Diane Maclagan, Bernd Sturmfels: "Introduction to Tropical Geometry". AMS, 2015, 363 pp

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What is tropical geometry about? Back in 2005 an influential paper by Richter-Gebert, Sturmfels and Theobald [11] answered that question in the following way: "Tropical algebraic geometry is the geometry of the tropical semiring $(\mathbb{R}, \min, +)$. Its objects are polyhedral cell complexes which behave like complex algebraic varieties." Let us look at plane algebraic curves and their tropicalizations to get an idea how this works. To this end consider a plane algebraic curve C, which arises as the vanishing locus of a single bivariate polynomial f (over an algebraically closed field $\mathbb{K}$ of characteristic zero). Instead of picking the complex numbers for $\mathbb{K}$, however, here it is more rewarding to take the field of formal *Puiseux series* with complex coefficients. These are the formal power series with rational exponents which share a common denominator. Puiseux series have been used for the resolution of singularities. As their special feature they admit a non-trivial valuation by sending Puiseux series to their smallest exponents. The tropicalization of the algebraic curve C now arises from applying the valuation map to C pointwise and coordinatewise. One then defines the tropical plane curve $\mathcal{T}(C)$ as the topological closure of the image of the valuation map in $\mathbb{R}^2$. The tropical curve $\mathcal{T}(C)$ is an unbounded one-dimensional polyhedral complex, equipped with integral weights, which still "knows" a lot about the original curve C. For instance, from $\mathcal{T}(C)$ one can see the Newton polygon and thus the degree of C. Moreover, the dimension of the space of cycles of $\mathcal{T}(C)$, seen as a planar graph, equals the arithmetic genus of C. Again by employing the valuation map, one can also *tropicalize* the polynomial f which defines the algebraic curve C. It is an essential feature that, via polyhedral combinatorics, one can obtain the tropical curve $\mathcal{T}(C)$ also directly from the tropicalization of f, which is a polynomial over the tropical semi-ring $(\mathbb{R}, \min, +)$; see [6].

To explain the concept let us look at the quadratic bivariate polynomial

$$f(x,y) \; = \; t + x - tx^2 + y + t^2y^2$$

with coefficients in the field $\mathbb{C}\{\{t\}\}$ of complex Puiseux series, which are power series in t with rational exponents. The curve C defined by f is a conic. To visualize the shape of C it is helpful, for a brief moment, to think of t as a small positive real number. Then f would become a bivariate real polynomial whose real locus is a hyperbola. However, we return to computing with Puiseux series. For any given $\xi \in \mathbb{C}\{\{t\}\}$ we can solve the equation $f(\xi, \eta) = 0$, and we arrive at

$$\eta \; = \; -\frac{1}{2}t^{-2} \pm \sqrt{\frac{1}{4}t^{-4} - t^{-2}\xi + t^{-1}\xi^2 - t^{-1}} \; .$$

For instance, we can substitute $\xi = t^{-3/2}$ and obtain

$$\eta \; = \; -\frac{1}{2}t^{-2} \pm \sqrt{-t^{-7/2} + \frac{5}{4}t^{-4} - t^{-1}} \; .$$

The square root of $-t^{-7/2} + 5/4t^{-4} - t^{-1}$ now is a Puiseux series whose lowest order term equals $\pm it^{-7/4}$, where i is the imaginary unit. This implies that $\eta = -1/2t^{-2}$, up to terms of higher order. Applying the valuation map to the point $(t^{-3/2}, -1/2t^{-2} \pm it^{-7/4} \pm \cdots)$ on the algebraic curve C gives the point $(-3/2, -2)$ on the tropical curve $\mathcal{T}(C)$. This tropical curve, which is a one-dimensional polyhedral complex in the plane such that each edge receives 1 as its weight, is shown in Figure 1 (left). The same figure displays a more interesting tropical curve on the right. There is software for computing with tropical curves and more general tropical varieties, e.g., `Gfan` [8] and `polymake` [7].

