Museums, Triangles and Algebraic Curves

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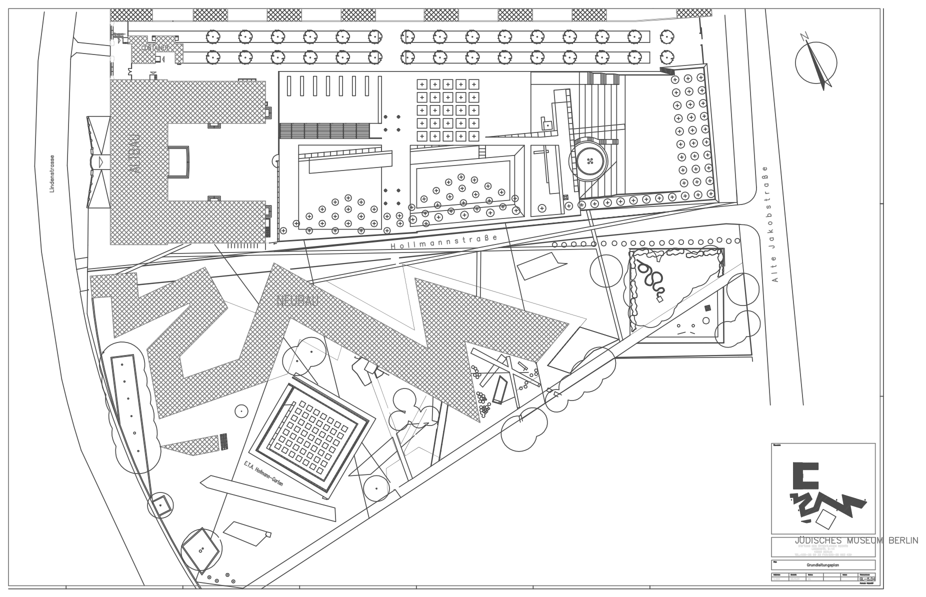
The Museum

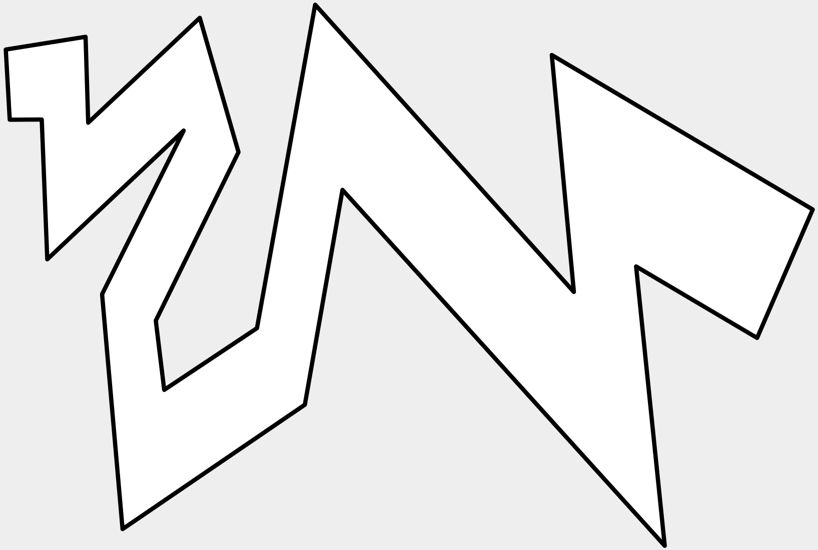

CONTAINER
ALTBAU
Lindenstrasse
Hollmannstraße
NEUBAU
Alte Jakobstraße
E.T.A. Hoffmann-Garten
JÜDISCHES MUSEUM BERLIN
Grundleitungsplan

