In[1]:=
```
scp = Function[{f, g}, Integrate[f[x] * g[x], {x, 0, 1}]];
v[n_] = Function[x, Cos[2 * Pi * n * x]];
w[n_] = Function[x, Sin[2 * Pi * n * x]];
```

In[4]:=
```
l2norm = Function[f, Sqrt[scp[f, f]]]
```

Out[4]=
$$\text{Function}[f, \sqrt{\text{scp}[f, f]}]$$

In[5]:=
```
funcminus[f_, g_] = Function[x, f[x] - g[x]];
l2dist[f_, g_] = l2norm[funcminus[f, g]]
```

Out[6]=
$$\sqrt{\int_0^1 (f[x] - g[x])^2 \, dx}$$

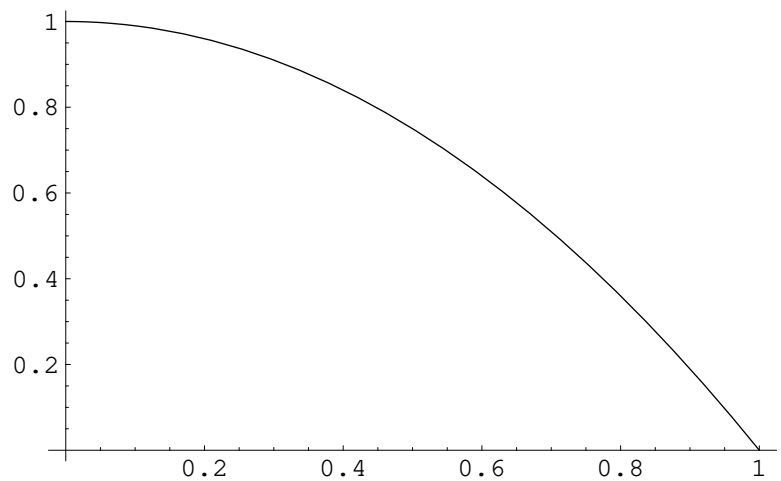
In[7]:=
```
vt[n_] = Function[x, v[n][x] / l2norm[v[n]]];
wt[n_] = Function[x, w[n][x] / l2norm[w[n]]];
```

In[9]:=
```
proj[n_, f_] = Function[x, scp[f, vt[0]] * vt[0][x] +
    Sum[scp[f, vt[k]] * vt[k][x] + scp[f, wt[k]] * wt[k][x], {k, 1, n}]]
```

Out[9]=
$$\text{Function}\left[x, \text{scp}[f, \text{vt}[0]]\, \text{vt}[0][x] + \sum_{k=1}^{n} (\text{scp}[f, \text{vt}[k]]\, \text{vt}[k][x] + \text{scp}[f, \text{wt}[k]]\, \text{wt}[k][x])\right]$$

In[10]:=
```
f = Function[x, 1 - x^2];
```

In[11]:=
```
Plot[f[x], {x, 0, 1}]
```

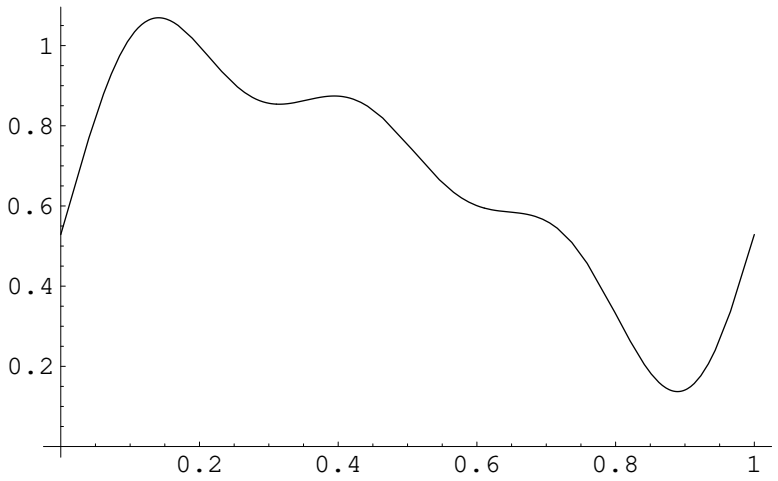


Out[11]=
```
- Graphics -
```

In[12]:=
```
f3 = proj[3, f]
```

Out[12]=
```
Function[x$, scp[Function[x, 1 - x^2], vt[0]] vt[0][x$] +
  Sum[(scp[Function[x, 1 - x^2], vt[k]] vt[k][x$] +
     scp[Function[x, 1 - x^2], wt[k]] wt[k][x$]), {k, 1, 3}]]
```

