Tropical bisectors and Voronoi diagrams

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Tropical distances

The tropical distance between $a, b \in \mathbb{R}^{d+1}$ is

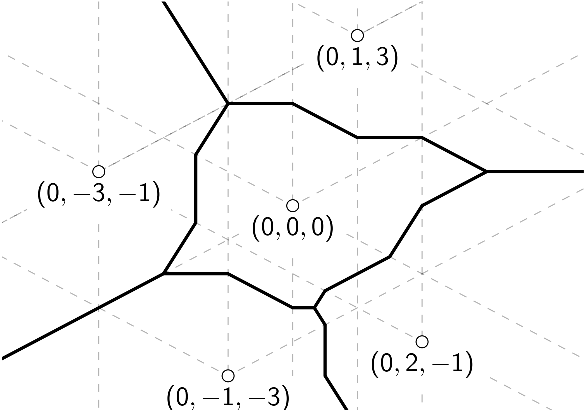
$$\begin{aligned} \operatorname{dist}(a,b) &= \max_{i\in[d+1]}(a_i - b_i) - \min_{j\in[d+1]}(a_j - b_j) \\ &= \max_{i,j\in[d+1]}(a_i - b_i - a_j + b_j) \ . \end{aligned}$$

- $\operatorname{dist}(a,b) \geq 0$
- $\operatorname{dist}(a,b) = \operatorname{dist}(b,a)$
- $\operatorname{dist}(a,b) + \operatorname{dist}(b,c) \geq \operatorname{dist}(a,c)$
- $\operatorname{dist}(\lambda a, \lambda b) = |\lambda| \operatorname{dist}(a,b)$ for all $\lambda \in \mathbb{R}$
- $\operatorname{dist}(a,b) = 0 \iff b - a \in \mathbb{R}\mathbf{1}$
- dist defines a metric on the *tropical projective torus* $\mathbb{R}^{d+1}/\mathbb{R}\mathbf{1} \approx \mathbb{R}^d$

(Tropical) Voronoi diagrams

Let $S \subset \mathbb{R}^{d+1}/\mathbb{R}\mathbf{1}$ be a finite set of *sites*.

- $\mathrm{Vor}_s(a) = \{\, x \mid \mathrm{dist}(x, a) \leq \mathrm{dist}(x, b)\ \forall b \neq a\}$ *Voronoi region*
- $\mathrm{Vor}(S)$ = decomposition of $\mathbb{R}^{d+1}/\mathbb{R}\mathbf{1}$ into Voronoi regions



Why should we care?

- Amini & Manjunath (2010): Riemann–Roch type theorem for A_n-sublattices; directed tropical distance function
- Alessandrini (2013); Jell, Scheiderer & Yu (2020): tropicalizing semi-algebraic sets
- Allamigeon, Benchimol, Gaubert & J (2018, 2021): complexity of linear programming
- Depersin, Gaubert & J (2017): tropical volume, tropical isodiametric inequality
- Loho & Schymura (2020): tropical Ehrhart theory
- Yoshida, Zhang & Zhang (2019): tropical principal component analysis
- Lin, Monod & Yoshida (2018+): tree spaces and phylogenetics

Unit balls and bisectors

Let dist be any metric on $\mathbb{R}^d$.

For

$$\mathbb{B}^d(a,r) \ := \ \left\{ x \in \mathbb{R}^d \ \middle| \ \mathrm{dist}(a,x) \leq r \right\}$$