Everything Starts With a Definition

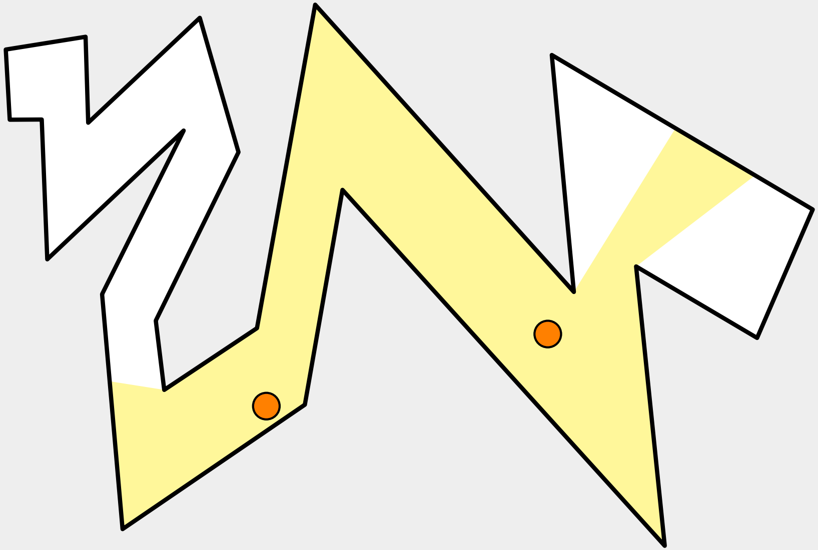
A simple polygon is a closed sequence of finitely many line segments without self-crossings.

- allows to dinstinguish between inside and outside
  - (polygonal version of) Jordan's Curve Theorem

