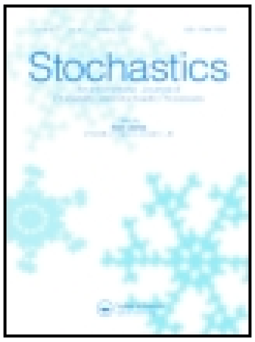

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Large Deviations from the McKean–Vlasov Limit for Weakly Interacting Diffusions

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A system of N diffusions on $\mathbb{R}^d$ in which the interaction is expressed in terms of the empirical measure is considered. The limiting behavior as $N \to \infty$ is described by a McKean–Vlasov equation. The purpose of this paper is to show that the large deviations from the McKean–Vlasov limit can be described by a generalization of the theory of Freidlin and Wentzell and to obtain a characterization of the action functional. In order to obtain this action functional we first obtain results on projective limits of large deviation systems, large deviations on dual vector spaces and a Sanov type theorem for vectors of empirical measures.

KEY WORDS: McKean–Vlasov limit, large deviations.

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1. INTRODUCTION

In this paper we consider a system of N diffusions on $\mathbb{R}^d$ interacting via their empirical measure which enters the drift vector. Such a system of *weakly interacting diffusions* can be described by N coupled Itô equations of the form

$$dx_k = \sigma(x_k)\, dw_k + b(x_k; \varepsilon_{\mathbf{x}})\, dt, \qquad k = 1, \ldots, N, \tag{1.1}$$

where $w_1, \ldots, w_N$ are independent Wiener processes and

$$\varepsilon_{\mathbf{x}} = N^{-1} \sum_{k=1}^{N} \delta_{x_k}$$

denotes the empirical measure of the particle configuration $\mathbf{x} = (x_1, \ldots, x_N) \in (\mathbb{R}^d)^N$. The random evolution (1.1) is associated with the N-particle generator

$$\mathscr{L}^{(N)} f(x_1, \ldots, x_N) = \sum_{k=1}^{N} \mathscr{L}_k(\varepsilon_{\mathbf{x}}) f(x_1, \ldots, x_N), \tag{1.2}$$

where $\mathscr{L}(\mu)$ is a diffusion operator on $\mathbb{R}^d$ with diffusion matrix $\{a^{ij}(x)\} = \sigma(x)\sigma^*(x)$ and drift vector $\{b^i(x; \mu)\} = b(x; \mu)$:

$$\mathscr{L}(\mu) = \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(\cdot)\, \partial^2/\partial x^i\, \partial x^j + \sum_{i=1}^{d} b^i(\cdot\,; \mu)\, \partial/\partial x^i.$$

In (1.2) the index k indicates that the corresponding operator acts on the variable x_k.

Given a solution $\mathbf{x}(t) = (x_1(t), \ldots, x_N(t))$ of (1.1), the associated *empirical process* $X_N(t) = \varepsilon_{\mathbf{x}(t)}$ is a Markov diffusion process with state space $\mathscr{M}$, the space of probability measures on $\mathbb{R}^d$. Several authors (see for example [17, 20, 25]) have studied the limiting behavior of such systems as the number of particles N tends to infinity. In particular, under quite general assumptions the sequence $(X_N(\cdot))$ of $\mathscr{M}$-valued processes was shown to converge in law to a deterministic $\mathscr{M}$-valued process $\mu(\cdot)$ provided that $X_N(0) \to \mu(0)$ (*McKean–Vlasov limit*). The measure-valued function $\mu(\cdot)$ can be characterized as a weak solution of the *McKean–Vlasov equation*

$$\dot{\mu}(t) = \mathscr{L}(\mu(t))^* \mu(t). \tag{1.3}$$

Here $\mathscr{L}(\mu)^*$ denotes the formal adjoint of $\mathscr{L}(\mu)$; $\dot{\mu}(t) = (d/dt)\mu(t)$.

The motivation for the present paper is to investigate long-time phenomena such as tunnelling and metastability for systems like (1.1). The appropriate approach to study such phenomena consists in an infinite dimensional generalization of the Freidlin–Wentzell theory [16]. Freidlin and Wentzell considered randomly perturbed (finite dimensional) dynamical systems in the case of small noisy disturbances. As an appropriate tool to investigate the long-time behavior of such systems, they developed a large deviation theory for them. Faris and Jona-Lasinio [15] have considered related questions for a nonlinear heat equation with Gaussian noise. In their model they obtain the large deviation results by representing the solution of the nonlinear system as the image under a continuous mapping of an infinite dimensional Ornstein–Uhlenbeck process and then using large deviation results for Gaussian processes due to Wentzell. Tanaka [31] has obtained a large deviation result for the empirical measures on $C([0, T]; \mathbb{R}^d)$ induced by a system of the form (1.1) in the special case in which $a^{ij}(x) = \delta_{ij}$, $b(x; \mu) = \int B(x, y)\mu(dy)$, and B is bounded and sufficiently smooth. In this case it is again possible to represent the solution of the nonlinear system by a continuous mapping from a system of N independent Brownian motions and then to use a result of Donsker and Varadhan. (See also the comment at the end of this section.)

The purpose of this paper is to build a framework in which to study large deviations from the McKean–Vlasov limit in the general case (1.1). In order to illustrate the analogy with the finite dimensional Freidlin–Wentzell theory, let us view $\mathscr{M}$ as an "infinite dimensional manifold". Then, in geometrical terms, the McKean–Vlasov equation (1.3) defines a dynamical system on $\mathscr{M}$ driven by the "vector field" $\mathscr{L}(\mu)^*\mu$. The Markov process X_N can be regarded as a random perturbation of this system. Indeed, the generator of X_N is of the form

$$G_N = G^{(1)} + N^{-1} G^{(2)},$$

where $G^{(1)}$ and $G^{(2)}$ act on "smooth" functions $F(\mu) = f(\langle \mu, g_1 \rangle, \ldots,$

$\langle\mu, g_r\rangle$) according to the formulae

$$G^{(1)}F(\mu)=\sum_{i=1}^{r}\frac{\partial f}{\partial x^i}(\langle\mu, g_1\rangle,\ldots,\langle\mu, g_r\rangle)\langle\mu, \mathscr{L}(\mu)g_i\rangle,$$

$$G^{(2)}F(\mu)=\frac{1}{2}\sum_{i,j=1}^{r}\frac{\partial^2 f}{\partial x^i\,\partial x^j}(\langle\mu, g_1\rangle,\ldots,\langle\mu, g_r\rangle)\langle\mu,(\nabla g_i, \nabla g_j)\rangle.$$

Here $\langle\mu, g\rangle=\int\mu(dx)g(x)$ and

$$(\nabla g, \nabla h)=\sum_{i,j=1}^{d} a^{ij}(\cdot)\frac{\partial g}{\partial x^i}\frac{\partial h}{\partial x^j}.$$

This means that $G^{(1)}$ is the generator of the (deterministic) flow on $\mathscr{M}$ caused by the "vector field" $\mathscr{L}(\mu)^*\mu$. The second order operator $G^{(2)}$ serves as an analogue of the second order partial differential operator in the theory of finite dimensional diffusions. Therefore, the investigation of weakly interacting diffusions for $N\to\infty$ leads to the study of a *weak noise limit* for Markov diffusions on $\mathscr{M}$.

Roughly speaking, our main result (Theorem 5.1) states that

$$\lim_{N\to\infty} N^{-1}\log \operatorname{Prob}(X_N(\cdot)\in A)=-\inf\{S(\mu(\cdot)):\mu(\cdot)\in A, \mu(0)=\nu\} \tag{1.4}$$

for all "regular" sets A of continuous paths $\mu(\cdot):[0, T]\to\mathscr{M}$ provided that $X_N(0)\to\nu$. The *action functional* S characterizes the difficulty of the passage of $X_N(t)$ near $\mu(t)$ in the time interval $[0, T]$. Indeed, according to (1.4), the probability of such a passage behaves like $\exp(-NS(\mu(\cdot)))$ as $N\to\infty$. The functional S will be shown to admit the representation

$$S(\mu(\cdot))=\int_0^T\|\dot\mu(t)-\mathscr{L}(\mu(t))^*\mu(t)\|^2_{\mu(t)}\,dt, \tag{1.5}$$

where

$$\|\vartheta\|^2_\mu=\frac{1}{2}\sup_f\frac{|\langle\vartheta, f\rangle|^2}{\langle\mu, |\nabla f|^2\rangle}$$

and

$$|\nabla f|^2 = \sum_{i,j=1}^{d} a^{ij}(\cdot)\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}.$$

By analogy with finite dimensional dynamical systems, nonlinear systems like (1.3) can exhibit a wide variety of qualitative behavior such as the existence of *multiple equilibria* (cf. [30]). To illustrate this, let us consider a *Curie–Weiss model* with continuous spin given by the N-particle Hamiltonian

$$\mathscr{H}_N(x_1, \ldots, x_N) = \sum_{k=1}^{N} U(x_k) + \frac{\theta}{4N}\sum_{k,l=1}^{N}(x_k - x_l)^2, \qquad (1.6)$$

$x_1, \ldots, x_N \in \mathbb{R}$, where $U(x) = x^4/4 - x^2/2$ is a double-well potential having two global minima and θ is a positive coupling constant ([9, 14]). The equilibrium distribution at temperature $\sigma^2 > 0$ is then given by

$$\pi(dx_1, \ldots, dx_N) = Z^{-1}\exp(-2\mathscr{H}_N(x_1, \ldots, x_N)/\sigma^2)\,dx_1 \ldots dx_N, \qquad (1.7)$$

where Z is a normalizing constant. A natural stochastic dynamics with prescribed equilibrium distribution (1.7) is defined by the Itô equations

$$dx_k = \sigma\,dw_k - \partial\mathscr{H}_N(x_1, \ldots, x_N)/\partial x_k\,dt, \qquad k = 1, \ldots, N. \qquad (1.8)$$

Since

$$\partial\mathscr{H}_N/\partial x_k = -(1-\theta)x_k + x_k^3 + \frac{\theta}{N}\sum_{l=1}^{N} x_l,$$

the stochastic model (1.8) can be viewed as a system of N anharmonic "oscillators" with internal noise and "mean-field" interaction. It fits in with our concept of weakly interacting diffusions for

$$d = 1, \quad \sigma(x) \equiv \sigma, \quad b(x; \mu) = -U'(x) - \theta\int(x-y)\mu(dy). \qquad (1.9)$$

The Curie–Weiss model (1.6)–(1.8) serves as a mean-field approximation for nearest neighbor Ising models. It exhibits a typical ferromagnetic behavior ([9], [7]). In [7] the associated McKean–Vlasov dynamics (1.3) was shown to undergo a bifurcation at a critical temperature σ_c^2. More precisely, for $\sigma \geqq \sigma_c$, (1.3) has exactly one equilibrium ν_0 which was shown to be globally stable and to correspond to mean magnetization zero. If σ becomes smaller than σ_c, then the stability of ν_0 will be lost and two new stable equilibria ν_+ and ν_- will appear which correspond, respectively, to positive and negative mean magnetization.

