Tropical Convexity

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❶ The Structure Theorem of Tropical Convexity

- tropical polytopes vs. $(\max, +)$-linear algebra
- explicit computations

❷ A Tropical Fermat–Weber Problem

`arXiv:2205.00036`

- an asymmetric distance
- tropical median consenus trees

Terminology

$(\mathbb{T}, \oplus, \odot)$ = tropical semiring (with respect to min)

- $\mathbb{T} := \mathbb{R} \cup \{\infty\}$, $\oplus :=$ min and $\odot := +$

A set $S \subset \mathbb{R}^d$ is a tropical cone if

$$\lambda \odot x \oplus \mu \odot y \in S \quad \text{for all } x, y \in S \text{ and } \lambda, \mu \in \mathbb{R} \; .$$

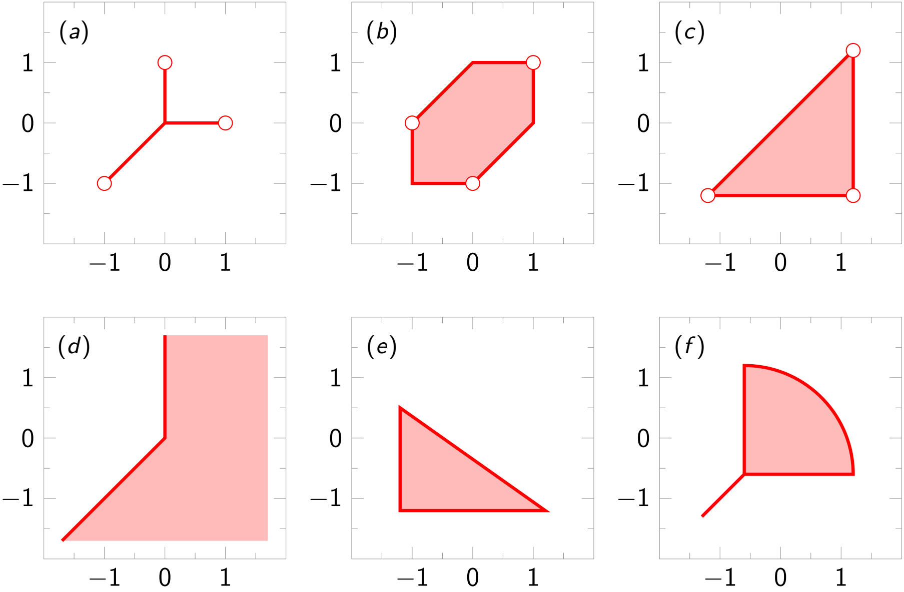
- tropical projective torus $\mathbb{R}^d/\mathbb{R}\mathbf{1}$

Definition (Develin & Sturmfels 2004)

A set $S' \subset \mathbb{R}^d/\mathbb{R}\mathbf{1}$ is tropically convex if it is the image of a tropical cone under the canonical projection $x \mapsto x + \mathbb{R}\mathbf{1}$.

- tropical polytope = finitely generated tropically convex set
- max-plus linear algebra:
  Cuninghame-Greene 1979, Baccelli et al. 2002, ...

Tropically convex sets in the plane $\mathbb{R}^3/\mathbb{R}\mathbf{1}$

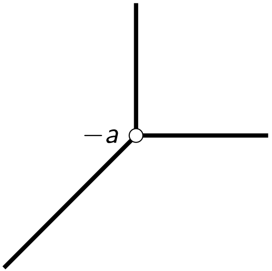


Tropical hyperplanes

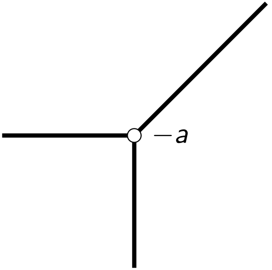
tropical linear form a with

$$a(x) \ = \ a_1 \odot x_1 \oplus \cdots \oplus a_d \odot x_d$$

$\mathbb{T}_{\min} = (\mathbb{R} \cup \{\infty\}, \min, +)$ $\mathbb{T}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$



$\mathcal{T}^{\min}(a)$



$\mathcal{T}^{\max}(a)$

- $\mathcal{T}^{\min}(a) = -\mathcal{T}^{\max}(-a)$

The structure theorem of tropical convexity

Let $V \in \mathbb{R}^{d \times n}$.