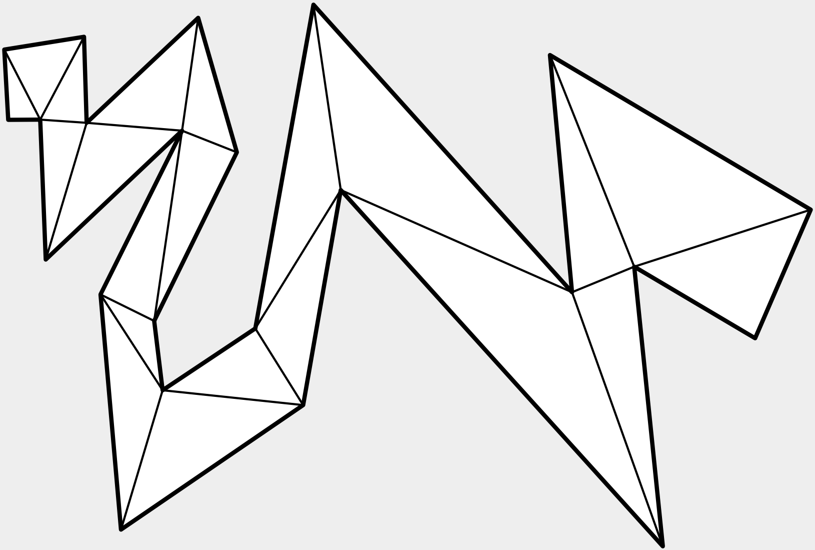
The Museum Is a Simple Polygon



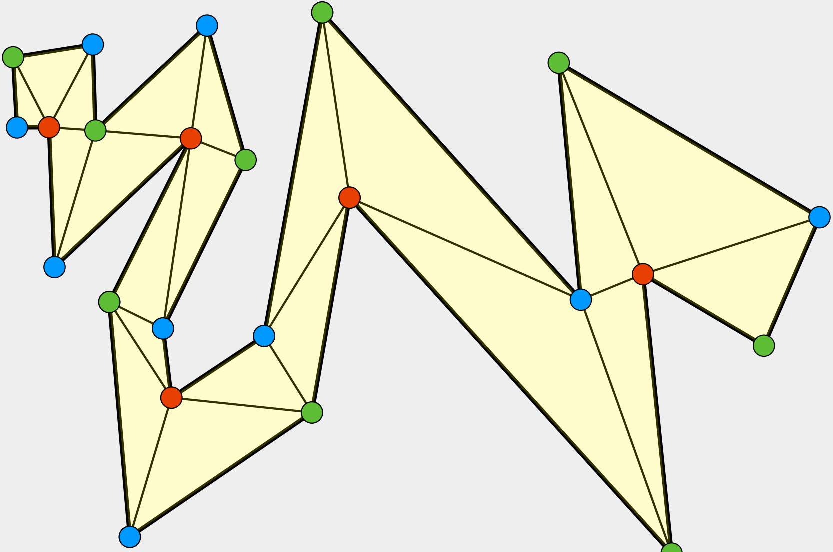
The Restricted View of a Guard



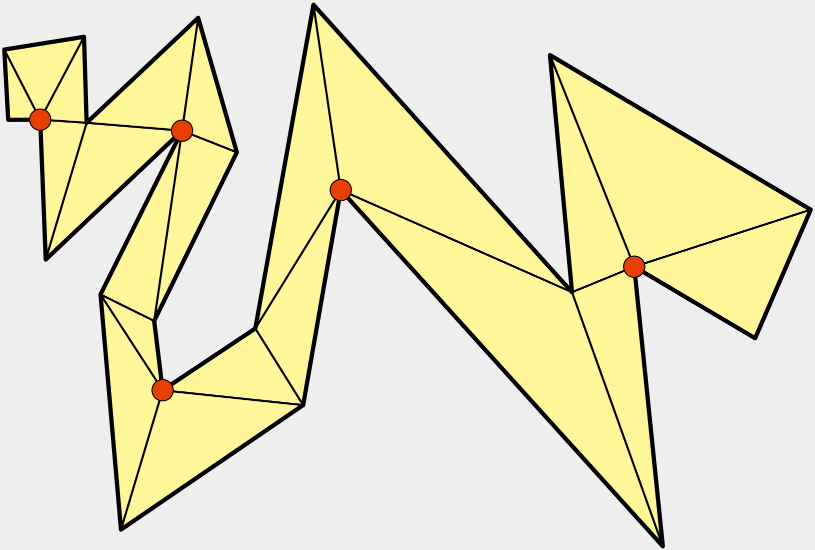
Subdividing Into Triangles



Coloring the Vertices (Triangle by Triangle)



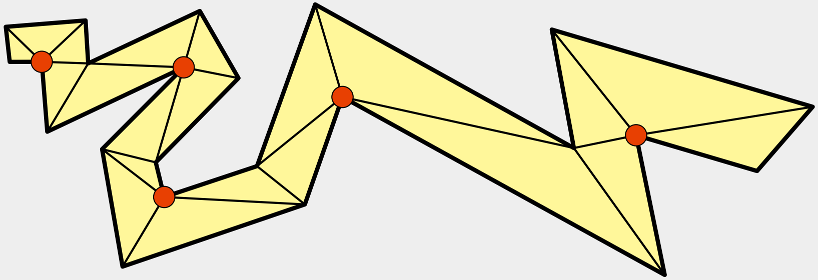
Choosing One Color



The Art Gallery Theorem

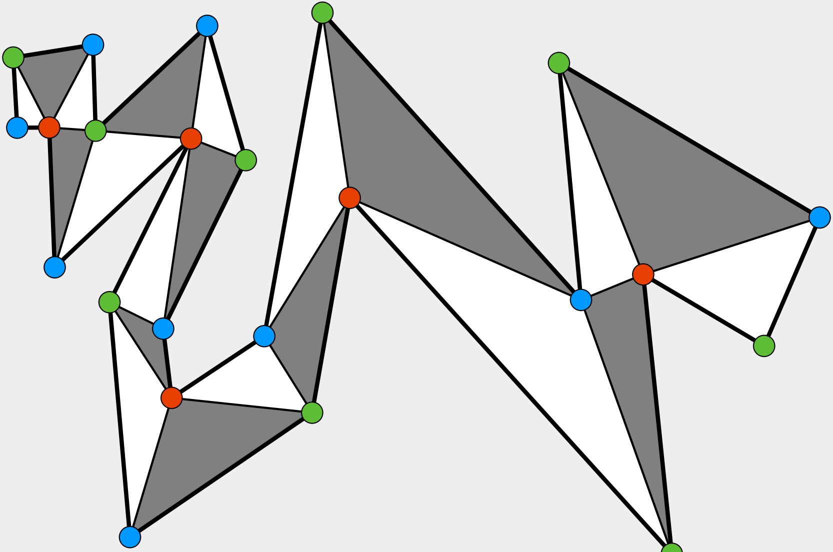
Theorem (Vašek Chvátal 1975; Steve Fisk 1978)