If N is large but finite, then the empirical process X_N will normally follow the path of the dynamical system (1.3) which is attracted by one of the equilibrium points ν_+ or ν_-. After that X_N will perform small fluctuations near this stable equilibrium. However, because of ergodicity, from time to time a transition from one stable equilibrium to the other one will occur via a large deviation. Such a transition is called a *tunnelling*. Following the ideas of Freidlin and Wentzell, we can introduce two quasipotentials which are defined by

$$Q_\pm(\nu) = \inf\{S_{T_1,T_2}(\mu(\cdot)) \colon -\infty \leqq T_1 < T_2 < \infty, \mu(T_1) = \nu_\pm, \mu(T_2) = \nu\},$$

where S_{T_1,T_2} stands for the functional (1.5) with integration over $[T_1, T_2]$ instead of $[0, T]$. The idea of using such quasipotentials will be developed in a future paper in which we will investigate the phenomenon of tunnelling between multiple stable equilibria for systems like (1.6)–(1.8) utilizing the relationship between the quasipotentials and the action functional associated to the equilibrium measures (1.7).

For Curie–Weiss models with two spins ± 1, the investigation of the associated measure-valued process leads to a large deviation problem for one-dimensional Markov jump processes (cf. [4]). On a formal level of rigour, Ruget (see [27]) investigated a slightly more complex situation by arranging the magnets (with spins ± 1) on a torus and allowing a "local" mean-field interaction which takes into account the geometrical structure of the system. This leads to an infinite dimensional extension of the Freidlin–Wentzell theory which is different from ours. Recently Comets [5] succeeded in carrying out Ruget's program with complete proofs.

We now sketch an outline of the development of our paper. Section 2 contains a list of frequently used symbols. In Section 3 we develop some abstract ideas concerning large deviation systems which may be of independent interest (projective limit of large deviation systems, large deviations on dual vector spaces). Relying on them, we prove a mild generalization of the Sanov theorem for vectors of empirical measures (Theorem 3.5). In Section 4 we study large deviations for the empirical process

$$X_N(t) = N^{-1} \sum_{k=1}^{N} \delta_{x_k(t)}$$

in which $x_1(t), \ldots, x_N(t)$ are independent temporally inhomogeneous diffusions on $\mathbb{R}^d$ all associated with the same time dependent diffusion operator (Theorem 4.5). That X_N satisfies the large deviation principle is essentially a straightforward consequence of our Sanov type theorem (see Lemma 4.6). But this theorem provides a rather abstract expression for the action functional. The greater part of Section 4 is therefore devoted to showing that the action functional admits a representation analogous to (1.5). Unfortunately we have not found a direct proof. For this reason, applying the results of Section 3, we will derive a second abstract expression for the action functional. After that the desired expression will be shown to be caught between these two. Many of the difficulties arising in this context originate in the fact that we allow the drift (and diffusion) coefficients to be strictly unbounded as in the case of the above Curie–Weiss model. In Section 5 we introduce the system of weakly interacting diffusions as a solution of a martingale problem equivalent to (1.1). As the main result of the paper we present the large deviation theorem for the associated empirical process (Theorem 5.1), cf. (1.4) and (1.5). The idea of its proof rests on the observation that locally, along a fixed path $\bar{\mu}(\cdot)$, the N-particle system (1.1) behaves for large N nearly as if it were a superposition of N independent copies of a diffusion process with diffusion matrix $\sigma(x)\sigma^*(x)$ and "frozen" drift vector $\bar{b}(x, t) = b(x; \bar{\mu}(t))$. This allows us to convert the large deviation result for independent diffusions into a local large deviation result for interacting diffusions. A combination of this local statement with some global exponential bounds then leads to the final result.

For systems like the above Curie–Weiss model, the approach of Tanaka [31] might also be applicable. But it seems to lead to similar difficulties as in our more general approach. First of all, since in (1.9) the drift coefficient $b(x;\mu)$ does not depend continuously upon μ in the topology of weak convergence of probability measures, the mapping which converts the empirical measure of N independent Wiener processes into the empirical process of our interacting system is not continuous in the topology of weak convergence of probability measures on $C([0,T];\mathbb{R})$. It is even not defined for all such measures. One must therefore work with a stronger topology which makes that mapping continuous and then derive the large deviation result for the empirical measures of the Wiener processes in this stronger topology! The action functional obtained in this way has essentially the same form as $S^{(1)}$ in Lemma 4.6 below. This again leads to the question how to derive the concrete representation (1.5) from such an abstract one.

2. FREQUENTLY USED NOTATION

$\bar{A}, \partial A$	Closure and boundary of the set A.
$a=\{a^{ij}(x,t)\}$	Diffusion matrix for independent diffusions (Section 4.2).
$a=\{a^{ij}(x)\}$	Diffusion matrix for weakly interacting diffusions (Section 5.1).
B_R	Ball in $\mathbb{R}^d$ with center 0 and radius R.
$b=\{b^i(x,t)\}$	Drift vector for independent diffusions (Section 4.2).
$b=\{b^i(x;\mu)\}$	Drift vector for weakly interacting diffusions (Section 5.1).
$C_b(X)$	Space of bounded continuous functions $X\to\mathbb{R}$ endowed with the topology of uniform convergence.
$C([0,T];X)$	Space of continuous functions $[0,T]\to X$. If X is a metric space, then $C([0,T];X)$ is equipped with the topology of uniform convergence.
C	$=C([0,T];\mathbb{R}^d)$.
$\mathscr{C}$	$=C([0,T];\mathscr{M})$.
$\mathscr{C}_R$	$=C([0,T];\mathscr{M}_R)$.
$\mathscr{C}_\infty$	$=C([0,T];\mathscr{M}_\infty)$ furnished with an "inductive" topology (Section 5.1).

$C_k^{2,1}$	$=C_k^{2,1}(\mathbb{R}^d\times[0,T])$. This is the space of continuous functions $\mathbb{R}^d\times[0,T]\to\mathbb{R}$ having compact support and being two times continuously differentiable with respect to the space variables and one times with respect to the time variable.
$\mathscr{D}$	Schwartz space of real test functions on $\mathbb{R}^d$.
$\mathscr{D}'$	Schwartz space of real distributions on $\mathbb{R}^d$.
$E_{x,s}$	Expectation with respect to $P_{x,s}$; $E_x=E_{x,0}$.
$\{\mathscr{L}_t; t\in[0,T]\}$	Family of diffusion operators on $\mathbb{R}^d$ (Section 4.2). (One-particle generator for a system of independent diffusions.)
$\{\mathscr{L}(\mu); \mu\in\mathscr{M}_\infty\}$	Family of diffusion operators on $\mathbb{R}^d$ (Section 5.1). (One-particle generator for weakly interacting diffusions.)
$\mathscr{L}^{(N)}$	N-particle generator for weakly interacting diffusions (Section 5.1).
$\mathscr{M}(X)$	Space of probability measures on a metric space X endowed with the Prokhorov metric (which induces on $\mathscr{M}(X)$ the weak topology).
$\mathscr{M}^{(N)}(X)$	Subspace of $\mathscr{M}(X)$ consisting of all empirical measures of N-particle configurations on X (Section 3.5).
$\mathscr{M}$	$=\mathscr{M}(\mathbb{R}^d)$.
$\mathscr{M}^{(N)}$	$=\mathscr{M}^{(N)}(\mathbb{R}^d)$.
$\mathscr{M}_R$	$=\{\mu\in\mathscr{M}:\int\varphi(x)\mu(dx)\leq R\}$ furnished with the subspace topology of $\mathscr{M}$.
$\mathscr{M}_\infty$	$=\{\mu\in\mathscr{M}:\int\varphi(x)\mu(dx)<\infty\}$ furnished with an "inductive" topology (Section 5.1).
$P_{x,s}$	Probability law on C of a diffusion starting at time s at x governed by $\{\mathscr{L}_t; t\in[0,T]\}$ (Section 4.2); $P_x=P_{x,0}$.
$P_{\mathbf{x},s}^{(N)}$	Probability law on $C([0,T];(\mathbb{R}^d)^N)$ of a system of N diffusions starting at time s at $\mathbf{x}$ (Sections 4.2 and 5.1); $P_{\mathbf{x}}^{(N)}=P_{\mathbf{x},0}^{(N)}$.
$\mathscr{P}_{\nu,s}^{(N)}$	Probability law on $\mathscr{C}$ (Section 4.2) or $\mathscr{C}_\infty$ (Section 5.1) of the empirical process starting at time s at ν associated to a system of N diffusions; $\mathscr{P}_\nu^{(N)}=\mathscr{P}_{\nu,0}^{(N)}$.
$\mathbb{R}^d$	d-dimensional Euclidean space; $\mathbb{R}=\mathbb{R}^1$.