In[13]:=
```
Plot[f3[x], {x, 0, 1}]
```



Out[13]=
```
- Graphics -
```

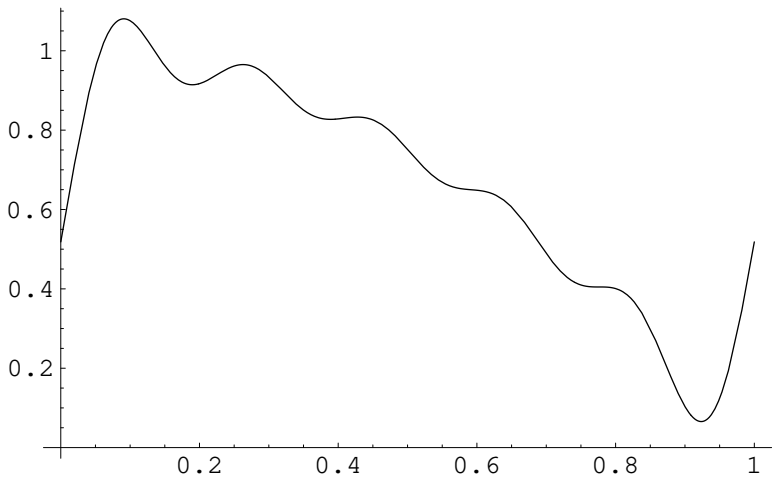
In[14]:=
```
N[l2dist[f3, f]]
```

Out[14]=
```
0.120071
```

In[15]:=
```
f5 = proj[5, f]
```

Out[15]=
```
Function[x$, scp[Function[x, 1 - x^2], vt[0]] vt[0][x$] +
  Sum[(scp[Function[x, 1 - x^2], vt[k]] vt[k][x$] +
     scp[Function[x, 1 - x^2], wt[k]] wt[k][x$]), {k, 1, 5}]]
```

In[16]:=
```
Plot[f5[x], {x, 0, 1}]
```



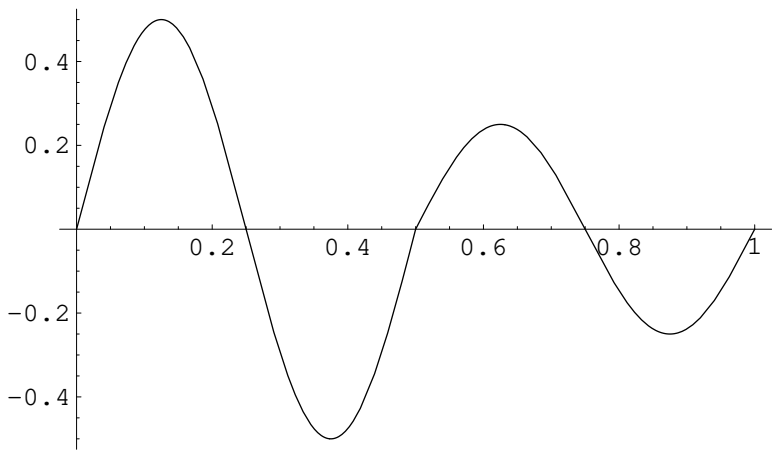
Out[16]=
```
- Graphics -
```

In[17]:=
```
N[l2dist[f5, f]]
```

Out[17]=
```
0.095896
```

In[18]:=
```
g = Function[x,
   If[x < 0.5, Sin[2 * Pi * x] * Cos[2 * Pi * x], 0.5 * Sin[2 * Pi * x] * Cos[2 * Pi * x]]];
```

