Polyhedral Computations With polymake

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polymake Overview

most recent version 3.2 of January 2018

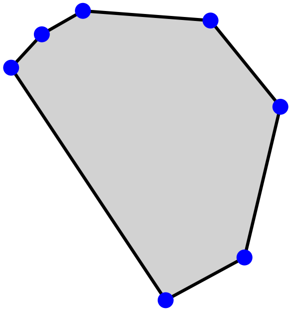
- software for research in mathematics
  - geometric combinatorics: convex polytopes, matroids, ...
  - linear/combinatorial optimization
  - toric/tropical geometry
  - combinatorial topology
- open source, GNU Public License
  - supported platforms: Linux, FreeBSD, MacOS X
  - about 150,000 uloc (C++, Perl, C, Java)
  - interfaces to many other software systems
- co-authored (since 1996) w/ Ewgenij Gawrilow
  - contributions by Benjamin Assarf, Simon Hampe, Katrin Herr, Silke Horn, Lars Kastner, Georg Loho, Benjamin Lorenz, Andreas Paffenholz, Julian Pfeifle, Thomas Rehn, Olivia Röhrig, Thilo Rörig, Benjamin Schröter, André Wagner and others

www.polymake.org

The Basic Definition

A (convex) polytope is the convex hull of finitely many points (in $\mathbb{R}^d$).

- = intersection of finitely many closed halfspaces (if bounded)
- = set of feasible points of a linear program (if bounded for all choices of linear objective functions)



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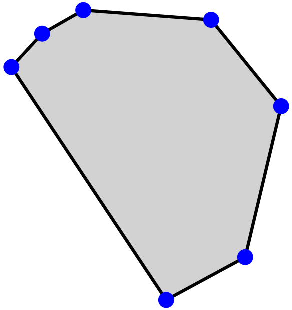

- conversion from points to inqualities (or vice versa) conceptually simple but still has its challenges

Example: Knapsack Problem

$$\begin{aligned} \max \quad & \sum_{i=1}^{d} u_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{d} w_i x_i \leq b \\ & x_i \in \mathbb{N} \qquad \text{for all } i \in [d] \end{aligned}$$

- $d = \#$ items
- u_i = utility of item i
- w_i = weight of item i
- b = total weight bound

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Algorithm Overview (Selection)

- convex polytopes, polyhedra and fans
  - convex hulls: `cdd`, `lrs`, `normaliz`, `ppl`, beneath-and-beyond
  - Voronoi diagrams, Delone decompositions
  - Hasse diagrams of face lattices
  - $\rightsquigarrow$ lattice polytopes/toric varieties
- optimization
- simplicial complexes
- tropical geometry
  - tropical hypersurfaces
  - tropical polytopes
- graphs, matroids, permutation groups, ...

Example: Max-Cut

- combinatorial optimization problem on $\Gamma = (V, E)$ finite graph

$$\max \sum_{s \in S,\, t \in T,\, \{s,t\} \in E} w(s,t)$$

- ◦ maximum over all partitions $S \sqcup T = V$
- ◦ w = weight function on E
- ◦ each cut $S \sqcup T$ gives rise to subset of E, which can be encoded by its characteristic vector
  - $\rightsquigarrow$ 0/1-polytope

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- goal: determine facets of the cut polytopes

Barahona & al. 1988; Avis, Imai & Ito 2008; Bonato & al. 2014; . . .

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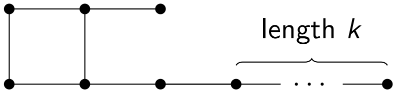
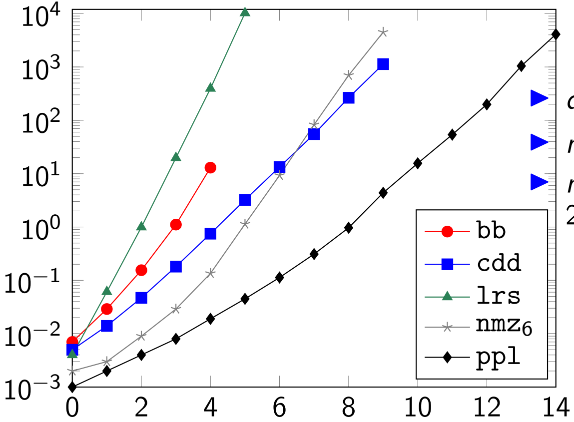
Facets of Cut Polytopes