S_ν, S	Action functional for the empirical process of independent (Section 4.2) and weakly interacting (Section 5.2) diffusions, respectively.
supp f	Support of the function f.
$\{U_{s,t}\}$	Semi-group of linear operators on $C_b(\mathbb{R}^d)$ generated by $\{P_{x,s}\}$ (Section 4.3).
δ_x	Dirac measure with unit mass at point x.
$\varepsilon_{\mathbf{x}}$	Empirical measure of the particle configuration $\mathbf{x}$ (Section 4.2).
φ	Lyapunov function for the system of weakly interacting diffusions (Section 5.1).
$\mathbb{1}_A$	Indicator function of the event A.
$\langle , \rangle$	Dual pairing: (i) If x belongs to a vector space and x^* to its algebraic dual, then $\langle x^*, x\rangle$ is the value of the linear functional x^* at x. (ii) If μ and f are a measure and a function on a measurable space, respectively, then $\langle \mu, f\rangle = \int f\, d\mu$ provided that the integral makes sense. (iii) If $f \in \mathscr{D}$ and $\vartheta \in \mathscr{D}'$, then $\langle \vartheta, f\rangle$ denotes the application of the test function f to the distribution ϑ.
$\langle\!\langle\ \rangle\!\rangle$	Quadratic characteristic (for martingales).
$\nabla_t, \| \ \|_t, (,)_t$	Gradient, norm, and inner product with respect to the Riemannian structure on $\mathbb{R}^d$ which is induced by the diffusion matrix $a(\cdot, t)$ (Section 4.2).
$\nabla, \| \ \|$	Gradient and norm with respect to the Riemannian structure on $\mathbb{R}^d$ which is induced by the time-independent diffusion matrix $a(\cdot)$ (Section 5.1).

3. ABSTRACT RESULTS ON LARGE DEVIATIONS

3.1 Large deviation systems

Let the following objects be given:

X a Hausdorff topological space;
(μ_N) a sequence of probability measures on X;
(γ_N) a sequence of positive numbers tending to infinity;
L a functional $X \to [0, \infty]$.

DEFINITION 3.1 (X, μ_N, γ_N) is said to be a *large deviation system* with *action functional* L, if the following conditions are satisfied:

i) for each open subset G of X

$$\liminf \gamma_N^{-1} \log \mu_N(G) \geqq -\inf_{x \in G} L(x); \tag{3.1}$$

ii) for each closed subset F of X

$$\limsup \gamma_N^{-1} \log \mu_N(F) \leqq -\inf_{x \in F} L(x); \tag{3.2}$$

iii) the sets $\Phi(s) = \{x \in X: L(x) \leqq s\}$, $s \geqq 0$, are compact.

(By convention, the infimum of the empty set equals $+\infty$.)

Some authors call the action functional *I-functional* (according to the notation of Donsker and Varadhan [10]), *rate function* [28], or *entropy function* [13]. For an introduction to the basic ideas of large deviation systems refer to [32] and for general properties of large deviation systems refer to [16, Chapter 3]. We only remark that the action functional is uniquely determined by (i)–(iii).

From (i) and (ii) we conclude that

$$\lim \gamma_N^{-1} \log \mu_N(A) = -\inf_{x \in A} L(x)$$

holds for all *regular* subsets A of X. Regularity of A means that A is a Borel set such that the infima of the action functional over the closure and the interior of A coincide. As a consequence of (iii), L is lower semi-continuous.

We mention that all assertions of this section have a straightforward extension to arbitrarily indexed families $\{\mu_\varepsilon; \varepsilon \in E\}$ and $\{\gamma_\varepsilon; \varepsilon \in E\}$, if convergence of sequences is replaced by convergence with respect to a filter on E.

3.2 The continuous image of a large deviation system

The following theorem is well known, at least for metrizable spaces ([16, Chapter 3, Theorem 3.1]). It is a simple consequence of our definitions.

THEOREM 3.2 *Let (X, μ_N, γ_N) be a large deviation system with action functional L. Let $f: X \to Y$ be a continuous map into a Hausdorff topological space Y and set $\nu_N = \mu_N \circ f^{-1}$. Then (Y, ν_N, γ_N) is a large deviation system with action functional*

$$L_f(y) = \min_{x \in f^{-1}(\{y\})} L(x), \qquad y \in Y.$$

3.3 The projective limit of large deviation systems

Let $(I, \leqq)$ be a right-filtering ordered set. Suppose that we are given a projective system (X_i, p_{ij}) of Hausdorff topological spaces, where p_{ii} is the identity map on X_i $(i \in I)$. Let $X = \varprojlim(X_i, p_{ij})$ be the corresponding limit (see e.g. [3]) and assume that it is non-empty. Denote by p_i the canonical projection $X \to X_i$ $(i \in I)$.

Let (μ_N) be a sequence of probability measures on X and define $\mu_{i,N} = \mu_N \circ p_i^{-1}$. Thus, for each N, μ_N can be regarded as a projective limit of the measures $\mu_{i,N}$, $i \in I$. Finally, let (γ_N) be a sequence of positive numbers with $\lim \gamma_N = \infty$.

THEOREM 3.3 *(X, μ_N, γ_N) is a large deviation system if and only if $(X_i, \mu_{i,N}, \gamma_N)$ is a large deviation system for each $i \in I$. The corresponding action functionals L and L_i are related to each other by*

$$L(x) = \sup_{i \in I} L_i(p_i(x)), \qquad x \in X, \tag{3.3}$$

and

$$L_i(z) = \min_{x \in p_i^{-1}(\{z\})} L(x), \qquad z \in X_i, \quad i \in I. \tag{3.4}$$

Proof (a) Suppose that (X, μ_N, γ_N) is a large deviation system with action functional L. Then Theorem 3.2 implies that $(X_i, \mu_{i,N}, \gamma_N)$ is a large deviation system with action functional (3.4) for each $i \in I$.

b) Suppose that $(X_i, \mu_{i,N}, \gamma_N)$ is a large deviation system with action functional L_i for each $i \in I$. Let L be defined by (3.3). We show that the conditions (i)–(iii) in Definition 3.1 are satisfied. Let $\Phi(s) = \{x \in X : L(x) \leqq s\}$ and define $\Phi_i(s)$, $i \in I$, accordingly. Applying

Theorem 3.2 to the continuous maps $p_{ij}: X_j \to X_i$ and noticing that $\mu_{i,N} = \mu_{j,N} \circ p_{ij}^{-1}$ $(i \leqq j)$, we see that $\{\Phi_i(s);\, i \in I\}$ is a projective system of subsets for each $s \geqq 0$. Together with (3.3) this gives

$$\Phi(s) = \underline{\lim}\, \Phi_i(s), \qquad s \geqq 0. \tag{3.5}$$

By assumption, the sets $\Phi_i(s)$, $i \in I$, $s \geqq 0$, are compact. Hence, by Tychonov's theorem, the sets $\Phi(s)$, $s \geqq 0$, are also compact. This proves condition (iii).

To check (i), let G be a non-empty open subset of X and choose $x \in G$ arbitrarily. Then we find some $i \in I$ and an open subset G_i of X_i such that $p_i(x) \in G_i$ and $p_i^{-1}(G_i) \subseteq G$. The assertion now follows from $\mu_N(G) \geqq \mu_{i,N}(G_i)$, $L(x) \geqq L_i(p_i(x))$, and

$$\liminf \gamma_N^{-1} \log \mu_{i,N}(G_i) \geqq -L_i(p_i(x)).$$

To verify condition (ii), let F be a non-empty closed subset of X and choose $s \geqq 0$ so that $F \cap \Phi(s) = \varnothing$. We need only show that

$$\limsup \gamma_N^{-1} \log \mu_N(F) \leqq -s. \tag{3.6}$$

Since F is closed, we have $F = \underline{\lim}\, \bar{F}_i$, where $\bar{F}_i$ denotes the closure of $F_i = p_i(F)$. Together with (3.5) this yields

$$F \cap \Phi(s) = \underline{\lim}\, \bar{F}_i \cap \Phi_i(s).$$

Since the sets $\bar{F}_i \cap \Phi_i(s)$, $i \in I$, are compact and $F \cap \Phi(s) = \varnothing$, there exists some $i \in I$ such that

$$\bar{F}_i \cap \Phi_i(s) = \varnothing$$

(see [3, Section 1.9.6]). Hence, applying (3.2) for the large deviation system $(X_i, \mu_{i,N}, \gamma_N)$, we obtain

$$\limsup \gamma_N^{-1} \log \mu_{i,N}(\bar{F}_i) \leqq -s.$$

Since $\mu_N(F) = \mu_{i,N}(\bar{F}_i)$, this yields (3.6). □

3.4 Large deviations on dual vector spaces

Let X be a real vector space and Y a non-empty subset of its algebraic dual X^*. We furnish X^* with the weak* topology $\sigma(X^*, X)$ induced by X (see e.g. [12]) and equip Y with the corresponding subspace topology. Let (μ_N) be a sequence of probability measures on Y, and let (γ_N) be a sequence of positive numbers with $\lim \gamma_N = \infty$. Denote by $\langle x^*, x \rangle$ the value of the linear functional $x^* \in X^*$ at point $x \in X$.

THEOREM 3.4 *Suppose that the following conditions are satisfied:*

i) for each $x \in X$, *the limit*

$$H(x) = \lim \gamma_N^{-1} \log \int_Y \exp(\gamma_N \langle x^*, x \rangle) \mu_N(dx^*)$$

exists and is finite;

ii) the function H *is Gâteaux differentiable, i.e. the real function* $t \mapsto H(x+ty)$ *is differentiable for every* $x, y \in X$.

Define

$$L(x^*) = \sup_{x \in X} [\langle x^*, x \rangle - H(x)], \qquad x^* \in X^*, \tag{3.7}$$

and suppose further that

iii) $\{x^* \in X^* : L(x^*) < \infty\} \subseteqq Y$.

Then (Y, μ_N, γ_N) *is a large deviation system. The corresponding action functional is the restriction of the functional* L *to* Y.