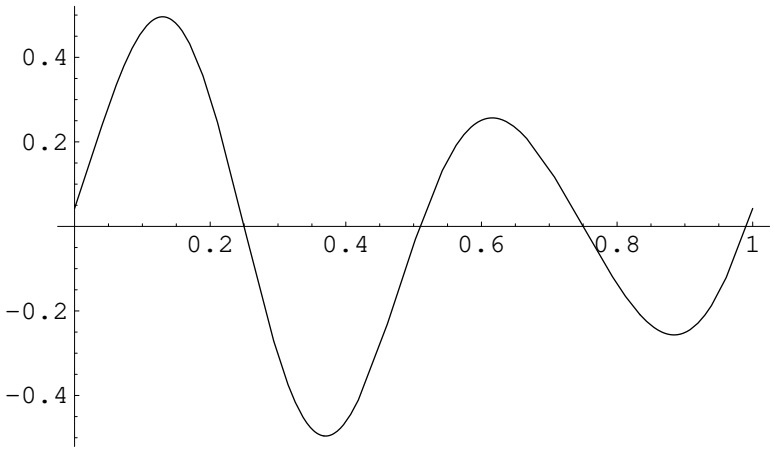
In[19]:=
```
Plot[g[x], {x, 0, 1}]
```



Out[19]=
```
- Graphics -
```

In[20]:=
```
g3 = proj[3, g];
```

In[21]:=
```
Plot[g3[x], {x, 0, 1}]
```



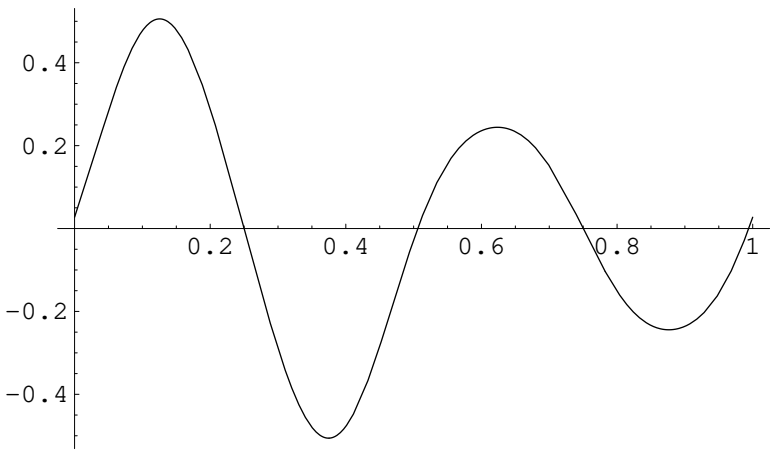
Out[21]=
```
- Graphics -
```

In[22]:=
```
N[l2dist[g3, g]]
```

Out[22]=
```
0.0125348
```

In[23]:=
```
g5 = proj[5, g];
```

In[24]:=
```
Plot[g5[x], {x, 0, 1}]
```



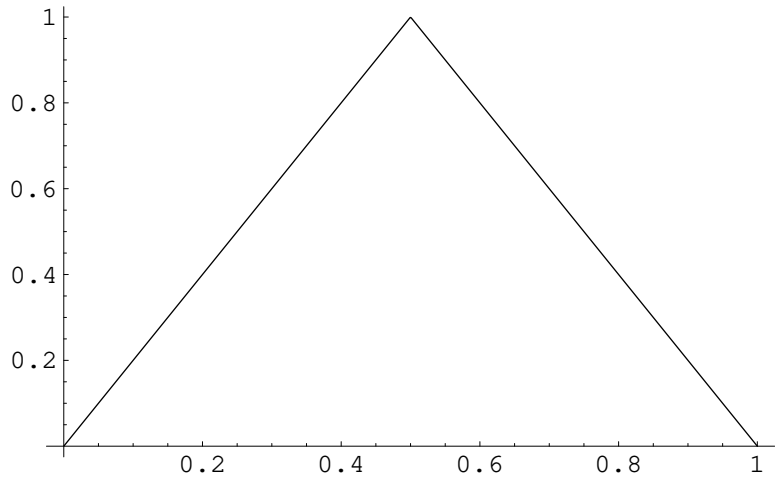
Out[24]=
```
- Graphics -
```

In[25]:=
```
N[l2dist[g5, g]]
```

Out[25]=
```
0.00649962
```

In[26]:=
```
h = Function[x, If[x < 0.5, 2 * x, 1 - 2 * (x - 0.5)]];
```

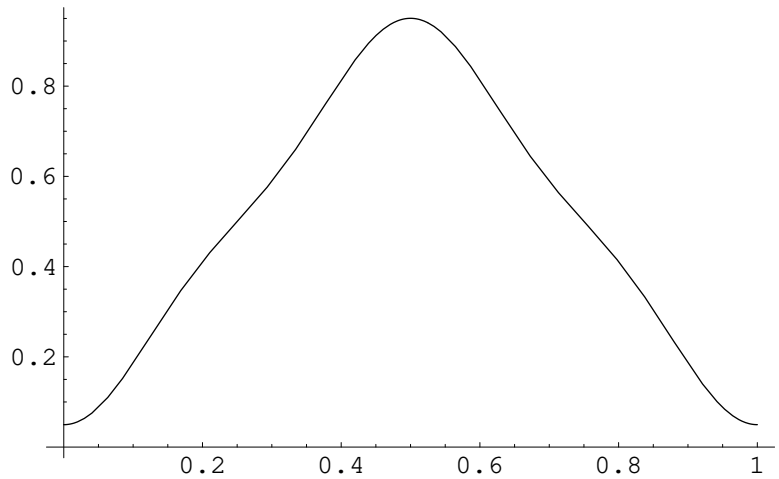
In[27]:=
```
Plot[h[x], {x, 0, 1}]
```



Out[27]=
```
- Graphics -
```

In[28]:=
```
h3 = proj[3, h];
```

In[29]:=
```
Plot[h3[x], {x, 0, 1}]
```



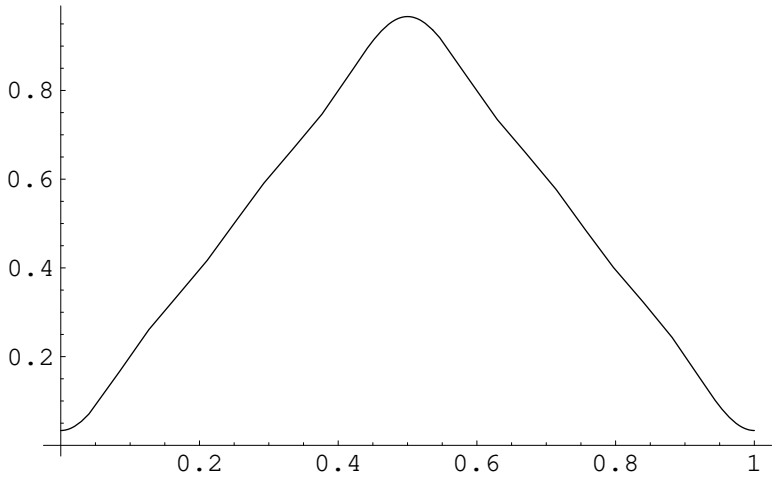
Out[29]=
```
- Graphics -
```

In[30]:=
```
N[l2dist[h3, h]]
```

Out[30]=
```
0.0138402
```

In[31]:=
```
h5 = proj[5, h];
```

In[32]:=
```
Plot[h5[x], {x, 0, 1}]
```