For a museum (or art gallery) which is modelled as a simple polygon with n vertices at most $\lfloor \frac{n}{3} \rfloor$ guards suffice.



here $n = 23$, $\lfloor \frac{n}{3} \rfloor = 7$, and 5 guards suffice

Three Colors for Vertices & Two Colors for Triangles

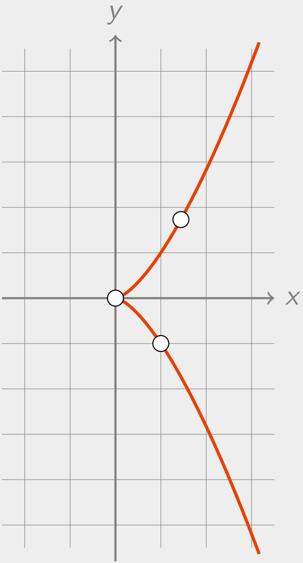


Example: Neil's Parabola

Wanted: all points (x, y) such that

$$x^3 - y^2 = 0$$

- $x = 0,\ y = 0$:
  $0^3 - 0^2 = 0 - 0 = 0$
- $x = 1,\ y = -1$:
  $1^3 - (-1)^2 = 1 - 1 = 0$
- $x = \sqrt[3]{3} \approx 1.4422,\ y = \sqrt{3} \approx 1.7321$:
  $(\sqrt[3]{3})^3 - (\sqrt{3})^2 = 3 - 3 = 0$

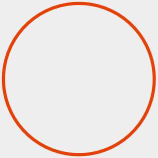


Plane Real Algebraic Curves (More Examples)

standard parabola

$x^2 - y$

circle

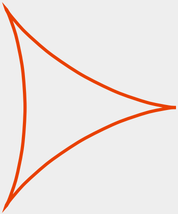


$x^2 + y^2 - 1$

Neil's parabola

$x^3 - y^2$

deltoid



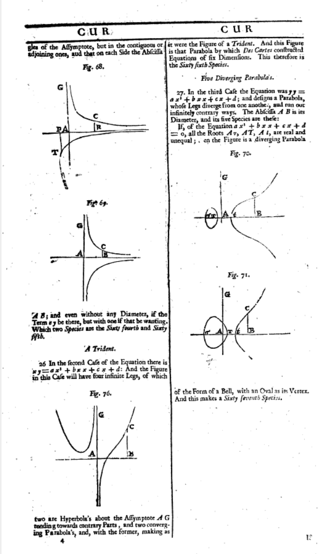
$(x^2 + y^2)^2 + 18(x^2 + y^2) - 27$

cardioid

$(x^2 + y^2 - 1)^2 - 4\big((x-1)^2 + y^2\big)$

Newton: Curves, Lexicon Technicum, London (1710)

Source: Google books



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gles of the Assymptote, but in the contiguous or adjoining ones, and that on each Side the Absciss *A B*; and even without any Diameter, if the Term *e y* be there, but with one if that be wanting. Which two *Species* are the *Sixty fourth* and *Sixty fifth*.

Fig. 68.

Fig. 69.

A Trident.

26 In the second Case of the Equation there is $x y = a x^3 + b x x + c x + d$: And the Figure in this Case will have four infinite Legs, of which two are Hyperbola's about the Assymptote *A* G tending towards contrary Parts, and two converging Parabola's, and, with the former, making as it were the Figure of a *Trident*. And this Figure is that Parabola by which *Des Cartes* constructed Equations of six Dimensions. This therefore is the *Sixty sixth Species*.

Fig. 76.