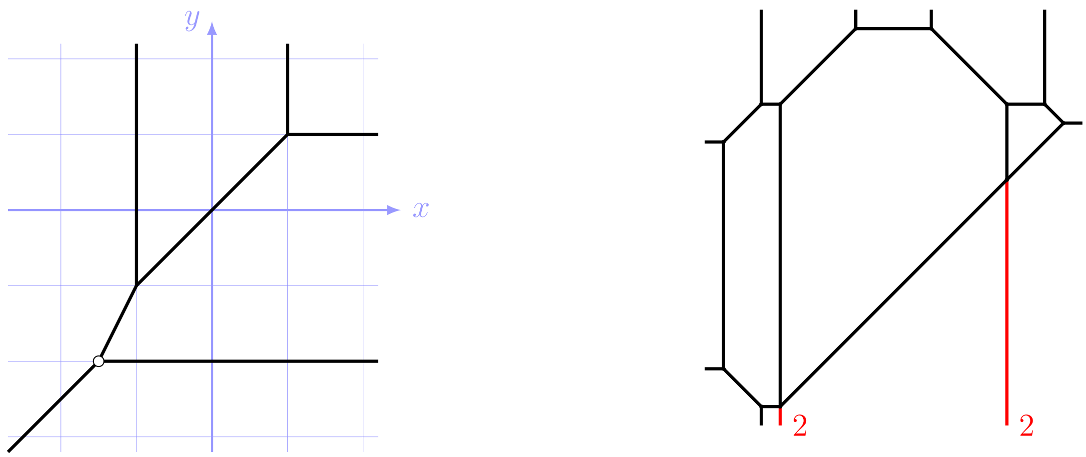

FIGURE 1. Left: tropical quadric $\mathcal{T}(C)$ discussed in the example with the point $(-3/2, -2)$ marked. Right: tropical hyperelliptic curve of genus three and degree six. Two of its unbounded edges have weight two.

The theory of tropical geometry actually started with a breakthrough. Mikhalkin had recognized [10] that tropical curves can replace complex algebraic curves in a geometric counting problem which occurred in Kontsevich's work on Gromov–Witten invariants of symplectic manifolds; see [9] and §1.7 in the book under review. Since then there has been a rapid development of tropical geometry into all possible directions. Interestingly, that development even works backward in time. For instance, a result of Bieri and Groves on the Krull dimension of a certain ring related to the logarithmic limit-set of an algebraic variety [2], which goes back as far as 1984, is now seen as an early contribution to tropical geometry. Furthermore, there is also a deep connection to optimization. This should not be too surprising, as the tropical addition min (or max; there is no global agreement among researchers in tropical geometry which one is to be preferred) is expressed in terms of an optimization problem. For instance, the optimization of discrete event systems as described in the monograph [1], is now seen as a part of *linear* tropical geometry (and this is discussed in the book under review in Chapter 5). One possible way to sum up the fruitful interaction between algebraic geometry and combinatorial optimization via tropical geometry is the following: Optimization gains a unified view on aspects which were previously perceived as unrelated, and algebraic geometry benefits from combinatorial methods which are inherently algorithmic.