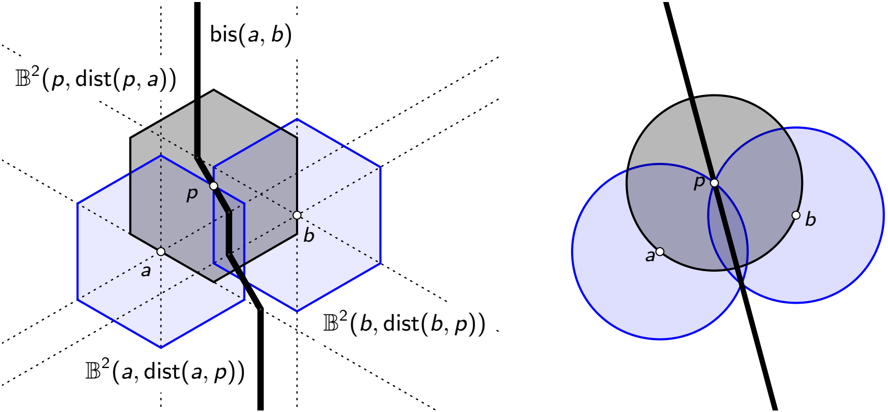
the set $\mathbb{B}^d := \mathbb{B}^d(0,1)$ is the *unit ball* with respect to dist.

Given a finite point set S, we define its *bisector*:

$$\mathrm{bis}(S) \ := \ \left\{ x \in \mathbb{R}^d \ \middle| \ \mathrm{dist}(a,x) = \mathrm{dist}(b,x) \text{ for } a,b \in S \right\} \ .$$

Bisectors of two points in the plane

left: tropical, right: Euclidean

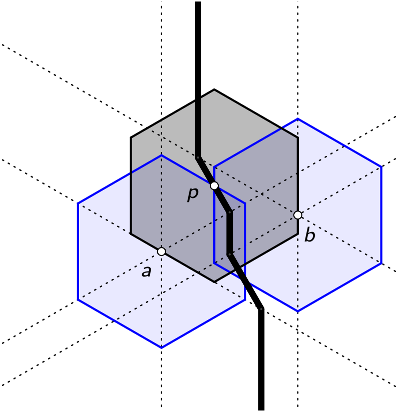


$\mathbb{R}^3/\mathbb{R}\mathbf{1} \approx \mathbb{R}^2$

Polyhedral norms

Let $K \subset \mathbb{R}^d$ be a convex body with $0 \in \operatorname{int} K$.

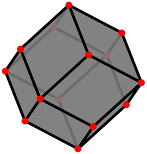
- $\operatorname{dist}_K(a, b) := \alpha$, such that
  $\alpha \geq 0$ and $b - a \in \partial(\alpha K)$
    - K is the unit ball with respect to dist_K
- if $K = -K$ then $\operatorname{dist}_K(0, \cdot)$ is a norm
- dist_K *convex distance function* $:\Longleftrightarrow$ K polytope
    - $\rightsquigarrow$ *polyhedral norm*



The tropical unit ball is a polytope

$$\begin{aligned}\mathbb{B}^d &= \left\{ x \in \mathbb{R}^{d+1}/\mathbb{R}\mathbf{1} \;\middle|\; \operatorname{dist}(x,0) \leq 1 \right\} \\ &= \bigcap_{i \neq j} \left\{ x \in \mathbb{R}^{d+1}/\mathbb{R}\mathbf{1} \;\middle|\; x_i - x_j \leq 1 \right\} \\ &= \frac{1}{2} \operatorname{conv}\left(\{\pm 1\}^{d+1} \setminus \{\pm \mathbf{1}\} \right) + \mathbb{R}\mathbf{1}\end{aligned}$$

- $d(d+1)$ facets
  - facet = choice of coordinates attaining maximum / minimum
- $2^{d+1} - 2$ vertices
  - vertex = (nontrivial) partition of coordinates into maxima / minima
- $\mathbb{B}^2$ = hexagon; $\mathbb{B}^3$ = rhombic dodecahedron
- zonotope



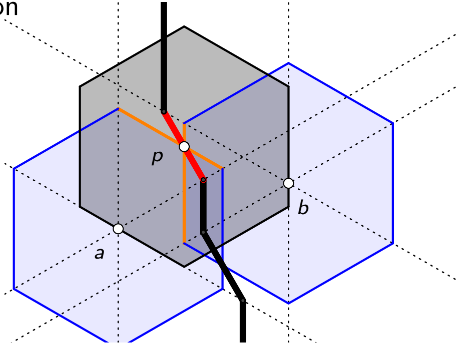
Polyhedral cells of bisectors

Let $K \subset \mathbb{R}^d$ be a polytope with $0 \in \operatorname{int} K$; *face fan* $\mathcal{F}(K)$.

