Technical University Berlin **WS 2025**
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6. Exercise Discrete Geometrie II

Deadline: 26.11.2025 (before the Exercise class)

Each answer should be sufficiently proven.

1. Exercise (Intersection of Varieties)

Sei $f, g \in \mathbb{C}[x_1, \ldots, x_k]$, show that,

1. $V(f) \cap V(g) = V(f, g)$,
2. $V(f) \cup V(g) = V(fg)$.

2. Exercise (Stereographic projection)

Let $f(x, y) := x^2 + y^2 - 1$ and $g(x, y) = y - t(x + 1)$ be polynomials in $\mathbb{Q}[x, y]$.

1. Show that
$$V(f) \cap V(g) = \{\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1-t^2}{1+t^2} \\ \frac{2t}{1+t^2} \end{pmatrix}\}.$$
with $t \in \mathbb{Q}$ such that $1 + t^2 \neq 0$.
2. Using this show that the integer solutions of $a^2 + b^2 = c^2$ are given by
$$a = k(m^2 - n^2), \quad b = k(2mn), \quad c = k(m^2 + n^2)$$
with $k, m, n \in \mathbb{Z}$.

3. Exercise (Real components of algebraic varieties)

Sketch the real components of the following hypersurfaces in $\mathbb{C}^2$:

1. $V(y - x^2)$A
2. $V(y^2 - x^3 + x)$
3. $V(y^2 - x^3)$
4. $V(y^2 - x^3 - x^2)$
5. $V((x^2 + y^2)^2 + 3x^2y - y^3)$
6. $V((x^2 + y^2)^3 - 4x^2y^2)$

Hint: If necessary, use a computer algebra system to plot the curves.