Today it is impossible to describe all features of tropical geometry on a few pages or even in a single book. Therefore, it is necessary to explain the approach taken by Maclagan and Sturmfels. Let I be an ideal in a polynomial ring R in, say, d indeterminates and with complex coefficients. If we assume I to be homogeneous, then each (generic) vector in $\mathbb{R}^d$ gives rise to a term order on R. This entails that such a *weight vector* is associated with a unique reduced Gröbner basis, defined by that term order. Conversely, for each reduced Gröbner basis, we can collect all its weight vectors. Those form a relatively open polyhedral cone. Taking all these cones and admitting non-generic weights, too, yields the *Gröbner fan* of the homogeneous ideal I. The monograph [13] by Sturmfels gives a comprehensive treatment of this topic, which includes the polyhedral geometry background, algorithms and the relationship to solving systems of polynomial equations. How is this related with tropical geometry? We return to our initial example of an algebraic curve C defined by a bivariate polynomial f. However, now we assume that f does have complex coefficients, which, by the way, are special Puiseux series whose valuations are zero. This situation is known as *constant coefficients*, and in this case the tropical curve $\mathcal{T}(C)$ can be recovered from the Gröbner fan of the principal ideal I spanned by the homogenization of f: the points on $\mathcal{T}(C)$ correspond to the non-generic weight vectors w such that the generalized initial ideal $\mathrm{in}_w(I)$ does not contain any monomial. This provides an approach to general tropical varieties (with constant coefficients) which was pioneered by Speyer and Sturmfels in their celebrated paper on the *tropical Grassmannians* [12]. The Chapters 2 and 3 of the book under review are devoted to developing the theory of tropical geometry from this point of view. Here the book greatly extends what was available in the literature previously, since it gives the first comprehensive treatment to rigorously include the non-constant coefficient case. Most importantly, this requires to replace the Gröbner fans by the more general *Gröbner complexes* in §2.5, and this also includes the key algorithm for computing a general tropical basis in §2.6 (which extends the method from [3] to non-constant coefficients). From this outset it becomes clear that the book is organized around the tropicalization of varieties with respect to a fixed embedding, and this allows for explicit computations.

Chapter 4 carries the title "Tropical Rain Forest". That name is motivated by the fact that the combinatorics of metric trees occur quite naturally in the context of tropical geometry. The core pieces of the chapter are the two sections §4.3 on the tropical Grassmannians and §4.4 on tropical linear spaces. Classically, the *Grassmannian* $\mathrm{Gr}(k,n)$ is an algebraic variety which parameterizes the k-dimensional subspaces of an n-dimensional vector space, over some field $\mathbb{K}$. For $\mathbb{K} = \mathbb{C}$ (so we are back to constant coefficients), Speyer and Sturmfels [12] studied a tropical analog. Due to the homogenization the case $k = 2$ corresponds to lines in projective $(n-1)$-space, and their tropicalizations are the trees with n marked leaves. Generic metric trees are those where each interior node has valence 3. They naturally occur in phylogenetics, a field in (not only computational) biology which seeks to derive ancestral relations among given species or individuals. These phylogenetic trees are the *tropical linear spaces*, corresponding to the points on the tropicalization of $\mathrm{Gr}(2,n)$. For higher values of k they generalize into (duals of) regular subdivisions of a class of convex polytopes, called *hypersimplices*. This part of the theory is closely related to matroid combinatorics (which is why §4.2 gives an introduction to matroids). The special case of tropical linear spaces for $k = 3$ is deferred until §5.4. The fifth chapter offers a geometric perspective on the linear algebra over the tropical semiring. Finally, Chapter 6 contains an in-depth discussion of the relationship between tropical and toric geometry [5]. This includes a detailed discussion of the moduli space $M_{0,n}$ of n labeled points on the projective line. Its tropicalization is the space of phylogenetic trees which, up to lineality, occurs as tropical $\mathrm{Gr}(2, n)$ in Chapter 4. The Deligne–Mumford compactification $\overline{M_{0,n}}$ of stable genus zero curves with n marked points is explicitly described in tropical and toric terms. This is a highlight in the book.

Who should read the book? Everybody who wants to learn what tropical geometry *can* be about. If you are unsure, look at the introductory chapter: it displays a multitude of connections of tropical geometry to other areas of mathematics. Who *can* read the book? The authors specifically mention the text books by Ziegler [14] (on polytopes) and Cox, Little, O'Shea [4] (on commutative algebra) as prerequisites. This background is certainly necessary, but the non-expert reader needs to be prepared to occasionally follow a few more links (given in the book). Sometimes it could have been useful to have even a few more references for more details on the historical context. What is missing in the book? Abstract tropical curves are only mentioned in passing, and the ramifications into optimization are only scratched. Further, there is barely any enumerative geometry, mirror symmetry, real algebraic geometry or patchworking. This is no complaint. To the contrary, these omissions help to give the book a clear shape. The result is a straight path to tropical geometry via combinatorial commutative algebra and polyhedral combinatorics. In this way, the book by Maclagan and Sturmfels will become a standard reference in the field for years to come.

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