- norm $\operatorname{dist}_K(0, \cdot)$ linear on each cone of $\mathcal{F}(K)$
- for $F : S \to \mathcal{F}(K)$ the set

$$\operatorname{bis}_F(S) \;=\; \operatorname{bis}(S) \cap \bigcap_{a \in S} a + F_a$$

is a polyhedron



Bisectors are polyhedral complexes

Let $K \subset \mathbb{R}^d$ be a polytope with $0 \in \operatorname{int} K$.

Proposition

The bisector $\operatorname{bis}(S)$ is a polyhedral complex with cells $\operatorname{bis}_F(S)$.

Proof.

We have

$$\operatorname{bis}_F(S) \cap \operatorname{bis}_G(S) \;=\; \operatorname{bis}_{F \cap G}(S) \;,$$

where $F \cap G : S \to \mathcal{F}(K)$, $a \mapsto F_a \cap G_a$.

Let $p \in \operatorname{bis}(S)$. Then for each $a \in S$ the point p lies in some face F_a of $p - \operatorname{dist}(a, p) \cdot K$. □

Weak and strong general position

Let $K \subset \mathbb{R}^d$ be a polytope with $0 \in \operatorname{int} K$.

A finite set $S \subset \mathbb{R}^d$ is in *(strong) general position*, if for every subset $T \subset S$ there are neighborhoods U_a of $a \in T$ such that for all $a' \in U_a$ and all maps $F : S \to \mathcal{F}(K)$ where F_a is a facet of K we have

$$\operatorname{bis}_F(T) = \emptyset \iff \operatorname{bis}_F(\{ a' \mid a \in T \}) = \emptyset \ .$$

The set S is in *weak general position* if $\operatorname{bis}(a, b)$ does not contain full-dimensional cells, for any distinct $a, b \in S$.

Proposition

The bisector of ℓ points in general position is either empty or pure of dimension $d + 1 - \ell$. In particular, the bisector of $d + 1$ points in general position is finite, and this is empty for more than $d + 1$ points.

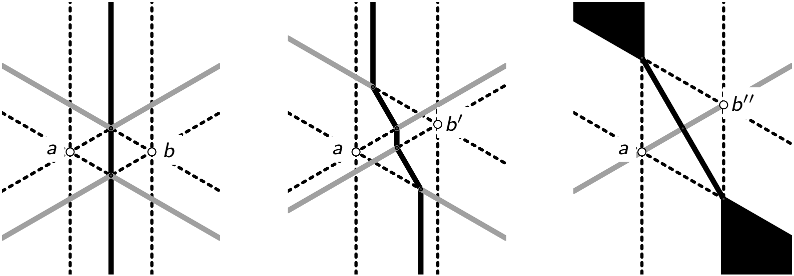
Tropical bisectors

Proposition

Let $a, b \in \mathbb{R}^{d+1}/\mathbb{R}\mathbf{1}$ be in weak general position. Then the homogeneous max-tropical Laurent polynomial

$$\phi(a,b) \; = \; \max\left(\max_{i,j\in[d+1]} (x_i - a_i - x_j + a_j), \max_{k,\ell\in[d+1]} (x_k - b_k - x_\ell + b_\ell) \right)$$

vanishes on the bisector $\mathrm{bis}(a,b)$.