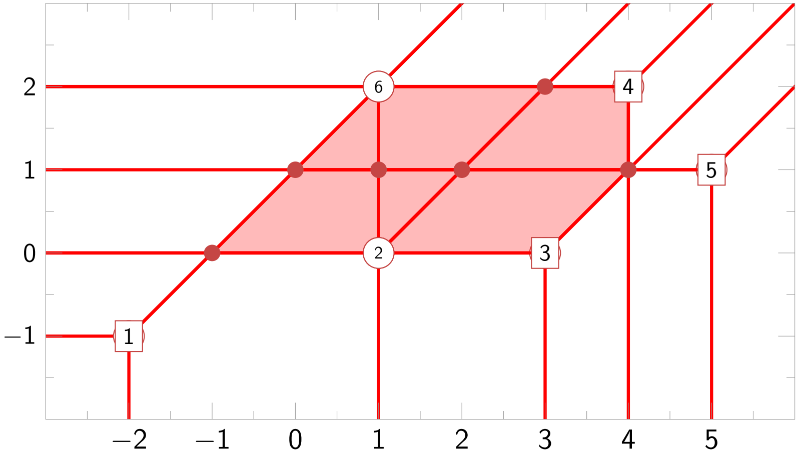
Theorem (Develin & Sturmfels 2004)

1. *The polyhedral decomposition* $\mathcal{S}_V$ *of* $\mathbb{R}^d/\mathbb{R}\mathbf{1}$*, which is formed by the regions of the* max*-tropical hyperplane arrangement* A_V*, is dual to the (lower) regular subdivision* $\Sigma(V)$*, where* V *is considered as a height function on the vertices of the ordinary polytope* $\Delta_{d-1} \times \Delta_{n-1}$*.*
2. *The* min*-tropical polytope* $\operatorname{tconv}(V)$ *agrees with the union of the bounded cells of the polyhedral complex* $\mathcal{S}_V$*.*

- Ardila & Develin 2007, Horn 2012: nonregular subdivisions
- Fink & Rincón 2015, J. & Loho 2016: $V \in \mathbb{T}^{d \times n}$

Example ($d = 3$, $n = 6$)

$$V = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 3 & 4 & 5 & 1 \\ -1 & 0 & 0 & 2 & 1 & 2 \end{pmatrix}$$



Computational classification

The known numbers of combinatorial types of regular triangulations of $\Delta_{d-1} \times \Delta_{n-1}$, up to symmetry:

$d \setminus n$	2	3	4	5	6	7
2	1	1	1	1	1	1
3		5	35	530	13 621	531 862
4			7 869	7 051 957		

- De Loera 1995: `PUNTOS`
- Rambau 2002–2020: `TOPCOM`
- Jordan, J. & Kastner 2018–2020: `MPTOPCOM`

Fermat–Weber sets

The asymmetric tropical distance in $\mathbb{R}^d\mathcal{H}$ is given by

$$\mathrm{dist}_\triangle(x,y) \;=\; \sum_{i\in[d]}(y_i - x_i) - d\min_{i\in[d]}(y_i - x_i) \;=\; \sum_{i\in[d]}(y_i - x_i) + d\max_{i\in[d]}(x_i - y_i) \;,$$

where $x, y \in \mathbb{R}^d\mathcal{H}$.

- restrict to $\mathcal{H} = \left\{ x \in \mathbb{R}^d \;\middle|\; \sum x_i = 0 \right\} \cong \mathbb{R}^d/\mathbb{R}\mathbf{1}$

Now pick finite subset $V \subset \mathcal{H}$.

Definition (asymmetric tropical Fermat–Weber set)

$$\mathrm{FW}(V) \;=\; \arg\min_{x\in\mathbb{R}^d} \textstyle\sum_{v\in V} \mathrm{dist}_\triangle(v,x)$$

- Lin & Yoshida 2018: symmetric tropical distance

Asymmetric tropical Fermat–Weber sets are tropical polytopes

Theorem (Comăneci & J. 2022+)

The Fermat–Weber set $\mathrm{FW}(V)$ *is dual to the cell of* $\Sigma(-V)$ *which contains* $(\frac{1}{d}\mathbf{1}, \frac{1}{n}\mathbf{1})$ *in its relative interior.*

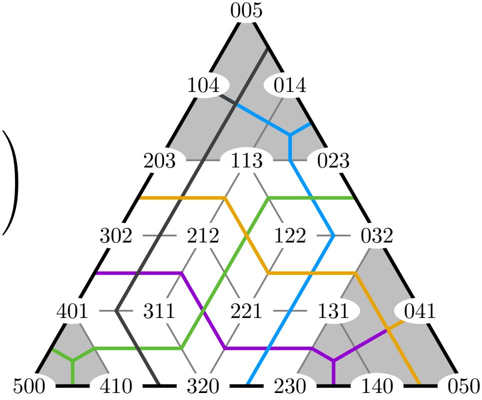
Moreover, $\mathrm{FW}(V)$ *is a bounded polytrope in* $\mathcal{H}$*, and it is contained in the max-tropical polytope* $-\operatorname{tconv}(-V)$.