variable dimension



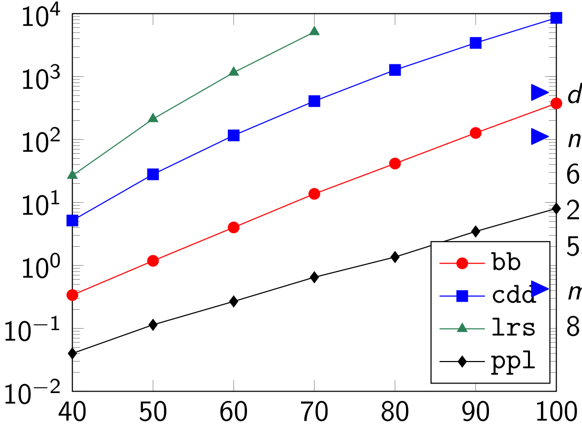


- $d = k + 6$
- $n = 2^{k+5} = \#$ cuts
- $m = 2d + 8 = 2k + 20$
  - Barahona 1983: facets known if no K_5-minor

Knapsack Integer Hulls

fixed dimension, variable right hand side

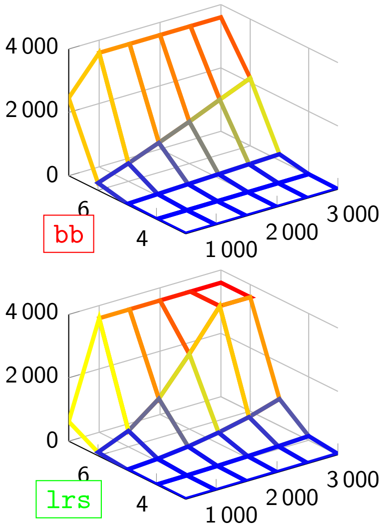
$a_1 = 2,\ a_2 = 3,\ a_i = a_{i-2} + a_{i-1}$
$F_d(b) = \{x \in \mathbb{R}^d_{\geq 0} \mid a^\top x \leq b\}$



- $d = 5$
- $n = 1366, 3173, 6509, 12182, 21245, 35025, 55157$
- $m = 12, 15, 12, 12, 8, 13, 15$

Voronoi Diagrams of Random Points in a Box

variable dimension, variable number of points



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Some Rules of Thumb

1. If you do not know anything about your input, try double description.
   - `cdd`, `ppl`, `nmz`
2. Do use double description for computing the facets of 0/1-polytopes.
   - `cdd`, `ppl`
3. On random input beneath-and-beyond often behaves very well.
   - `bb`
4. Use reverse search for partial information and non-degenerate input.
   - `lrs`

Selected References

- Benjamin Assarf, Ewgenij Gawrilow, Katrin Herr, Michael Joswig, Benjamin Lorenz, Andreas Paffenholz, and Thomas Rehn, Computing convex hulls and counting integer points with `polymake`, Math. Program. Comput. **9** (2017), no. 1, 1–38. MR 3613012

- Ewgenij Gawrilow and Michael Joswig, `polymake`: a framework for analyzing convex polytopes, Polytopes—combinatorics and computation (Oberwolfach, 1997), DMV Sem., vol. 29, Birkhäuser, Basel, 2000, pp. 43–73. MR 1785292 (2001f:52033)

- ______, Flexible object hierarchies in `polymake`, Proceedings of the 2nd International Congress of Mathematical Software (Andrés Iglesias and Nobuki Takayama, eds.), 2006, 1.–3. September 2006, Castro Urdiales, Spanien, pp. 219–221.