Proof (1) Suppose for the moment that the theorem is already proved in the case $Y = X^*$. Then it is also true in the general case $Y \subsetneqq X^*$. Indeed, let ν_N be the image of the measure μ_N with respect to the canonical injection $Y \to X^*$. Then we can apply Theorem 3.4 to the measures ν_N instead of μ_N in order to see that (X^*, ν_N, γ_N) is a large deviation system with action functional L. Taking into account assumption (iii), it then follows that (Y, μ_N, γ_N) is a large deviation system with action functional $L|_Y$. We may and will therefore assume in the following that $Y = X^*$.

2) The theorem holds if X is finite dimensional (see [19, Section 1] or [16, Chapter 5, Section 1]). Applying Theorem 3.3, we shall

reduce the infinite dimensional case to the finite dimensional one. Assume therefore that X is infinite dimensional and let V be an arbitrary finite dimensional linear subspace of X. Denote by q_V the canonical projection $X^* \to V^*$ and set $\mu_{V,N} = \mu_N \circ q_V^{-1}$. ($q_V(x^*)$ is the restriction of the linear functional $x^*: X \to \mathbb{R}$ to the domain V.) Assumption (i) yields

$$H(v) = \lim \gamma_N^{-1} \log \int_{V^*} \exp(\gamma_N \langle v^*, v \rangle) \mu_{V,N}(dv^*), \qquad v \in V.$$

Since V is finite dimensional, we can therefore apply Theorem 3.4 to the measures $\mu_{V,N}$ (instead of μ_N). Thus, $(V^*, \mu_{V,N}, \gamma_N)$ is a large deviation system with action functional

$$L_V(v^*) = \sup_{v \in V} [\langle v^*, v \rangle - H(v)], \qquad v^* \in V^*. \tag{3.8}$$

3) Let $\mathscr{V}$ be the system of all finite dimensional linear subspaces of X. $\mathscr{V}$ is right-filtering with respect to the order $\subseteq$. For $V \in \mathscr{V}$, we furnish the algebraic dual V^* with the weak* topology and set $X_V = V^*$. Given $V, W \in \mathscr{V}$ with $V \subseteq W$, denote by P_{VW} the canonical projection $W^* \to V^*$. Then (X_V, p_{VW}) is a projective system of topological spaces. Its projective limit can be topologically identified with X^* via the homeomorphism.

$$q: X^* \ni x^* \mapsto (x^*|_V)_{V \in \mathscr{V}} \in \varprojlim (X_V, p_{VW}).$$

Furthermore, for each N, $\mu_N \circ q^{-1}$ is the projective limit of the projective family $(\mu_{V,N})_{V \in \mathscr{V}}$ of probability measures. We can therefore apply Theorem 3.3 in order to check that (X^*, μ_N, γ_N) is a large deviation system with action functional

$$L(x^*) = \sup_{V \in \mathscr{V}} L_V(q_V(x^*)), \qquad x^* \in X^*. \tag{3.9}$$

4) It remains to show that the functional (3.9) coincides with (3.7). But this easily follows from (3.8) and the definition of q_V. □

In the case when X is the space of bounded continuous functions on a Polish space E and Y is the space of probability measures on E, a result similar to Theorem 3.4 was stated in [19] and has been applied to the study of large deviations for the occupation time measure and local times of Markov processes [18].

3.5 A Sanov type theorem

Let $X, Y_1, \ldots, Y_r$ be Polish spaces. Denote by $C_b(X)$ the space of bounded continuous functions on X equipped with the supremum norm. Let $\mathcal{M}(X)$ be the space of probability measures on X endowed with the topology of weak convergence. Given a natural number N, set

$$\mathcal{M}^{(N)}(X) = \left\{ N^{-1} \sum_{k=1}^{N} \delta_{x_k} : x_1, \ldots, x_N \in X \right\},$$

where δ_x is the Dirac measure on X with unit mass at x. Let $\{P_x; x \in X\}$ be a Feller continuous family of probability measures on $Y = Y_1 \times \cdots \times Y_r$. (Feller continuity means that the integral $\int F(y) P_x(dy)$ depends continuously on x for each $F \in C_b(Y)$.) Given $N \geqq 1$ and

$$\nu = N^{-1} \sum_{k=1}^{N} \delta_{x_k} \in \mathcal{M}^{(N)}(X), \tag{3.10}$$

denote by $Q_\nu^{(N)}$ the probability law of the vector of empirical measures

$$\left(N^{-1} \sum_{k=1}^{N} \delta_{y_k^{(1)}}, \ldots, N^{-1} \sum_{k=1}^{N} \delta_{y_k^{(r)}} \right)$$

under $P_{x_1} \otimes \cdots \otimes P_{x_N}$, where $y_k = (y_k^{(1)}, \ldots, y_k^{(r)})$. More precisely, $Q_\nu^{(N)}$ is the image of the measure $P_{x_1} \otimes \cdots \otimes P_{x_N}$ with respect to the map

$$Y^N \ni (y_1, \ldots, y_N) \mapsto$$

$$\left(N^{-1} \sum_{k=1}^{N} \delta_{y_k^{(1)}}, \ldots, N^{-1} \sum_{k=1}^{N} \delta_{y_k^{(r)}} \right) \in \mathcal{M}(Y_1) \times \cdots \times \mathcal{M}(Y_r).$$

The measure $Q_\nu^{(N)}$ does not depend upon the concrete representation (3.10) of ν.

The next theorem is a mild generalization of Sanov's theorem for empirical measures on Polish spaces (cf. [6, 10, 21]). Sanov's result corresponds to $r=1$ and $P_x \equiv P$. We will present here a short proof relying on Theorem 3.4. In principle it must also be possible to derive this generalization along the lines of [10, 19, or 21].

THEOREM 3.5 *Given* $\nu_N \in \mathcal{M}^{(N)}(X)$, $N \geq 1$, *and* $\nu \in \mathcal{M}(X)$, *suppose that* $\nu_N \to \nu$ *in* $\mathcal{M}(X)$. *Then* $(\mathcal{M}(Y_1) \times \cdots \times \mathcal{M}(Y_r), Q_{\nu_N}^{(N)}, N)$ *is a large deviation system with action functional*

$$L_\nu(\mu_1, \ldots, \mu_r) = \sup_{(f_1, \ldots, f_r) \in C_b(Y_1) \times \cdots \times C_b(Y_r)} \left[\sum_{i=1}^{r} \int_{Y_i} \mu_i(dz) f_i(z) - \int_X \nu(dx) \log \int_Y P_x(dy^{(1)}, \ldots, dy^{(r)}) \exp\left(\sum_{i=1}^{r} f_i(y^{(i)}) \right) \right], \tag{3.11}$$

$$(\mu_1, \ldots, \mu_r) \in \mathcal{M}(Y_1) \times \cdots \times \mathcal{M}(Y_r).$$

Proof For $i=1, \ldots, r$, we will view $\mathcal{M}(Y_i)$ as a topological subspace of $C_b(Y_i)^*$, the algebraic dual of $C_b(Y_i)$ endowed with the weak* topology. Set $E = C_b(Y_1) \times \cdots \times C_b(Y_r)$ and equip its algebraic dual E^* with the weak* topology. Since the topological vector space E^* can be identified with $C_b(Y_1)^* \times \cdots \times C_b(Y_r)^*$, $\mathcal{M}(Y_1) \times \cdots \times \mathcal{M}(Y_r)$ can and will be viewed as a topological subspace of E^*.

Using the representation

$$\nu_N = N^{-1} \sum_{k=1}^{N} \delta_{x_k^{(N)}}$$

and the Feller continuity of $\{P_x; x \in X\}$ and remembering the definition of $Q_{\nu_N}^{(N)}$, we obtain for all $(f_1, \ldots, f_r) \in C_b(Y_1) \times \cdots \times C_b(Y_r)$:

$$H_\nu(f_1, \ldots, f_r) := \lim N^{-1} \log \int Q_{\nu_N}^{(N)}(d\mu_1, \ldots, d\mu_r) \times \exp\left(N \sum_{i=1}^{r} \langle \mu_i, f_i \rangle \right)$$

$$= \lim N^{-1} \log \prod_{k=1}^{N} \int_Y P_{x_k^{(N)}}(dy_k^{(1)}, \ldots, dy_k^{(r)}) \exp\left(\sum_{i=1}^{r} f_i(y_k^{(i)})\right)$$

$$= \lim \int \nu_N(dx) \log \int P_x(dy^{(1)}, \ldots, dy^{(r)}) \exp\left(\sum_{i=1}^{r} f_i(y^{(i)})\right)$$

$$= \int \nu(dx) \log \int P_x(dy^{(1)}, \ldots, dy^{(r)}) \exp\left(\sum_{i=1}^{r} f_i(y^{(i)})\right),$$

where $\langle \mu_i, f_i \rangle = \int f_i(z) \mu_i(dz)$. Hence, conditions (i) and (ii) of Theorem 3.4 are satisfied. It remains to check condition (iii). To this end, let $f^* = (f_1^*, \ldots, f_r^*)$ be an element of E^* and suppose that

$$L_\nu(f^*) := \sup_{(f_1, \ldots, f_r) \in E} \left[\sum_{i=1}^{r} \langle f_i^*, f_i \rangle - H_\nu(f_1, \ldots, f_r) \right] < \infty. \tag{3.12}$$

We must show that f_i^* belongs to $\mathcal{M}(Y_i)$ for $i = 1, \ldots, r$. According to the Daniell–Stone theorem, this is certainly true, if the following conditions are satisfied for $i = 1, \ldots, r$:

a) $\langle f_i^*, f \rangle \geqq 0$ for $f \geqq 0$;

b) $\langle f_i^*, \mathbb{1} \rangle = 1$;

c) $\langle f_i^*, f^{(n)} \rangle \to 0$ as $f^{(n)} \downarrow 0$ pointwise

(see e.g. [1]). Suppose that $f \in C_b(Y_i)$, $f \geqq 0$, and $\langle f_i^*, f \rangle < 0$. Then

$$\lambda \langle f_i^*, f \rangle - \int \nu(dx) \log P_x(dy^{(1)}, \ldots, dy^{(r)}) \exp(\lambda f(y^{(i)}))$$

$$\to \infty \quad \text{as} \quad \lambda \to -\infty$$

in contradiction to (3.12). This yields (a). In a similar way one proves (b). To verify (c), suppose that $f^{(n)} \in C_b(Y_i)$, $f^{(n)} \downarrow 0$ pointwise, and $\langle f_i^*, f^{(n)} \rangle \geqq c > 0$ for all n. Then for arbitrary $\lambda > 0$:

$$\lambda \langle f_i^*, f^{(n)} \rangle - \int \nu(dx) \log \int P_x(dy^{(1)}, \ldots, dy^{(r)}) \exp(\lambda f^{(n)}(y^{(i)})) \leqq L_\nu(f^*).$$

