

ADDENDA: ESSENTIALS OF TROPICAL COMBINATORICS

MICHAEL JOSWIG

ABSTRACT. This document collects errors, comments and updates to the book “[Essentials of Tropical Combinatorics](#)”, published in 2021 by the American Mathematical Society as volume 219 of the series Graduate Studies in Mathematics.

CHAPTER 1: TROPICAL HYPERSURFACES

Theorem 1.13. The bijection β is between the *proper* faces of the dome $\mathcal{D}(F)$ and the bounded faces of the extended Newton polyhedron $\widetilde{\mathcal{N}}(F)$. Further, Claim (2) must read “Via orthogonal projection the bounded (or lower) faces of $\widetilde{\mathcal{N}}(F)$ correspond to the cells of the dual subdivision $\mathcal{S}(F)$ of $\text{supp}(F)$.”

CHAPTER 2: FIELDS OF POWER SERIES AND TROPICALIZATION

Section 2.2. I defined the polytope algebra as the semiring $(\mathfrak{P}_d, \oplus, \odot)$, where \oplus denotes the joint convex hull of polytopes in \mathbb{R}^d , and \odot is the Minkowski sum. This deviates from the definition in [McM89], which uses valuations on polytopes. The two notions are related but different. Thanks to Benjamin Schröter for pointing this out.

Theorem 2.11. The proof invokes Lemma 2.4 to show $\text{trop}(g)(Y) = F(p_1 + Y, p_2 + Y, \dots, p_d + Y)$. However, that claim is a direct consequence of the definition of $\text{trop}(g)$ and the properties of g . Thanks to Benjamin Schröter for pointing this out.

CHAPTER 4: PRODUCTS OF TROPICAL POLYNOMIALS AND THE CAYLEY TRICK

Lemma 4.25. There is a sign error. The final sentence of the claim should read: “In particular, $R(G)$ is a simplex of maximal dimension $m + d - 2$ if and only if G is a spanning tree of $K_{m,d}$.” Thanks to Andrei Comăneci for this correction.

Date: October 8, 2024.

CHAPTER 5: TROPICAL CONVEXITY

Section 5.1. After the definition of the stratum $\mathbb{T}^d(Z)$ it was claimed that $v \in \mathbb{T}^d(Z)$ if and only if $\text{supp}(v) \cap Z = \emptyset$. However, we have $v \in \mathbb{T}^d(Z)$ if and only if $\text{supp}(v)$ and Z partition the set $[d]$. So it is necessary to also require $\text{supp}(v) \cup Z = [d]$. Thanks to Benjamin Schröter.

Observation 5.5. The partial ordering defined on $\mathbb{L}_{\geq 0}^d$ is sometimes called “dominance ordering” in the literature; see, e.g., [FK11].

Proposition 5.22. This result requires the additional assumption that the tropically convex set is nonempty.

CHAPTER 6: COMBINATORICS OF TROPICAL POLYTOPES

Theorem 6.5. It is worth noting that the duality claimed and proved in Theorem 6.5 can also be viewed in the following slightly more abstract way. The envelope $\mathcal{E}(V)$ is an unbounded polyhedron, with vertices; so it is pointed. Hence $\mathcal{E}(V)$ is projectively equivalent to a bounded polytope, say P . The polytope P has a dual polytope Q ; see Appendix A.2. The dimension of P is the dimension of $\mathcal{E}(V)$, which is $d \cdot n$. In this way, the k -dimensional faces of P are in bijection with the $dn - k - 1$ -faces of Q . The duality in Theorem 6.5 now amounts to restricting the duality between P and Q to the faces in the boundary of $\mathcal{E}V$, seen as a subcomplex of the boundary ∂P . For instance, the case ($n = d = 2$) discussed in Example 6.1 and displayed in Fig. 6.1 is dual to a triangulation of the unit square into two triangles.

Figure 6.3. Left: The arc from the circled node 1 to the square node 2 should have weight -1 (instead of 0). This is the coefficient in the first row and second column of the matrix $V[G]$. Right: The nodes in the graph describing the projection should be squares (not circles), since they correspond to the rows of the matrix V . Thanks to Aenne Benjes.

Example 6.22. The graph on three nodes has arcs $(1, 2)$, $(2, 3)$, and $(3, 2)$, which is a subgraph of the graph in Figure 6.3 (right). The nodes correspond to the rows and columns of the square matrix $V \odot V[G]$ from Example 6.8. Thanks to John Sabol.

CHAPTER 7: TROPICAL HALF-SPACES

Equation (7.8) and below. The indices i and j lie in the set $[d] = \{1, 2, \dots, d\}$ (not in $[k]$). The same error occurs in the displayed formula before Equation (7.10). Thanks to Aenne Benjes.

CHAPTER 10: MATROIDS AND TROPICAL LINEAR SPACES

Definition 10.27. The middle term in Equation (10.12) must read $\pi(\rho + ik) + \pi(\rho + j\ell)$. Thanks to Florian Frickenstein.

Proposition 10.30. For any matrix $A \in \mathbb{T}^{d \times n}$ the map π_A is a tropical Plücker vector (or trivial), but not necessarily a finite one.

Theorem 10.36. In the proof of the implication '(3) \Rightarrow (1)' it must read $\rho \in \binom{[n]}{d-2}$ (rather than $\rho \in \binom{[n]}{d}$).