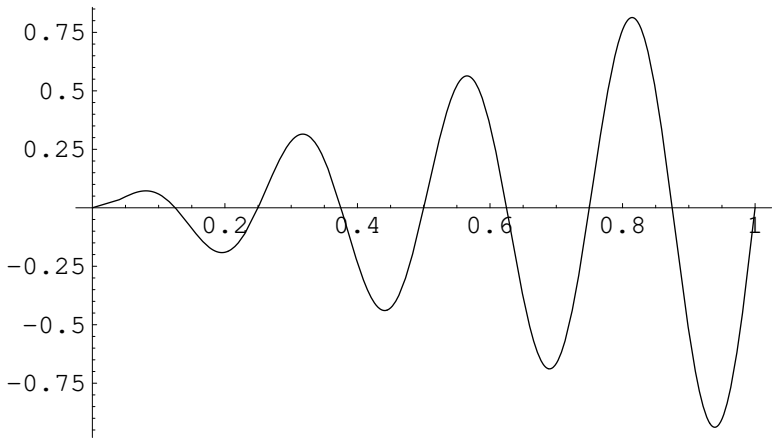
Out[32]=
```
- Graphics -
```

In[33]:=
```
N[l2dist[h5, h]]
```

Out[33]=
```
0.00775542
```

In[34]:=
```
s = Function[x, x * Sin[8 * Pi * x]];
```

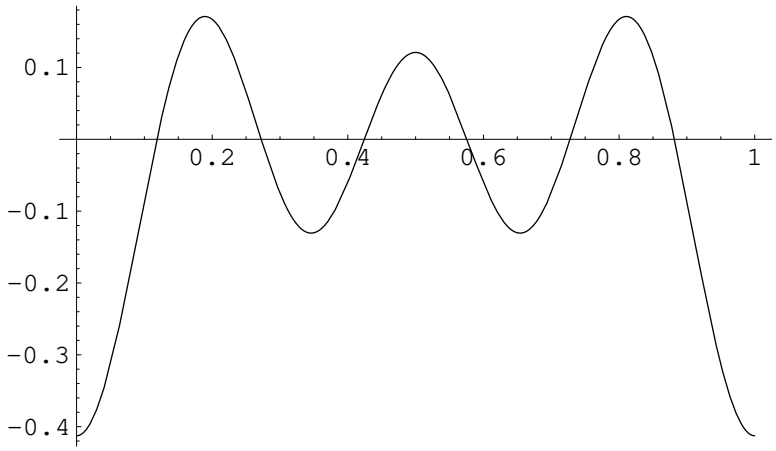
In[35]:=
```
Plot[s[x], {x, 0, 1}]
```



Out[35]=
```
- Graphics -
```

In[36]:=
```
s3 = proj[3, s];
```

In[37]:=
```
Plot[s3[x], {x, 0, 1}]
```



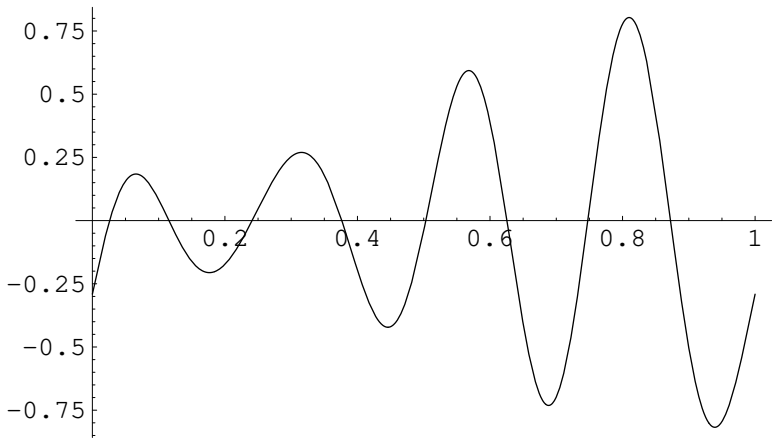
Out[37]=
```
- Graphics -
```

In[38]:=
```
N[l2dist[s3, s]]
```

Out[38]=
```
0.372712
```

In[39]:=
```
s5 = proj[5, s];
```

In[40]:=
```
Plot[s5[x], {x, 0, 1}]
```



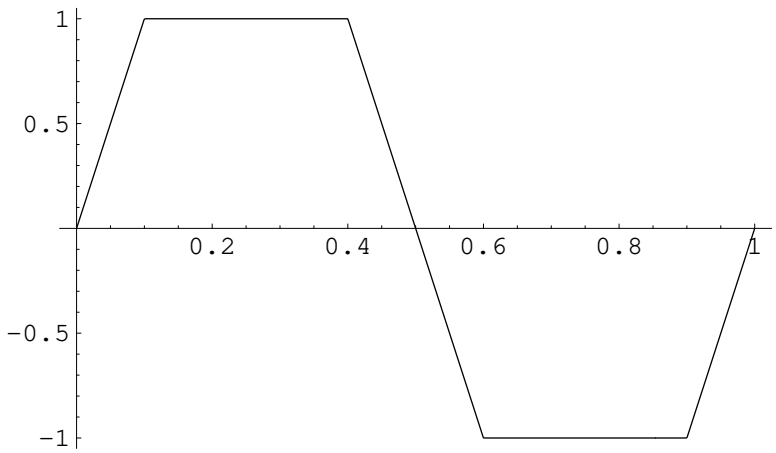
Out[40]=
```
- Graphics -
```

In[41]:=
```
N[l2dist[s5, s]]
```

Out[41]=
```
0.0609023
```

In[42]:=
```
r = Function[x, If[x < 0.1, 10 * x,
    If[x < 0.4, 1, If[x < 0.6, -10 * (x - 0.5), If[x < 0.9, -1, 10 (x - 1)]]]]];
```

In[43]:=
```
Plot[r[x], {x, 0, 1}]
```

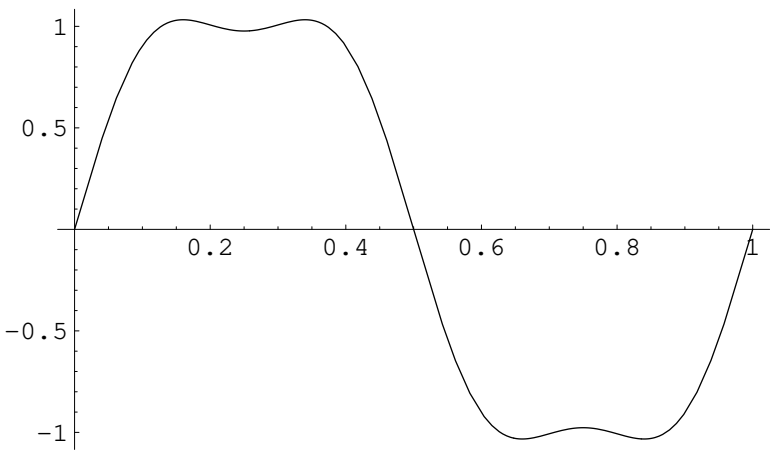


Out[43]=
```
- Graphics -
```

In[44]:=
```
r3 = proj[3, r];
```

In[45]:=
```
Plot[r3[x], {x, 0, 1}]