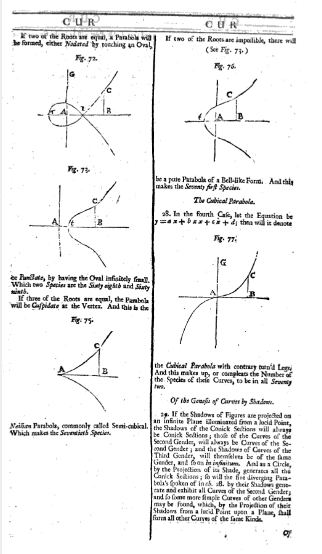
Five Diverging Parabola's.

27. In the third Case the Equation was $y y = a x^3 + b x x + c x + d$; and designs a Parabola, whose Legs diverge from one another, and run out infinitely contrary ways. The Absciss *A B* is its Diameter, and its five Species are these:

If, of the Equation $a x^3 + b x x + c x + d = 0$, all the Roots $A\tau$, AT, $A\,t$, are real and unequal; then the Figure is a diverging Parabola of the Form of a Bell, with an Oval as its Vertex. And this makes a *Sixty seventh Species*.

Fig. 70.

Fig. 71.



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If two of the Roots are equal, a Parabola will be formed, either *Nodated* by touching an Oval, or *Punctate*, by having the Oval infinitely small. Which two *Species* are the *Sixty eighth* and *Sixty ninth*.

Fig. 72.

Fig. 73.

If three of the Roots are equal, the Parabola will be *Cuspidate* at the Vertex. And this is the *Neilian* Parabola, commonly called Semi-cubical. Which makes the *Seventieth Species*.

Fig. 75.

If two of the Roots are impossible, there will (See *Fig.* 73.)

Fig. 76.

be a pure Parabola of a Bell-like Form. And this makes the *Seventy first Species*.

The Cubical Parabola.

28. In the fourth Case, let the Equation be $y = a x + b x x + c x + d$; then will it denote the *Cubical Parabola* with contrary turn'd Legs; And this makes up, or compleats the Number of the Species of these Curves, to be in all *Seventy two*.

Fig. 77.

Of the Genesis of Curves by Shadows.

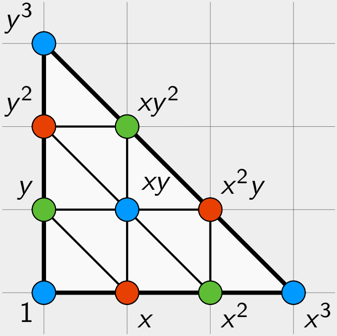
29. If the Shadows of Figures are projected on an infinite Plane illuminated from a lucid Point, the Shadows of the Conick Sections will always be Conick Sections; those of the Curves of the Second Gender, will always be Curves of the Second Gender; and the Shadows of Curves of the Third Gender, will themselves be of the same Gender, and so on *in infinitum*. And as a Circle, by the Projection of its Shade, generates all the Conick Sections; so will the five diverging Parabola's spoken of in *ch.* 28. by their Shadows generate and exhibit all Curves of the Second Gender; and so some more simple Curves of other Genders may be found, which, by the Projection of their Shadows from a lucid Point upon a Plane, shall form all other Curves of the same Kinds.

Of

The Recipe

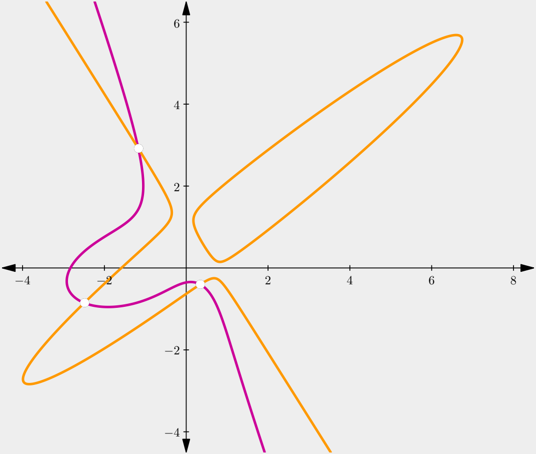
How to create a curve from a (special kind of) triangulation