Letting $n \to \infty$, we obtain $\lambda c \leqq L_\nu(f^*) < \infty$, which is not true for large λ. □

Let us add some comment. The assertion *that* $(\mathcal{M}(Y_1)\times\cdots\times\mathcal{M}(Y_r), Q^{(N)}_{\nu_N}, N)$ is a large deviation system could be reduced to the case $r=1$ by considering the emprical measure of the random vectors $y_1,\ldots,y_N$ instead of studying vectors of empirical measures and applying then Theorem 3.2 to the canonical projection $\mathcal{M}(Y_1\times\cdots\times Y_r)\to\mathcal{M}(Y_1)\times\cdots\times\mathcal{M}(Y_r)$. But as a result, we would get a representation of the action functional L which is different from (3.11). The question would then arise to show that both representations coincide.

4. LARGE DEVIATIONS FOR INDEPENDENT DIFFUSIONS

4.1 Preliminaries on distribution-valued functions

We denote by $\mathcal{D}$ the Schwartz space of test functions $\mathbb{R}^d\to\mathbb{R}$ having compact support and possessing continuous derivatives of all orders. We endow $\mathcal{D}$ with the usual inductive topology. Let $\mathcal{D}'$ be the corresponding space of real distributions. For each compact set $K\subset\mathbb{R}^d$, $\mathcal{D}_K$ will denote the subspace of $\mathcal{D}$ consisting of all test functions the support of which is contained in K. Finally, let $\langle\vartheta, f\rangle$ denote the application of the test function f to the distribution ϑ.

DEFINITION 4.1 Let I be an interval of the real line. A map $\vartheta(\cdot): I\to\mathcal{D}'$ is called absolutely continuous if for each compact set $K\subset\mathbb{R}^d$ there exist a neighborhood U_K of 0 in $\mathcal{D}_K$ and an absolutely continuous function $H_K: I\to\mathbb{R}$ such that

$$|\langle\vartheta(u), f\rangle-\langle\vartheta(v), f\rangle|\leqq|H_K(u)-H_K(v)| \tag{4.1}$$

for all $u, v\in I$ and $f\in U_K$.

LEMMA 4.2 *Assume that the map $\vartheta(\cdot): I\to\mathcal{D}'$ is absolutely continuous. Then the real function $\langle\vartheta(\cdot), f\rangle$ is absolutely continuous for each $f\in\mathcal{D}$. Moreover, the derivative in the distribution sense*

$$\dot{\vartheta}(t)=\lim_{h\to 0} h^{-1}[\vartheta(t+h)-\vartheta(t)]$$

exists for Lebesgue almost all $t\in I$.

Proof Let U_K and H_K be as in Definition 4.1. Since U_K is absorbing in $\mathscr{D}_K$ for each K and $\bigcup_K \mathscr{D}_K = \mathscr{D}$, the first part of the assertion is immediate from (4.1). For each natural number n, let K_n denote the closed ball in $\mathbb{R}^d$ with center 0 and radius n. Let N_n be the null set of points at which the function H_{K_n} is not differentiable. For each $f \in \mathscr{D}$, denote by $N(f)$ the null set on which the function $\langle \vartheta(\cdot), f \rangle$ is not differentiable. Fix an arbitrary countable dense subset $\mathscr{D}^{(s)}$ of $\mathscr{D}$. To prove the second part of our assertion, it is enough to check that $\vartheta(\cdot)$ is differentiable in the distribution sense on $I \backslash N$, where

$$N = \bigcup_n N_n \cup \bigcup_{f \in \mathscr{D}^{(s)}} N(f).$$

To this end, choose $t \in I \backslash N$ arbitrarily. It then follows from (4.1) that for each n and all sufficiently small $|h|$,

$$h^{-1}[\vartheta(t+h) - \vartheta(t)] \in (1 + |\dot{H}_{K_n}(t)|) U^0_{K_n},$$

where

$$U^0_{K_n} = \{ \ell \in \mathscr{D}'_{K_n} : |\langle \ell, f \rangle| \leq 1 \text{ for all } f \in U_{K_n} \}$$

is the polar set of U_{K_n} in $\mathscr{D}'_{K_n}$, the dual of $\mathscr{D}_{K_n}$. By the Banach–Alaoglu theorem and the separability of $\mathscr{D}_{K_n}$, $U^0_{K_n}$ is sequentially compact in $\mathscr{D}'_{K_n}$ (furnished with the weak* topology), see e.g. [26, Chap. 3]. Thus, $h^{-1}[\vartheta(t+h) - \vartheta(t)]$ is sequentially compact in each $\mathscr{D}'_{K_n}$ and, consequently, also in $\mathscr{D}'$ for $h \to 0$. Furthermore, since $t \notin N$, the finite limit

$$\lim_{h \to 0} \langle h^{-1}[\vartheta(t+h) - \vartheta(t)], f \rangle$$

exists for all $f \in \mathscr{D}^{(s)}$. This proves that $\vartheta(\cdot)$ is differentiable in the distribution sense at point t. □

Let $C_k^\infty(\mathbb{R}^d \times [s, t])$ be the space of functions $\mathbb{R}^d \times [s, t] \to \mathbb{R}$ having compact support and possessing continuous derivatives of all orders. Given a function $f: \mathbb{R}^d \times [s, t] \to \mathbb{R}$, we will write $f(u)(x) = f(x, u)$ and $\dot{f}(u)(x) = (\partial/\partial u) f(x, u)$, $(x, u) \in \mathbb{R}^d \times [s, t]$.

LEMMA 4.3 (Integration by parts formula) *For each absolutely continuous map* $\vartheta(\cdot):[s,t]\to\mathscr{D}'$ *and each* $f\in C_k^\infty(\mathbb{R}^d\times[s,t])$ it holds

$$\langle\vartheta(t),f(t)\rangle-\langle\vartheta(s),f(s)\rangle=\int_s^t du\langle\dot\vartheta(u),f(u)\rangle+\int_s^t du\langle\vartheta(u),\dot f(u)\rangle. \tag{4.2}$$

Proof At least formally,

$$\begin{aligned}
&\langle\vartheta(t),f(t)\rangle-\langle\vartheta(s),f(s)\rangle\\
&\quad=\langle\vartheta(t)-\vartheta(s),f(t)\rangle+\langle\vartheta(s),f(t)-f(s)\rangle\\
&\quad=\int_s^t du\langle\dot\vartheta(u),f(t)\rangle+\int_s^t dv\langle\vartheta(s),\dot f(v)\rangle\\
&\quad=\int_s^t du\langle\dot\vartheta(u),f(u)\rangle+\int_s^t dv\langle\vartheta(v),\dot f(v)\rangle\\
&\qquad+\int_s^t du\langle\dot\vartheta(u),f(t)-f(u)\rangle-\int_s^t dv\langle\vartheta(v)-\vartheta(s),\dot f(v)\rangle,
\end{aligned}$$

where the integrals

$$\int_s^t du\langle\dot\vartheta(u),f(t)-f(u)\rangle=\int_s^t du\int_u^t dv\langle\dot\vartheta(u),\dot f(v)\rangle$$

and

$$\int_s^t dv\langle\vartheta(v)-\vartheta(s),\dot f(v)\rangle=\int_s^t dv\int_s^v du\langle\dot\vartheta(u),\dot f(v)\rangle$$

coincide by Fubini's theorem. This yields (4.2). To make these computations rigorous, we must verify among other things that

$$\left\langle\vartheta,\int_s^t du\,\dot f(u)\right\rangle=\int_s^t du\langle\vartheta,\dot f(u)\rangle \tag{4.3}$$

for all $\vartheta\in\mathscr{D}'$. But this follows from the fact that the integral $\int_s^t du\,\dot f(u)$ is the limit in $\mathscr{D}$ of Riemannian sums. Since the function $\langle\dot\vartheta(u),\dot f(v)\rangle$

is measurable in u and continuous in v, it is measurable in (u, v). There exist some compact set $K \subset \mathbb{R}^d$ and some $c > 0$ such that $\dot{f}(v) \in cU_K$ for all v, where U_K is taken from Definition 4.1. Therefore (4.1) and Lemma 4.2 imply that

$$\int_s^t du \langle \vartheta(u), \dot{f}(v) \rangle = c \operatorname{Var} H_K < \infty$$

for all $v \in [s, t]$, where $\operatorname{Var} H_K$ denotes the variation of H_K. Hence

$$\int_s^t \int_s^t du\, dv \left| \langle \vartheta(u), \dot{f}(v) \rangle \right| < \infty, \tag{4.4}$$

which justifies the application of Fubini's theorem. By (4.3) and (4.4),

$$\int_s^t du \left| \langle \vartheta(u), f(t) - f(u) \rangle \right| < \infty.$$

Together with

$$\int_s^t du \left| \langle \vartheta(u), f(t) \rangle \right|$$

(see Lemma 4.2) this implies the existence of the first integral on the right of (4.2). The existence of the second one is obvious from the continuity of $u \mapsto \langle \vartheta(u), \dot{f}(u) \rangle$. □

