–1387.
[2] C. Breuil, B. Conrad, F. Diamond, and R. Taylor, *On the modularity of elliptic curves over $\mathbb{Q}$: wild 3-adic exercises*, J. Amer. Math. Soc. 14(4) (2001), 843–939.
[3] E. González-Jiménez and J. González, *Modular curves of genus 2*, Math. Comp. 72(241) (2003), 397–418.
[4] E. González-Jiménez *New Modular Curves*. In progress.
[5] A. Wiles, *Modular elliptic curves and Fermat's Last Theorem*, Ann. Math. 141 (1995), 443–551.