Moduli of Tropical Curves — Data and Algorithms

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Res. Math. Sci. (2015), **2**:4

❶ Lattice polygons

- Height functions, triangulations and tropical curves
- Moduli spaces
- Theoretical and computational results

❷ Computations

- Secondary fans
- `TOPCOM`, `Gfan` and `MPTOPCOM`

❸ The Data

- `polymake`
- `https://github.com/micjoswig/TropicalModuliData/`

Lattice Polygons and Height Functions

Let $P \subseteq \mathbb{R}^2$ be a lattice polygon with lattice points $A := P \cap \mathbb{Z}^2$.

Any (height) function $h : A \to \mathbb{R}$ yields

- regular subdivision Δ of A and [upper/lower hull]
- tropical polynomial [min/max]

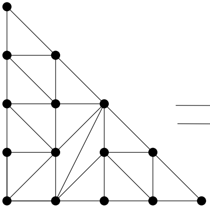
$$F(X,Y) \;:=\; \bigoplus_{(i,j)\in A} h(i,j) \odot X^{\odot i} Y^{\odot j}$$

With $g := \#(\operatorname{int} P \cap \mathbb{Z}^2)$ the tropical hypersurface $\mathcal{C} := \mathcal{T}(F)$ is a plane tropical curve of genus g.

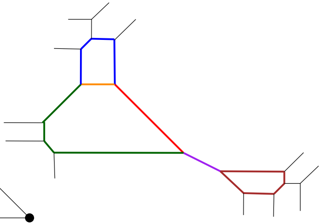
A natural length function on the edges turns $\mathcal{C}$ into a planar metric graph, which is dual Δ.

Unimodular Triangulation, Tropical Quartic, and Skeleton

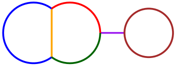
An example of genus $g = 3$



P, A and Δ



$\mathcal{C}$



G

(Berkovich) skeleton $G = \mathcal{C}$ without ends and nodes of degree 2

- $\mathcal{C}$ is smooth $\iff$ Δ is a unimodular triangulation
- in this case: G is a 3-regular plane multigraph of genus g with $2g-2$ nodes and $3g-3$ edges

Problem: for given Δ, A, G or g determine possible moduli = edge lengths

A Zoo of Moduli Spaces

$$
\begin{array}{ccccccccccc}
 & & & & \mathcal{M}_P & \subseteq & \mathcal{M}_g^{\text{planar}} & \subseteq & \mathcal{M}_g \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & \text{trop}(\mathcal{M}_P) & \subseteq & \text{trop}(\mathcal{M}_g^{\text{planar}}) & \subseteq & \text{trop}(\mathcal{M}_g) \\
 & & & & \cup\!\mid & & \cup\!\mid & & \parallel \\
\mathbb{M}_\Delta & \subseteq & \mathbb{M}_{P,G} & \subseteq & \mathbb{M}_P & \subseteq & \mathbb{M}_g^{\text{planar}} & \subseteq & \mathbb{M}_g
\end{array}
$$

- Abramovich, Caporaso & Payne 2015; ...;
  Chan, Gelatius & Payne (2020+); Allcock, Corey & Payne (2019+)
- Castryck & Voight 2009
- $\mathbb{M}_g^{\text{planar}} = \bigcup_P \mathbb{M}_P$ is a finite union
  - Scott 1976; Lagarias & Ziegler 1991
  - Koelmann, Haase & Schicho 2009; Castryck 2011: algorithm
  - Birkmeyer, Gathmann & Schmitz 2017: planarity test

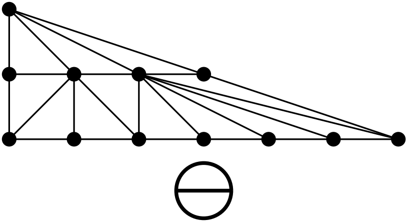
Very Small Genus

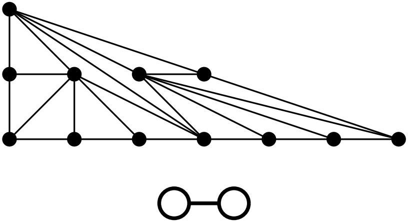
Genus 1 (elliptic curves)

- triangle $3\Delta_2$ with $g = 1$ interior lattice point
- skeleton G = circle; length = tropical j-invariant

Genus 2 (hyperelliptic curves ...)