4.2 The large deviation result

Let $\mathcal{M} = \mathcal{M}(\mathbb{R}^d)$ be the space of probability measures on $\mathbb{R}^d$ $(d \geqq 1)$ equipped with the Prokhorov metric which induces on $\mathcal{M}$ the topology of weak convergence [2]. Fix $T > 0$ arbitrarily. We denote by $C = C([0, T]; \mathbb{R}^d)$ and $\mathscr{C} = C([0, T]; \mathcal{M})$ the spaces of continuous maps from $[0, T]$ to $\mathbb{R}^d$ and $\mathcal{M}$, respectively, and furnish them with the topology of uniform convergence. Usually the elements of C and $\mathscr{C}$ will be denoted by $x(\cdot)$ and $\mu(\cdot)$, respectively. Let $\mathbb{S}^d$ be the space of symmetric non-negative definite $d \times d$ real matrices. Given maps $a: \mathbb{R}^d \times [0, T] \to \mathbb{S}^d$ and $b: \mathbb{R}^d \times [(0, T] \to \mathbb{R}^d$, let us introduce the diffusion operators

$$\mathscr{L}_t = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(\cdot, t)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(\cdot, t)\frac{\partial}{\partial x^i}, \tag{4.5}$$

$t \in [0, T]$, where $a^{ij}(x, t)$ and $b^i(x, t)$, $1 \leqq i, j \leqq d$, are the components of the *diffusion matrix* $a(x, t)$ and the *drift vector* $b(x, t)$, respectively. We impose the following assumptions on $\mathscr{L}_t$:

(*A.1*) The matrix $a(x, t)$ is strictly positive definite for all $(x, t) \in \mathbb{R}^d \times [0, T]$. The (possibly unbounded) coefficients a^{ij} and b^i, $1 \leqq i, j \leqq d$, are locally Hölder continuous on $\mathbb{R}^d \times [0, T]$.

(*A.2*) The martingale problem for $\{\mathscr{L}_t; t \in [0, T]\}$ is well posed. In other words, for each $(x, s) \in \mathbb{R}^d \times [0, T]$ there exists exactly one probability measure $P_{x,s}$ on C having the following properties:

i) $P_{x,s}(x(u) = x, u \in [0, s]) = 1$;

ii) $f(x(t), t) - \int_s^t \left(\frac{\partial}{\partial u} + \mathscr{L}_u\right) f(x(u), u)\, du, \qquad t \in [s, T]$.

is a local $P_{x,s}$-martingale after time s (with respect to the canonical filtration on C) for all continuous functions $f: \mathbb{R}^d \times [0, T] \to \mathbb{R}$ possessing continuous spatial derivatives up to the second order and a continuous time derivative of first order.

Remark 4.4 The Hölder continuity in Assumption (A.1) will only be used in the proof of Lemma 4.11 below. For the purposes of Section 5 we must be able to handle the case in which the drift coefficients are continuous but not Hölder continuous (at least with respect to the time variable). In Section 4.5 we will show how one can relax in a simple way the Hölder continuity of the drift vector (but not of the diffusion matrix). In our preliminary technical report [8] we imposed instead of (A.1) the weaker condition that the diffusion matrix is non-degenerate and both the drift and diffusion coefficients are merely continuous. The proof of Lemma 4.11 below given in [8] without assuming Hölder continuity follows the same lines as the proof presented here, but it requires in addition the application of Sobolev space techniques and Krylov bounds.

The family $\{P_{x,s}; (x, s) \in \mathbb{R}^d \times [0, T]\}$ defines a strong Markov-

Feller diffusion process on $\mathbb{R}^d$ ([29, Theorem 10.1.1 and Corollary 11.1.5]). Let us consider N independent copies of this process. This leads to the family of probability measures

$$P_{\mathbf{x},s}^{(N)} = P_{x_1,s} \otimes \cdots \otimes P_{x_N,s}, \quad x = (x_1, \ldots, x_N) \in (\mathbb{R}^d)^N, \quad s \in [0, T],$$

on the product space C^N. Canonically identifying $C^N = C([0, T]; \mathbb{R}^d)^N$ with $C([0, T]; (\mathbb{R}^d)^N)$, we will sometimes view $P_{\mathbf{x},s}^{(N)}$ as a measure on $C([0, T]; (\mathbb{R}^d)^N)$. Given $\mathbf{x} = (x_1, \ldots, x_N) \in (\mathbb{R}^d)^N$, set

$$\varepsilon_{\mathbf{x}} = N^{-1} \sum_{k=1}^{N} \delta_{x_k},$$

where δ_z denotes the Dirac measure at point $z \in \mathbb{R}^d$; $\varepsilon_{\mathbf{x}}$ is the *empirical measure* of the N-particle configuration $\mathbf{x}$. The set consisting of the empirical measures of all N-particle configurations will be denoted by $\mathscr{M}^{(N)}$. The *empirical process* associated to N independent copies of the above diffusion process is given by a family $\{\mathscr{P}_\nu^{(N)}; \nu \in \mathscr{M}^{(N)}\}$ of probability measures on $\mathscr{C}$. If $\nu = \varepsilon_{\mathbf{x}}$ for some $\mathbf{x} \in (\mathbb{R}^d)^N$, then $\mathscr{P}_\nu^{(N)}$ is the probability law of the process $\varepsilon_{\mathbf{x}(\cdot)}$ under $P_{\mathbf{x}}^{(N)} = P_{\mathbf{x},0}^{(N)}$. In other words, $\mathscr{P}_\nu^{(N)}$ is the image of the measure $P_{\mathbf{x}}^{(N)}$ with respect to the map

$$C^N \ni (x_1(\cdot), \ldots, x_N(\cdot)) \mapsto \left(t \mapsto N^{-1} \sum_{k=1}^{N} \delta_{x_k(t)} \right) \in \mathscr{C}. \tag{4.6}$$

Since the family $\{P_{\mathbf{x}}^{(N)}; \mathbf{x} \in (\mathbb{R}^d)^N\}$ is invariant with respect to permutations of the initial configuration $\mathbf{x} = (x_1, \ldots, x_N)$, the measures $\mathscr{P}_\nu^{(N)}$ are well defined.

In this section we will present a theorem on large deviations for the probability laws $\mathscr{P}_\nu^{(N)}$ of our empirical process as N tends to infinity. Before stating the result, we need some further notation.

Let ∇_t, $(\ ,\ )_t$, and $|\ |_t$ be, respectively, the Riemannian gradient, the inner product, and the Riemannian norm in the tangent space of the Riemannian structure on $\mathbb{R}^d$ induced by the diffusion matrix $a(\cdot, t)$. In global Euclidean coordinates $x^1, \ldots, x^d$,

$$(\nabla_t f)^i = \sum_{j=1}^{d} a^{ij}(\cdot, t) \frac{\partial f}{\partial x^j}, \qquad i = 1, \ldots, d,$$

$$(X, Y)_t = \sum_{i,j=1}^{d} a_{ij}(\cdot, t) X^i Y^j, \qquad |X|_t = (X, X)_t^{1/2},$$

where $\{a_{ij}(x, t)\}$ is the inverse of the matrix $\{a^{ij}(x, t)\}$. In particular,

$$|\nabla_t f|_t^2 = \sum_{i,j=1}^{d} a^{ij}(\cdot, t) \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j}.$$

(Of course, if $a(\cdot, t)$ is not sufficiently smooth, then there is not really a Riemannian structure associated with $a(\cdot, t)$. But the above formulae still makes sense in this case.)

For each $\mu \in \mathscr{M}$ and $t \in [0, T]$ we introduce a normed linear space

$$T_{\mu,t} = \{\vartheta \in \mathscr{D}' : \|\vartheta\|_{\mu,t} < \infty\},$$

where the norm $\| \; \|_{\mu,t}$ is defined by

$$\|\vartheta\|_{\mu,t}^2 = \frac{1}{2} \sup_{f \in \mathscr{D}_{\mu,t}} \frac{|\langle \vartheta, f \rangle|^2}{\langle \mu, |\nabla_t f|_t^2 \rangle}, \qquad \vartheta \in \mathscr{D}'. \tag{4.7}$$

Here $\mathscr{D}_{\mu,t} = \{f \in \mathscr{D} : \langle \mu, |\nabla_t f|_t^2 \rangle \neq 0\}$. For each $\vartheta \in \mathscr{D}'$,

$$\|\vartheta\|_{\mu,t}^2 = \sup_{f \in \mathscr{D}} [\langle \vartheta, f \rangle - \tfrac{1}{2} \langle \mu, |\nabla_t f|_t^2 \rangle]. \tag{4.8}$$

Indeed, replacing the function f in (4.8) by $c \cdot f$ and taking the supremum at first over all $c \in \mathbb{R}$ and then over $f \in \mathscr{D}_{\mu,t}$, we see that the expressions on the right of (4.7) and (4.8) coincide.

For each $\nu \in \mathscr{M}$ we introduce a functional $S_\nu : \mathscr{C} \to [0, \infty]$ by setting

$$S_\nu(\mu(\cdot)) = \int_0^T \|\dot{\mu}(t) - \mathscr{L}_t^* \mu(t)\|_{\mu(t),t}^2 \, dt, \tag{4.9}$$

if $\mu(0) = \nu$ and $\mu(\cdot)$ is absolutely continuous (in the sense of Definition 4.1) and $S_\nu(\mu(\cdot)) = \infty$ otherwise. Here $\mathscr{L}_t^*$ is the formal adjoint of $\mathscr{L}_t$ defined by (4.5). The operator $\mathscr{L}_t^*$ acts on $\mathscr{D}'$. We remark that the measurability of the integrand in (4.9) is a consequence of the fact that it suffices to take the supremum on the right of (4.8) over a countable dense subset of $\mathscr{D}$ and that, by Lemma 4.2, $\langle \mu(\cdot), f \rangle$ is absolutely continuous and $\langle \dot{\mu}(t), f \rangle = (d/dt)\langle \mu(t), f \rangle$ almost everywhere for each $f \in \mathscr{D}$.

We are now ready to formulate our large deviation result.

