Optimization and Tropical Geometry: **6. Divisors on Curves, Riemann-Roch and Chip Firing Games**

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Graphs: basic definitions

Let G be a finite undirected graph with nodes $V = V(G)$ and edges $E = E(G)$; the number of nodes is denoted by $n = |V|$.

Multiple edges are allowed but no loops.

For $k \geq 2$ the graph G is k-edge-connected if $G - W$ is connected for every set W of at most $k - 1$ edges of G.

- 1-edge-connected = connected
- G is k-edge-connected iff every cut has at least k edges
- trivial graph with one node and no edges k-edge-connected for all k

Usually, we pick a fixed ordering on V, and we may assume that $V = [n]$.

- throughout we assume that G is connected

A chip-firing game

On a graph G we consider the following game for a single player.

- initial configuration: integer number of chips on each node
- a node is in debt if that number is negative
- move = firing a node v =
  - either send one chip to each neighbor of v
  - or receive one chip from each neighbor of v
- configuration is winning if no node is in debt

Question

Does the player have a winning strategy?

The genus of a (connected) graph

Definition

The number

$$g \;=\; |E(G)| - |V(G)| + 1 \;=\; |E(G)| - n + 1$$

is the genus of the graph G.

- $g = 0 \iff G$ is a tree
- $g =$ first Betti number of G, seen as 1-dimensional simplicial complex

Lemma

Let G be planar with set of regions L.
Then $g = |E| - |V| + 1 = |L| - 1$, which is the number of bounded regions.

Laplacian of a graph

Let $G = (V, E)$ be a graph with n nodes and adjacency matrix $A \in \mathbb{N}^{n \times n}$.

Further let δ be the $n \times n$-diagonal matrix with $\delta_{vv} = \deg v$.

Definition

The Laplacian of G is the matrix $\Delta = \delta - A$.

Exercise

Show that Δ is symmetric of rank $n - 1$, and that $\ker \Delta$ is spanned by $\mathbf{1}$.

Graph divisors

Let $G = (V, E)$ be a finite connected graph.

Definition (Group of divisors)

$\mathrm{Div}(G)$ = free abelian group on V

- divisor D = formal linear combination $D = \sum_{v \in V} a_v \cdot v$ with $a_v \in \mathbb{Z}$
- notation: $D(v) := a_v$
- partial order $D \geq D' :\Longleftrightarrow D(v) \geq D'(v)$ for all $v \in V$
- divisor E effective $:\Longleftrightarrow E \geq 0$
- $\mathrm{Div}_+(G) :=$ set of effective divisors
- $\deg D := \sum_{v \in V} D(v)$ degree
- $\mathrm{Div}^k(G) := \{D \in \mathrm{Div}(G) \mid \deg D = k\}$

Jacobian of a graph

- $\mathcal{M}(G) := \mathrm{Hom}(V, \mathbb{Z})$
  $= \text{abelian group of integer-valued functions on vertices}$
  $\cong \mathrm{Div}(G)^*$

Definition

$D \in \mathrm{Div}(G)$ principal $:\Longleftrightarrow$ exists $f \in \mathcal{M}(G)$ with $D = \Delta[f]$

- D principal $\implies \deg D = 0$
- $\mathrm{Prin}(G) :=$ set of principal divisors

Definition (Jacobian)

$\mathrm{Jac}(G) := \mathrm{Div}^0(G) / \mathrm{Prin}(G)$

- Bacher, de la Harpe & Nagnibeda 1997:
  $\mathrm{Jac}(G)$ finite of order $= \#(\text{spanning trees in } G)$

Linear systems of divisors

Definition

- equivalence of divisors

$$D \sim D' \;:\Longleftrightarrow\; D - D' \in \mathrm{Prin}(G)$$

- linear system associated to D

$$|D| \;:=\; \{E \in \mathrm{Div}(G) \,|\, E \geq 0,\, E \sim D\}$$

- dimension

$$\begin{array}{ll} r(D) \geq k & \text{iff } |D - E| = \emptyset \text{ for all } E \in \mathrm{Div}_+^k(G) \\ r(D) = -1 & \text{iff } |D| = \emptyset \end{array}$$

Chip-firing, revisited

Consider the chip-firing game on a (connected) graph G of genus g.

Let $D, D' \in \mathrm{Div}(G)$.

Lemma

We have $D \sim D'$ iff there is a sequence of moves which transforms the configuration corresponding to D into the configuration corresponding to D'.

- $r(D) \geq k \iff$ there is a winning strategy after subtracting k chips from some vertex

Theorem (Baker & Norine 2007)

For $k \geq g$ and all $D \in \mathrm{Div}^k(G)$ there is a winning strategy (with initial configuration D).
For $k < g$ exists $D \in \mathrm{Div}^k(G)$ without winning strategy.

The Riemann–Roch theorem for graphs

The canonical divisor, K, on G is

$$K \ = \ \sum_{v \in V} (\deg(v) - 2) v \ \ .$$

- $\deg(K) = 2|E| - 2|V| = 2g - 2$

Theorem (Baker & Norine 2007)

Let G be a graph, and let D be any divisor on G.
Then

$$r(D) - r(K - D) \ = \ \deg(D) + 1 - g \ \ .$$

The classical theorem of Riemann and Roch

Let X be a (compact) Riemann surface of genus g.

- $H_1(X, \mathbb{C}) \cong \mathbb{C}^{2g}$
- $\mathrm{Div}(X) :=$ abelian group on points of X
- meromorphic function $f \in \mathcal{M}(X)$ yields principal divisor (f) of zeros and poles (with their signed orders)
- $r(D) := \dim_{\mathbb{C}}(\{h \in \mathcal{M}(X) \colon (h) + D \geq 0\})$
- canonical divisor K obtained from "global meromorphic 1-form"; unique up to linear equivalence

Theorem (Riemann 1857; Roch 1865)

$$r(D) - r(K - D) \;=\; \deg(D) + 1 - g \quad .$$

References

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