Section 10.7. For polynomials with real exponents I picked the term ‘‘Hahn polynomials’’ since they form special Hahn series. Yet that name is perhaps badly chosen as Hahn polynomials already have a meaning in the context of hypergeometric functions. See recent work of Geist and Miller [GM22] on real-exponent polynomial rings.

As another key example, it should have been pointed out that covector decompositions of tropical polytopes are precisely the tight spans of regular subdivisions of products of simplices.

Example 10.56. The tight span construction for general polyhedral subdivisions is erroneously attributed to [HJJS09]; in fact, this occurs in [HJ08].

Section 10.8. On line 2 of page 315 the domain of the map $\pi : \binom{[n]}{d} \rightarrow \{0, \infty\}$ is missing.

Example 10.69. It is not fully correct that this continues the Example 5.12. The latter discusses the tropical line segment spanned by the points $(0, 0, 1)$ and $(0, 1, 4)$, whereas Example 10.69 uses the points $(0, 1, 0)$ and $(0, 4, 1)$. These two pairs of points differ by swapping the second and third coordinates, which is an isometry (with respect to the tropical distance function) and which entails isomorphisms of the polyhedral complexes arising. Thanks to Ruriko Yoshida for pointing this out.

APPENDIX A.4 SECONDARY CONES AND SECONDARY FANS

The first sentence of the last paragraph must read ‘‘The set of all secondary cones forms a complete fan . . .’’. Thanks to Florian Frickenstein.

BIBLIOGRAPHY

Some articles marked as preprints in the book have now been published: [AHL20] \rightarrow [AHL21]; [BBRS20] \rightarrow [BBRS24]; [CE20] \rightarrow [CE21]; [CJS19] \rightarrow [CJS22]; [FO20] \rightarrow [FO22]; [JS20] \rightarrow [JS22]; [JSY18] \rightarrow [JSY22]; [LV19] \rightarrow [LV20]; [MRZ21] \rightarrow [MRZ22]

- [AHL21] Nima Arkani-Hamed, Thomas Lam, and Marcus Spradlin, *Positive configuration space*, *Comm. Math. Phys.* **384** (2021), no. 2, 909–954. MR 4259378 3
- [BBS24] Dominik Bendle, Janko Böhm, Yue Ren, and Benjamin Schröter, *Massively parallel computation of tropical varieties, their positive part, and tropical Grassmannians*, *J. Symbolic Comput.* **120** (2024), Paper No. 102224, 28. MR 4583112 3
- [CE21] Freddy Cachazo and Nick Early, *Minimal kinematics: An all k and n peek into $\text{Trop}^+ G(k, n)$* , *SIGMA Symmetry Integrability Geom. Methods Appl.* **17** (2021), Paper No. 078, 22. MR 4303678 3
- [CJS22] Francisco Criado, Michael Joswig, and Francisco Santos, *Tropical bisectors and Voronoi diagrams*, *Found. Comput. Math.* **22** (2022), 1923–1960. 3
- [FK11] Stefan Felsner and Kolja Knauer, *Distributive lattices, polyhedra, and generalized flows*, *European J. Combin.* **32** (2011), no. 1, 45–59. MR 2727459 2
- [FO22] Alex Fink and Jorge Alberto Olarte, *Presentations of transversal valuated matroids*, *J. Lond. Math. Soc. (2)* **105** (2022), no. 1, 24–62. MR 4411319 3
- [GM22] Nathan Geist and Ezra Miller, *Global dimension of real-exponent polynomial rings*, 2022, Preprint [arXiv:2109.04924](https://arxiv.org/abs/2109.04924). 3
- [HJ08] Sven Herrmann and Michael Joswig, *Splitting polytopes*, *Münster J. Math.* **1** (2008), 109–141. MR 2502496 3
- [HJS09] Sven Herrmann, Anders Jensen, Michael Joswig, and Bernd Sturmfels, *How to draw tropical planes*, *Electron. J. Combin.* **16** (2009), no. 2, Special volume in honor of Anders Björner, Research Paper 6, 26. MR 2515769 3
- [JS22] Michael Joswig and Benjamin Schröter, *Parametric shortest-path algorithms via tropical geometry*, *Math. Oper. Res.* **47** (2022), no. 3, 2065–2081. 3
- [JSY22] Philipp Jell, Claus Scheiderer, and Josephine Yu, *Real tropicalization and analytification of semialgebraic sets*, *Int. Math. Res. Not. IMRN* (2022), no. 2, 928–958, [Initially appeared as 2021, no. 24, 19178–19208]. MR 4366008 3
- [LV20] Georg Loho and László A. Végh, *Signed tropical convexity*, 11th Innovations in Theoretical Computer Science Conference, LIPIcs. Leibniz Int. Proc. Inform., vol. 151, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2020, pp. Art. No. 24, 35. MR 4048127 3
- [McM89] Peter McMullen, *The polytope algebra*, *Adv. Math.* **78** (1989), no. 1, 76–130. MR 1021549 (91a:52017) 1
- [MRZ22] Guido Montúfar, Yue Ren, and Leon Zhang, *Sharp bounds for the number of regions of maxout networks and vertices of Minkowski sums*, *SIAM J. Appl. Algebra Geom.* **6** (2022), no. 4, 618–649. MR 4521653 3

TECHNISCHE UNIVERSITÄT BERLIN, CHAIR OF DISCRETE MATHEMATICS/GEOMETRY, AND MAX-PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES, LEIPZIG
Email address: joswig@math.tu-berlin.de