- $(\frac{1}{d}\mathbf{1}, \frac{1}{n}\mathbf{1})$ = vertex barycenter of $\Delta_{d-1} \times \Delta_{n-1}$
- sharp upper bound for $\dim \mathrm{FW}(V)$
- efficient algorithms

Example ($d = 3$, $n = 5$)

$$V = \begin{pmatrix} 14 & 13 & 11 & 10 & 3 \\ -7 & -14 & -13 & 1 & -3 \\ -7 & 1 & 2 & -11 & 0 \end{pmatrix}$$

$$\mathrm{FW}(V) = \left\{(9, -6, -3)^\top\right\}$$



mixed subdivision of $5 \cdot \Delta_2$ (not fine)

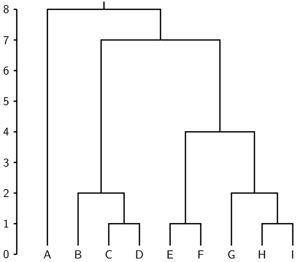
Ultrametrics and equidistant trees

- dissimilarity map = symmetric $\ell \times \ell$-matrix $D = (\delta_{ij})$ with $\delta_{ii} = 0$
- ultrametric if additionally $\delta_{ij} \geq 0$ and $\delta_{ik} \leq \max(\delta_{ij}, \delta_{jk})$ for all $i, j, k \in [\ell]$
- Billera, Holmes & Vogtmann 2001: space of equidistant trees $\mathcal{T}_\ell$
  - ultrametric $\iff$ equidistant tree
  - Lin, Sturmfels, Tang & Yoshida 2017: employ tropical convexity
- Ardila & Klivans 2006: D is an ultrametric $\iff$ D corresponds to a point in the Bergman fan of the complete graph K_n
  - $d = \binom{\ell}{2}$
- Speyer 2005: tropical linear spaces

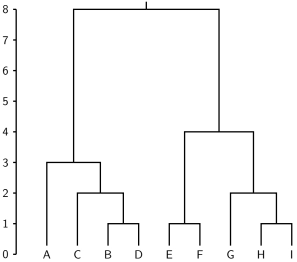
Theorem (Comăneci & J. 2022+)

Let $V \subset \mathcal{T}_\ell$ be finite.
Then the tropical polytope FW(V) *is contained in $\mathcal{T}_\ell$.*
Moreover, any two trees in FW(V) *share the same tree topology.*

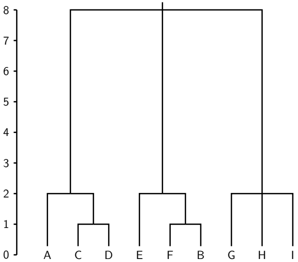
Three trees (a),(b),(c) and tropical median consensus (d)



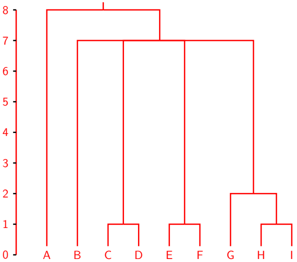
(a)



(b)



(c)



(d)

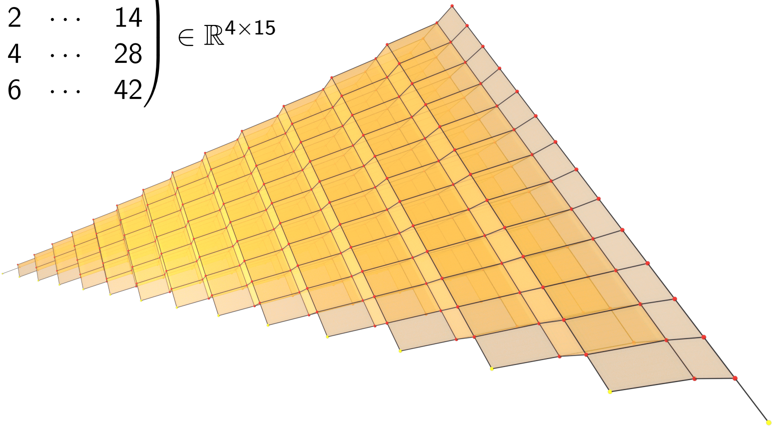
Conclusion

- tropical polytopes come with a natural cell decomposition:
  - gives access to methods from polyhedral combinatorics
  - key to understanding "extremal" examples

- tropical median consensus trees are nice:
  - fast algorithm via transportation
  - robust, Pareto and co-Pareto on triplets, ...

Epilogue

$$V = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 2 & \cdots & 14 \\ 0 & 2 & 4 & \cdots & 28 \\ 0 & 3 & 6 & \cdots & 42 \end{pmatrix} \in \mathbb{R}^{4\times 15}$$



Happy Birthday, Bernd!