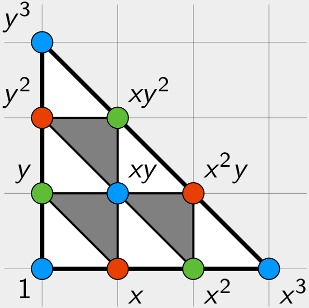
- take a convex lattice polygon P
- triangulate, using all interior lattice points
- **suppose that** these vertices can be 3-colored such that the two vertices of each edge receive distinct colors
- replace lattice point (i, j) by $x^i y^j$
- pick one real number per color
  - e.g., -12, 7, -30
- add up to form polynomial



$$-12(1 + x^3 + xy + y^3) + 7(x + x^2y + y^2) - 30(x^2 + y + xy^2)$$

Two Curves with Three Points of Intersection





number of white triangles
- number of black triangles
= 3

$$-12(1 + x^3 + xy + y^3) + 7(x + x^2y + y^2) - 30(x^2 + y + xy^2) = 0$$

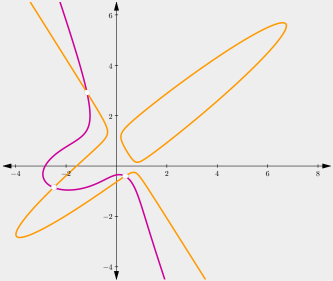
$$490(1 + x^3 + xy + y^3) - 890(x + x^2y + y^2) + 20(x^2 + y + xy^2) = 0$$

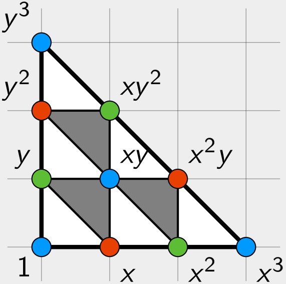
Lower Bounds for the Number of Points of Intersection

Theorem (Soprunova & Sottile, 2006)

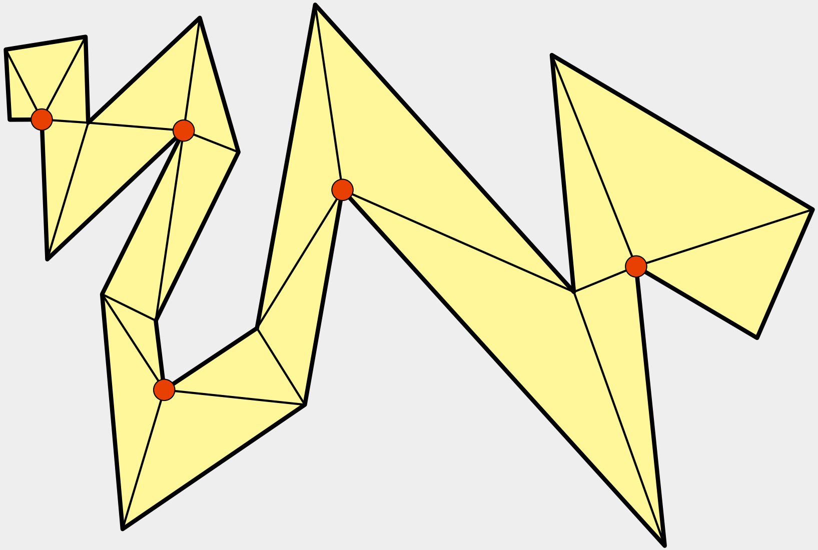
Let C and D be two curves constructed from a triangulation Δ according to the recipe.
Then C and D intersect in at least $\sigma(\Delta)$ many points, ...
provided that certain additional conditions are satisfied.