THEOREM 4.5 *Given $\nu_N \in \mathscr{M}^{(N)}$ and $\nu \in \mathscr{M}$, suppose that $\nu_N \to \nu$ in $\mathscr{M}$. Then $(\mathscr{C}, \mathscr{P}^{(N)}_{\nu_N}, N)$ is a large deviation system with action functional S_ν.*

The proof of this theorem will be given in the next two sections. The assertion *that* $(\mathscr{C}, \mathscr{P}^{(N)}_{\nu_N}, N)$ is a large deviation system is a simple consequence of Theorem 3.5 (see Lemma 4.6 below). The difficulty consists in showing that the action functional has the form (4.9). To this end, we will apply the results of Section 3 in order to obtain two different expressions $S^{(1)}_\nu$ and $S^{(2)}_\nu$ for the action functional (Lemmas 4.6 and 4.7). The functional S_ν will then be shown to be caught between $S^{(1)}_\nu$ and $S^{(2)}_\nu$ (Lemmas 4.9 and 4.10).

4.3 Two representations of the action functional

We will denote by $E_{x,s}$ the expectation with respect to the probability measure $P_{x,s}$ defined in the preceding section. We will write E_x and P_x instead of $E_{x,0}$ and $P_{x,0}$, respectively. In the following, $C_b(C)$ and $\mathscr{M}(C)$ will stand for the space of bounded continuous functions and the space of probability measures on $C = C([0, T]; \mathbb{R}^d)$, respectively. Given $P \in \mathscr{M}(C)$, let $\pi(t; P) = P \circ x(t)^{-1}$, $t \in [0, T]$, denote the associated one-dimensional distributions.

LEMMA 4.6 *Given $\nu_N \in \mathscr{M}^{(N)}$, $N \geqq 1$, and $\nu \in \mathscr{M}$, suppose that $\nu_N \to \nu$ in $\mathscr{M}$. Then $(\mathscr{C}, \mathscr{P}^{(N)}_{\nu_N}, N)$ is a large deviation system with action functional*

$$S^{(1)}_\nu(\mu(\cdot)) = \min_{P \in \mathscr{M}(C):\, \pi(\cdot\,; P) = \mu(\cdot)} L^{(1)}_\nu(P), \qquad \mu(\cdot) \in \mathscr{C}, \tag{4.10}$$

where

$$L^{(1)}_\nu(P) = \sup_{F \in C_b(C)} [\langle P, F \rangle - \langle \nu, \log E_{\cdot}\, e^F \rangle], \qquad P \in \mathscr{M}(C). \tag{4.11}$$

Proof Given $N \geqq 1$ and

$$\mu = N^{-1} \sum_{k=1}^{N} \delta_{x_k} \in \mathscr{M}^{(N)},$$

let $Q_\mu^{(N)}$ denote the image of the measure $P_{x_1} \otimes \cdots \otimes P_{x_N}$ with respect to the map

$$C^N \ni (x_1(\cdot), \ldots, x_N(\cdot)) \mapsto N^{-1} \sum_{k=1}^{N} \eta_{x_k(\cdot)} \in \mathcal{M}(C), \tag{4.12}$$

where $\eta_{x(\cdot)}$ is the Dirac measure on C with unit mass at $x(\cdot)$. The map (4.6) is the composition of the map (4.12) and the map

$$\mathcal{M}(C) \ni P \mapsto \pi(\cdot\,; P) \in \mathscr{C}. \tag{4.13}$$

Therefore $\mathscr{P}_\mu^{(N)}$ is the image of the measure $Q_\mu^{(N)}$ with respect to (4.13). The map (4.13) is continuous. For, suppose that $P_n \to P$ in $\mathcal{M}(C)$. Then

$$\langle \pi(t; P_n), f \rangle \to \langle \pi(t; P), f \rangle \tag{4.14}$$

for each $t \in [0, T]$ and each $f \in C_b(\mathbb{R}^d)$. Moreover, the tightness criterion for probability measures on C ([2, Theorem 8.2]) implies that, for each $\kappa > 0$,

$$P_n(\omega(\delta; x(\cdot)) > \kappa) \to 0 \quad \text{as} \quad \delta \to 0, \tag{4.15}$$

uniformly in n, where $\omega(\cdot\,; x(\cdot))$ is the modulus of continuity of the path $x(\cdot) \in C$. Combining (4.14) with (4.15), we find that the convergence in (4.14) is uniform in $t \in [0, T]$ and, therefore, $\pi(\cdot\,; P_n) \to \pi(\cdot\,; P)$ in $\mathscr{C}$.

For $r = 1$, $X = \mathbb{R}^d$, and $Y = C$, Theorem 3.5 says that $(\mathcal{M}(C), Q_{\nu_N}^{(N)}, N)$ is a large deviation system with action functional $L_\nu^{(1)}$ defined by (4.11). Together with Theorem 3.2 this yields that $(\mathscr{C}, \mathscr{P}_{\nu_N}^{(N)}, N)$ is a large deviation system with action functional (4.10). □

Let us introduce the two-parameter semi-group $\{U_{s,t}; 0 \leqq s \leqq t \leqq T\}$ of linear operators acting on $C_b(\mathbb{R}^d)$ according to

$$U_{s,t} f(z) = \int_C f(x(t)) P_{z,s}(dx(\cdot)). \tag{4.16}$$

LEMMA 4.7 *Given $\nu_N \in \mathcal{M}^{(N)}$, $N \geqq 1$, and $\nu \in \mathcal{M}$, suppose that $\nu_N \to \nu$ in*

$\mathcal{M}$. *Then* $(\mathcal{C}, \mathcal{P}^{(N)}_{\nu_N}, N)$ *is a large deviation system with action functional*

$$S^{(2)}_\nu(\mu(\cdot)) = \sup_{0 \leq t_1 < \cdots < t_r \leq T} L^{t_1,\ldots,t_r}_\nu(\mu(t_1),\ldots,\mu(t_r)), \tag{4.17}$$

$$\mu(\cdot) \in \mathcal{C},$$

where

$$L^{t_1,\ldots,t_r}_{\mu_0}(\mu_1,\ldots,\mu_r) = \sum_{i=1}^{r} \sup_{f \in \mathcal{D}} [\langle \mu_i, f\rangle - \langle \mu_{i-1}, \log U_{t_{i-1},t_i} e^f\rangle], \tag{4.18}$$

$$\mu_0, \mu_1, \ldots, \mu_r \in \mathcal{M}.$$

Proof We first derive a large deviation result for the finite dimensional distributions of $\mathcal{P}^{(N)}_{\nu_N}$. Given $N \geq 1$, $0 = t_0 \leq t_1 < \cdots < t_r \leq T$, $x \in \mathbb{R}^d$, and

$$\mu = N^{-1} \sum_{k=1}^{N} \delta_{x_k} \in \mathcal{M}^{(N)},$$

let $p_x(t_1,\ldots,t_r)$ and $\pi^{(N)}_\mu(t_1,\ldots,t_r)$ denote the probability law of $(x(t_1),\ldots,x(t_r))$ under P_x and the probability law of $(\mu(t_1),\ldots,\mu(t_r))$ under $\mathcal{P}^{(N)}_\mu$. Since $\{P_x; x \in \mathbb{R}^d\}$ is Feller continuous, the same is true for $\{p_x(t_1,\ldots,t_r); x \in \mathbb{R}^d\}$. It follows from the definition of $\mathcal{P}^{(N)}_\mu$ that $\pi^{(N)}_\mu(t_1,\ldots,t_r)$ is the image of the measure

$$p_{x_1}(t_1,\ldots,t_r) \otimes \cdots \otimes p_{x_N}(t_1,\ldots,t_r)$$

with respect to the map

$$((\mathbb{R}^d)^r)^N \ni (y_1,\ldots,y_N) \mapsto \left(N^{-1}\sum_{k=1}^{N} \delta_{y^{(1)}_k},\ldots, N^{-1}\sum_{k=1}^{N} \delta_{y^{(r)}_k}\right) \in \mathcal{M}^r,$$

where $y_k = (y^{(1)}_k,\ldots,y^{(r)}_k) \in (\mathbb{R}^d)^r$. We can therefore apply Theorem 3.5 for $X = Y_1 = \cdots = Y_r = \mathbb{R}^d$, in order to obtain that $(\mathcal{M}^r, \pi^{(N)}_{\nu_N}(t_1,\ldots,t_r), N)$ is a large deviation system with action functional

$$L^{t_1,\ldots,t_r}_\nu(\mu_1,\ldots,\mu_r) = \sup_{f_1,\ldots,f_r \in C_b(\mathbb{R}^d)} \left[\sum_{i=1}^{r} \langle \mu_i, f_i\rangle - H(f_1,\ldots,f_r)\right], \tag{4.19}$$

$$\mu_1,\ldots,\mu_r \in \mathcal{M},$$

where

$$H(f_1,\dots,f_r)=\int \nu(dx)\int p_x(t_1,\dots,t_r)(dx_1,\dots,dx_r)$$

$$\times\exp\left(\sum_{i=1}^{r} f_i(x_i)\right).$$

We next show that the functional (4.19) coincides with (4.18) for $\mu_0=\nu$. An application of the Markov property of

$$\{P_{x,s};(x,s)\in\mathbb{R}^d\times[0,T]\}$$

yields

$$H(f_1,\dots,f_r)=\langle \nu,\log U_{t_0,t_1}e^{f_1}U_{t_1,t_2}e^{f_2}\dots U_{t_{r-1},t_r}e^{f_r}\rangle.$$

Abbreviate

$$h(f_2,\dots,f_r)=\log U_{t_1,t_2}e^{f_2}\dots U_{t_{r-1},t_r}e^{f_r}.$$

Then the functional (4.19) (with $\mu_0=\nu$) has the form

$$L_{\mu_0}^{t_1,\dots,t_r}(\mu_1,\dots,\mu_r)$$

$$=\sup_{f_1,\dots,f_r}\left[\sum_{i=1}^{r}\langle\mu_i,f_i\rangle-\langle\mu_0,\log U_{t_0,t_1}e^{f_1}\dots U_{t_{r-1},t_r}e^{f_r}\rangle\right]$$

$$=\sup_{f_1,\dots,f_r}\Big[\langle\mu_1,f_1+h(f_2,\dots,f_r)\rangle$$

$$-\langle\mu_0,\log U_{t_0,t_1}\exp(f_1+h(f_2,\dots,f_r))\rangle$$

$$+\sum_{i=2}^{r}\langle\mu_i,f_i\rangle-\langle\mu_1,\log U_{t_1,t_2}e^{f_2}\dots U_{t_{r-1},t_r}e^{f_r}\rangle\Big]$$

$$=\sup_{f\in C_b(\mathbb{R}^d)}[\langle\mu_1,f\rangle-\langle\mu_0,\log U_{t_0,t_1}e^{f}\rangle]$$

$$+\sup_{f_2,\dots,f_r}\left[\sum_{i=2}^{r}\langle\mu_i,f_i\rangle-\langle\mu_1,\log U_{t_1,t_2}e^{f_2}\dots U_{t_{r-1},t_r}e^{f_r}\rangle\right].$$

Hence, we successively arrive at

$$L_{\mu_0}^{t_1,\ldots,t_r}(\mu_1,\ldots,\mu_r)=\sum_{i=1}^{r}\sup_{f\in C_b(\mathbb{R}^d)}[\langle\mu_i,f\rangle-\langle\mu_{i-1},\log U_{t_{i-1},t_i}e^f\rangle].$$

Taking into account the continuity of the operators $U_{s,t}$, this gives (4.18).