```

```
NIntegrate::slwcon :
 Numerical integration converging too slowly; suspect one
   of the following: singularity, value of the integration
   being 0, oscillatory integrand, or insufficient
   WorkingPrecision. If your integrand is oscillatory try
   using the option Method->Oscillatory in NIntegrate. More...

NIntegrate::ncvb :
 NIntegrate failed to converge to prescribed accuracy after
   7 recursive bisections in x near x = 0.09765625`. More...

NIntegrate::slwcon :
 Numerical integration converging too slowly; suspect one
   of the following: singularity, value of the integration
   being 0, oscillatory integrand, or insufficient
   WorkingPrecision. If your integrand is oscillatory try
   using the option Method->Oscillatory in NIntegrate. More...

NIntegrate::ncvb :
 NIntegrate failed to converge to prescribed accuracy after
   7 recursive bisections in x near x = 0.09765625`. More...

NIntegrate::ncvb :
 NIntegrate failed to converge to prescribed accuracy after
   7 recursive bisections in x near x = 0.90234375`. More...

General::stop : Further output of NIntegrate::ncvb will
   be suppressed during this calculation. More...

NIntegrate::slwcon :
 Numerical integration converging too slowly; suspect one
   of the following: singularity, value of the integration
   being 0, oscillatory integrand, or insufficient
   WorkingPrecision. If your integrand is oscillatory try
   using the option Method->Oscillatory in NIntegrate. More...

General::stop : Further output of NIntegrate::slwcon will
   be suppressed during this calculation. More...
```



Out[45]=
```
- Graphics -
```

In[46]:=
```
l2dist[r3, r]
```

```
Integrate::ilim :
 Invalid integration variable or limit(s) in {Integrate`Y/(2 π), 0, 1}. More...

Integrate::ilim :
 Invalid integration variable or limit(s) in {Integrate`Y/(2 π), 0, 1}. More...

Integrate::ilim :
 Invalid integration variable or limit(s) in {Integrate`Y/(2 π), 0, 1}. More...

General::stop : Further output of Integrate::ilim will
   be suppressed during this calculation. More...
```