- three triangulations of one triangle suffice
- all metric graphs are realizable as plane tropical curves





Theoretical Result

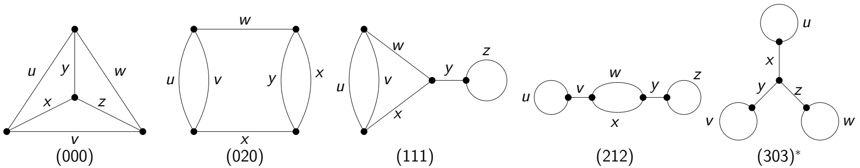
Theorem (Brodsky, J., Morrison & Sturmfels 2015)

For all $g \geq 2$ there exists a lattice polygon P with g interior lattice points and a unimodular triangulation Δ such that $\mathbb{M}_\Delta$ has the dimension

$$\dim(\mathbb{M}_g^{\text{planar}}) \quad = \quad \dim(\mathbb{M}_\Delta) \quad = \quad \begin{cases} 3 & \text{if } \ g = 2, \\ 6 & \text{if } \ g = 3, \\ 16 & \text{if } \ g = 7, \\ 2g+1 & \text{otherwise.} \end{cases}$$

(Semi-)Computational Result

Five trivalent planar graphs of genus 3



Theorem (Brodsky, J., Morrison & Sturmfels 2015)

A graph in $\mathbb{M}_3$ *arises from a smooth tropical quartic iff it is not of type* (303)*, with edge lengths satisfying, up to symmetry:*

(000) *realizable* $\iff$ $\max\{x, y\} \leq u$, $\max\{x, z\} \leq v$ *and* $\max\{y, z\} \leq w$, *where* ...

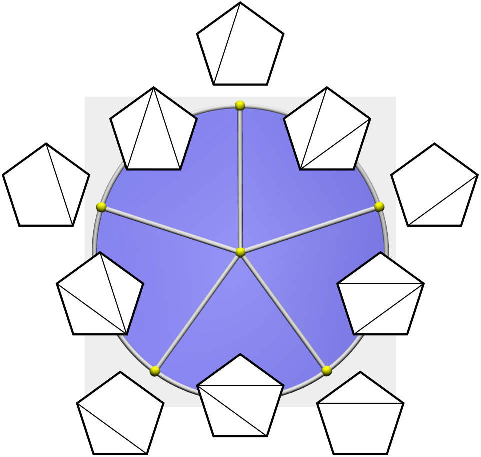
(020) *realizable* $\iff$ $v \leq u$, $y \leq z$, *and* ...

(111) *realizable* $\iff$ $w < x$ *and* ...

(212) *realizable* $\iff$ $w < x \leq 2w$.

Secondary Fans

Let $A \subset \mathbb{R}^d$ be configuration of n points.



- height functions inducing fixed subdivision form (relatively open) polyhedral cone
- considering all height functions yields polyhedral fan of dimension $n - d - 1$

Software for Computing Secondary Fans

`TOPCOM` 0.17.8 [Rambau 2000–2019]

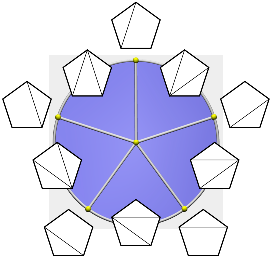
- reduce to oriented matroids
- (breadth-first) search through flip-graph

`Gfan` 0.6.2 [Jensen 2005–2017]

- (breadth-first) search through dual graph of secondary fan

`MPTOPCOM` 1.2 [Jordan, J. & Kastner 2019-2020]

- parallel reverse search via `mts` framework



Numbers and Dimensions of Moduli Cones

Non-hyperelliptic case

- Genus 3: 1278 (regular unimodular) triangulations (up to symmetry) of triangle T_4

$G \setminus$ dim	3	4	5	6	$\#\Delta$'s
(000)	18	142	269	144	573
(020)		59	216	175	450
(111)		10	120	95	225
(212)			15	15	30
total	18	211	620	429	1278

- Genus 4: three polygons with $5941 + 1278 + 20 = 7239$ triangulations
- Genus 5: four polygons with
  $147{,}908 + 968 + 508 + 162 = 149{,}546$ triangulations
- Coles & al. (2019+): skeleta for genus 6 and 7

switch to demo

https://github.com/micjoswig/TropicalModuliData/blob/main/2015-Moduli+of+tropical+plane+curves/demo.ipynb

Conclusion

- data available at

  `https://github.com/micjoswig/TropicalModuliData/`

  - with explanations and examples
  - use fork and pull-requests for building your own work on top

- future additions
  - database access via `polyDB` (note that this will not replace this repo as here we additionally have the original code and examples)
  - versioning and DOI