$$\sigma(\Delta) = |\#(\text{white triangles}) - \#(\text{black triangles})|$$

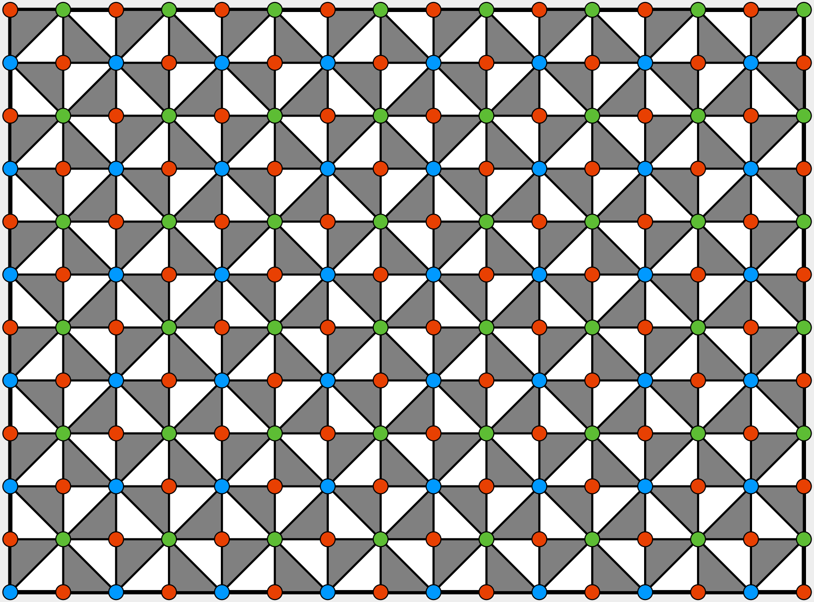




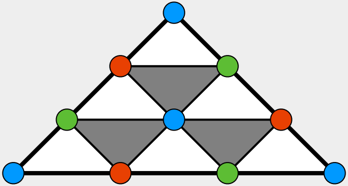
Recall: How to Guard a Museum



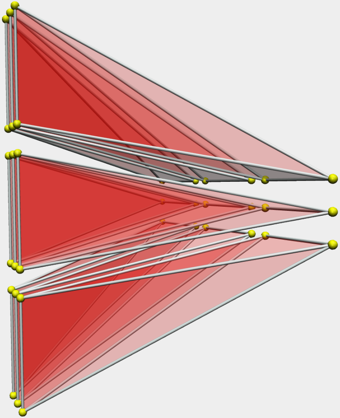
Chek Lap Kok Floor Pattern



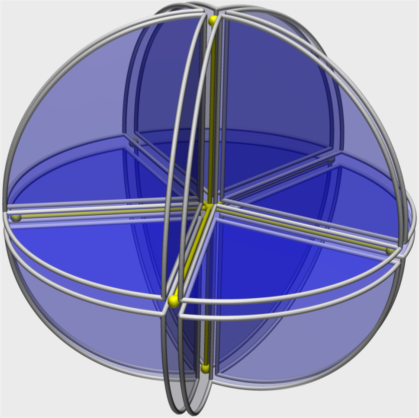
Chocolate



What Comes Next?



triangulation in 3 dimensions



secondary fan of $C_4 * C_5$

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The "Additional Conditions"

- $P \subset \mathbb{R}^d_{\geq 0}$: lattice d-polytope with N lattice points,
  Δ induced by height function λ **generic coefficients**

  $$\phi_P : (\mathbb{C}^\times)^d \to \mathbb{CP}^{N-1} : t \mapsto [t^v \mid v \in P \cap \mathbb{Z}^d] ,$$

  - toric variety $X_P =$ (Zariski) closure of image
  - real part $Y_P = X_P \cap \mathbb{RP}^{N-1}$, lift Y_P^+ to $\mathbb{S}^{N-1}$ must be oriented
- s-deformation $s.Y_P$ (for $s \in (0,1]$) = closure of the image of

  $$s.\phi_P : (\mathbb{C}^\times)^d \to \mathbb{CP}^{N-1} : t \mapsto [s^{\lambda(v)}\, t^v \mid v \in P \cap \mathbb{Z}^d]$$

- Wronski projection

  $$\pi : \begin{array}{ccc} \mathbb{CP}^{N-1} \setminus E & \to & \mathbb{CP}^d \\ [x_v \mid v \in P \cap \mathbb{Z}^d] & \mapsto & [\sum_{v \in c^{-1}(i)} x_v \mid i = 0, 1, \ldots, d\,] \end{array}$$

must avoid
$$E = \left\{ x \in \mathbb{CP}^{N-1} \mid \sum_{v \in c^{-1}(i)} x_v = 0 \quad \text{for } i = 0, 1, \ldots, d \right\}$$

S&S 2006