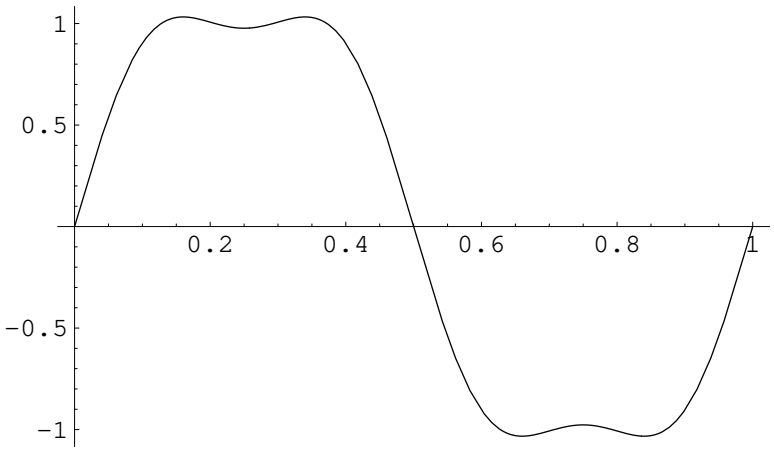
Out[46]=
```
0
```

In[47]:=
```
r5 = proj[5, r];
```

In[48]:=
```
Plot[r5[x], {x, 0, 1}]
```

```
NIntegrate::slwcon :
 Numerical integration converging too slowly; suspect one
   of the following: singularity, value of the integration
   being 0, oscillatory integrand, or insufficient
   WorkingPrecision. If your integrand is oscillatory try
   using the option Method->Oscillatory in NIntegrate. More...

NIntegrate::ncvb :
 NIntegrate failed to converge to prescribed accuracy after
   7 recursive bisections in x near x = 0.09765625`. More...

NIntegrate::slwcon :
 Numerical integration converging too slowly; suspect one
   of the following: singularity, value of the integration
   being 0, oscillatory integrand, or insufficient
   WorkingPrecision. If your integrand is oscillatory try
   using the option Method->Oscillatory in NIntegrate. More...

NIntegrate::ncvb :
 NIntegrate failed to converge to prescribed accuracy after
   7 recursive bisections in x near x = 0.09765625`. More...

NIntegrate::ncvb :
 NIntegrate failed to converge to prescribed accuracy after
   7 recursive bisections in x near x = 0.90234375`. More...

General::stop : Further output of NIntegrate::ncvb will
   be suppressed during this calculation. More...

NIntegrate::slwcon :
 Numerical integration converging too slowly; suspect one
   of the following: singularity, value of the integration
   being 0, oscillatory integrand, or insufficient
   WorkingPrecision. If your integrand is oscillatory try
   using the option Method->Oscillatory in NIntegrate. More...

General::stop : Further output of NIntegrate::slwcon will
   be suppressed during this calculation. More...
```

```
NIntegrate::ploss :
 Numerical integration stopping due to loss of precision. Achieved
   neither the requested PrecisionGoal nor AccuracyGoal; suspect
   one of the following: highly oscillatory integrand or the true
   value of the integral is 0. If your integrand is oscillatory
   try using the option Method->Oscillatory in NIntegrate. More…

NIntegrate::ploss :
 Numerical integration stopping due to loss of precision. Achieved
   neither the requested PrecisionGoal nor AccuracyGoal; suspect
   one of the following: highly oscillatory integrand or the true
   value of the integral is 0. If your integrand is oscillatory
   try using the option Method->Oscillatory in NIntegrate. More…

NIntegrate::ploss :
 Numerical integration stopping due to loss of precision. Achieved
   neither the requested PrecisionGoal nor AccuracyGoal; suspect
   one of the following: highly oscillatory integrand or the true
   value of the integral is 0. If your integrand is oscillatory
   try using the option Method->Oscillatory in NIntegrate. More…

General::stop : Further output of NIntegrate::ploss will
   be suppressed during this calculation. More…
```



Out[48]= - Graphics -