Example: double circumcenters

tropical norm

in $\mathbb{R}^4/\mathbb{R}\mathbf{1} \approx \mathbb{R}^3$ consider $S = \{a, b, c, d\}$ where

$$a = (0, 2, 3, 3),\ b = (0, 4, 2, 2),\ c = (2, 4, 1, 1),\ d = (4, 0, 2, 2)$$

- $\mathrm{bis}(S)$ contains $x = (0, 0, 1, -1)$ and $y = (0, 0, -1, 1)$:
  - indeed, both x and y are at distance 4 from all the a_i's:

$$\begin{aligned} a - x &= (0, 2, 2, 4), & a - y &= (0, 2, 4, 2), \\ b - x &= (0, 4, 1, 3), & b - y &= (0, 4, 3, 1), \\ c - x &= (2, 4, 0, 2), & c - y &= (2, 4, 2, 0), \\ d - x &= (4, 0, 1, 3), & d - y &= (4, 0, 3, 1). \end{aligned}$$

- S not in weak general position, since contained in $\{x \mid x_3 - x_4 = 0\}$
- yet $\mathrm{bis}(S \pm \epsilon)$ still has at least two circumcenters
- hence there are sites in general position for which this happens

Topology of bisectors in polyhedral norms

Let $K \subset \mathbb{R}^d$ be a centrally symmetric d-polytope.

Theorem (Icking et al. (1999) for $d = 3$; Criado, J & Santos)

Let $a, b \in \mathbb{R}^d$ be in weak general position.
Then $\operatorname{bis}(a, b)$ *is homeomorphic to* $\mathbb{R}^{d-1}$.

Theorem (Criado, J & Santos)

Let $S = \{a, b, c\} \subset \mathbb{R}^d$ be in weak general position.

1. $\operatorname{bis}(S) = \emptyset$ *or pure of dimension* $d - 2$
   - *if $d = 2$ then either* $\operatorname{bis}(S) = \emptyset$ *or single point*
2. *if $H_S(a), H_S(b), H_S(c) \neq \emptyset$, then for $j \in \{0, \ldots, d-3\}$:*

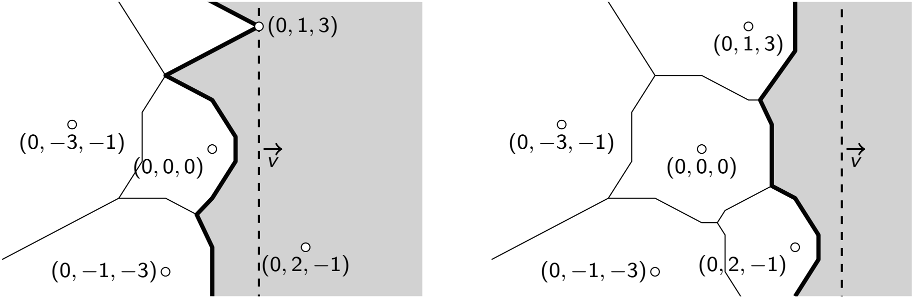
$$\tilde{\beta}_j(\operatorname{bis}(S)) \; = \; \tilde{\beta}_j(H_S(a)) + \tilde{\beta}_j(H_S(b)) + \tilde{\beta}_j(H_S(c))$$

Beach line algorithm

Fortune (1987)

Theorem (Criado, J & Santos)

The beach line algorithm computes a tropical Voronoi diagram of n sites in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$ *in* $O(n \log n)$ *time and* $O(n)$ *space.*



Polytrope partitions

Theorem (Criado, J & Santos)

Let $S \subset \mathbb{R}^{d+1}/\mathbb{R}\mathbf{1}$ be a finite set in weak general position. Then each tropical Voronoi region of S is the star convex union of finitely many (possibly unbounded) semi-polytropes.

Theorem (Criado, J & Santos)

There is a randomized incremental algorithm for computing tropical Voronoi diagrams of n sites in $\mathbb{R}^{d+1}/\mathbb{R}\mathbf{1}$ in general position with expected time complexity $O(n^d \log n)$ and space complexity $O(n^d)$, for d constant.

Conclusion

- topological results for bisectors in general polyhedral norms
- structural and algorithmic results on tropical Voronoi diagrams
- proof of concept implementation (and visualization) in `polymake`

Criado, Joswig & Santos: Tropical bisectors and Voronoi diagrams, *Found. Comput. Math.* (2021), published online: 07 Sep 2021