We already know from Lemma 4.6 that $(\mathscr{C},\mathscr{P}_{\nu_N}^{(N)},N)$ is a large deviation system. Let $S_\nu^{(2)}$ denote its action functional. It remains to show that $S_\nu^{(2)}$ has the form (4.17). Given $N\geqq 1$ and $\mu\in\mathscr{M}^{(N)}$, let $\tilde{\mathscr{P}}_\mu^{(N)}$ denote the image of the measure $\mathscr{P}_\mu^{(N)}$ with respect to the continuous imbedding $\mathscr{C}\to\mathscr{M}^{[0,T]}$. Here $\mathscr{M}^{[0,T]}$ is the space of all maps $[0,T]\to\mathscr{M}$ furnished with the product topology. By Theorem 3.2, $(\mathscr{M}^{[0,T]},\tilde{\mathscr{P}}_{\nu_N}^{(N)},N)$ is a large deviation system. Its action functional $\tilde{S}_\nu^{(2)}$ coincides with $S_\nu^{(2)}$ on $\mathscr{C}$ and equals $+\infty$ on $\mathscr{M}^{[0,T]}\backslash\mathscr{C}$. But $(\mathscr{M}^{[0,T]},\tilde{\mathscr{P}}_{\nu_N}^{(N)},N)$ can be canonically identified with the projective limit of the large deviation systems $(\mathscr{M}^r,\pi_{\nu_N}^{(N)}(t_1,\ldots,t_r),N)$. The index set of this projective system consists of all finite sets $\{t_1,\ldots,t_r\}$ of points from $[0,T]$ with $\subseteq$ as order relation. We can therefore apply Theorem 3.3 in order to obtain

$$\tilde{S}_\nu^{(2)}(\mu(\cdot))=\sup_{0\leqq t_1<\cdots<t_r\leqq T}L_\nu^{t_1,\ldots,t_r}(\mu(t_1),\ldots,\mu(t_r)),$$

$$\mu(\cdot)\in\mathscr{M}^{[0,T]}.$$

This proves (4.17). □

4.4 Coincidence of the action functional with S_ν

In Lemma 4.6 and Lemma 4.7 we have got two different expressions $S_\nu^{(1)}$ and $S_\nu^{(2)}$ for the action functional of $(\mathscr{C},\mathscr{P}_{\nu_N}^{(N)},N)$. To finish the proof of Theorem 4.5, it will be enough to show that, for each $\nu\in\mathscr{M}$, the functional S_ν defined by (4.9) satisfies

$$S_\nu^{(1)}\geqq S_\nu\geqq S_\nu^{(2)}. \tag{4.20}$$

The present section is mainly devoted to the proof of (4.20). Let $C_k^{2,1}=C_k^{2,1}(\mathbb{R}^d\times[0,T])$ denote the set of continuous real functions on $\mathbb{R}^d\times[0,T]$ having compact support and possessing continuous spatial derivatives of first and second order and a continuous time derivative of first order. We first derive the following representation of the functional S_ν.

LEMMA 4.8 *Given $\nu \in \mathcal{M}$ and $\mu(\cdot) \in \mathscr{C}$, suppose that $\mu(0) = \nu$. Then*

$$S_\nu(\mu(\cdot)) = \sup_{f \in C_k^{2,1}} I(\mu(\cdot); f), \tag{4.21}$$

where

$$I(\mu(\cdot); f) = \langle \mu(T), f(T) \rangle - \langle \mu(0), f(0) \rangle$$
$$- \int_0^T \left\langle \mu(t), \left(\frac{\partial}{\partial t} + \mathscr{L}_t\right) f(t) + \tfrac{1}{2} |\nabla_t f(t)|_t^2 \right\rangle dt. \tag{4.22}$$

Proof Fix $\nu \in \mathcal{M}$ and $\mu(\cdot) \in \mathscr{C}$ arbitrarily and assume that $\mu(0) = \nu$.

1) Given $0 \leqq s < t \leqq T$ and $f \in C_k^{2,1}(\mathbb{R}^d \times [s, t])$, set

$$l_{s,t}(f) = \langle \mu(t), f(t) \rangle - \langle \mu(s), f(s) \rangle - \int_s^t \left\langle \mu(u), \left(\frac{\partial}{\partial u} + \mathscr{L}_u\right) f(u) \right\rangle du \tag{4.23}$$

and

$$I_{s,t}(f) = l_{s,t}(f) - \tfrac{1}{2} \int_s^t \langle \mu(u), |\nabla_u f(u)|_u^2 \rangle \, du. \tag{4.24}$$

Clearly $I_{0,T}(f) = I(\mu(\cdot); f)$. Let $(\hbar_n)$ be a sequence of smooth functions from $[0, T]$ to $[0, 1]$ such that $\hbar_n(u) = 1$ for all $u \in [s, t]$ and all n and $\hbar_n \downarrow \mathbb{1}_{[s,t]}$, where $\mathbb{1}_{[s,t]}$ is the indicator function of the interval $[s, t]$. Then one easily checks that

$$I_{0,T}(\hbar_n f) \to I_{s,t}(f)$$

for all $f \in C_k^{2,1}$. Therefore

$$\sup_{f \in C_k^{2,1}} I_{s,t}(f) \leqq \sup_{f \in C_k^{2,1}} I(\mu(\cdot); f) \tag{4.25}$$

for $0 \leqq s < t \leqq T$. From (4.23) and (4.24) we conclude that

$$c \cdot l_{s,t}(f) - \frac{c^2}{2} \int_s^t \langle \mu(u), |\nabla_u f(u)|_u^2 \rangle \, du \leqq \sup_{g \in C_k^{2,1}} I_{s,t}(g)$$

for all $f \in C_k^{2,1}$ and $c \in \mathbb{R}$. Taking on the left-hand side the supremum over all $c \in \mathbb{R}$, we arrive at

$$|l_{s,t}(f)|^2 \leqq 2 \sup_{g \in C_k^{2,1}} I_{s,t}(g) \int_s^t \langle \mu(u), |\nabla_u f(u)|_u^2 \rangle \, du$$

$$\leqq 2 \sup_{g \in C_k^{2,1}} I(\mu(\cdot); g) \int_s^t \langle \mu(u), |\nabla_u f(u)|_u^2 \rangle \, du. \tag{4.26}$$

Here we have also used (4.25).

2) We next show that

$$\sup_{f \in C_k^{2,1}} I(\mu(\cdot); f) \leqq S_\nu(\mu(\cdot)). \tag{4.27}$$

To this end, put an arbitrary smooth function $f: \mathbb{R}^d \times [0, T] \to \mathbb{R}$ with compact support and suppose that $S_\nu(\mu(\cdot)) < \infty$. Then a combination of (4.8), (4.9), and Lemma 4.3 yields

$$S_\nu(\mu(\cdot)) \geqq \int_0^T [\langle \dot{\mu}(t) - \mathscr{L}_t^* \mu(t), f(t) \rangle - \tfrac{1}{2} \langle \mu(t), |\nabla_t f(t)|_t^2 \rangle] \, dt$$

$$= \langle \mu(T), f(T) \rangle - \langle \mu(0), f(0) \rangle$$

$$- \int_0^T \left\langle \mu(t), \left(\frac{\partial}{\partial t} + \mathscr{L}_t \right) f(t) + \tfrac{1}{2} |\nabla_t f(t)|_t^2 \right\rangle dt$$

$$= I(\mu(\cdot); f),$$

and we arrive at (4.27).

3) After these preliminaries we start to prove (4.21). Because of (4.27) we can assume without loss of generality that

$$\sup_{f \in C_k^{2,1}} I(\mu(\cdot); f) < \infty. \tag{4.28}$$

Let $L^2(s,t)$ be the Hilbert space of all measurable maps $h: \mathbb{R}^d \times [s,t] \to \mathbb{R}^d$ with finite norm

$$\|h\| = \left\{ \int_s^t \langle \mu(u), |h(u)|_u^2 \rangle \, du \right\}^{1/2}$$

and inner product

$$[h_1, h_2] = \int_s^t \langle \mu(u), (h_1(u), h_2(u))_u \rangle \, du.$$

Denote by $L^2_\nabla(s,t)$ the closure in $L^2(s,t)$ of the linear subset L consisting of all maps $(x,u) \mapsto \nabla_u f(x,u)$, $f \in C_k^{2,1}(\mathbb{R}^d \times [s,t])$. Since $f(x,u) \mapsto \nabla_u f(x,u)$ is a one-to-one correspondence between $C_k^{2,1}(\mathbb{R}^d \times [s,t])$ and L, $l_{s,t}$ can be viewed as a linear functional on L. (Actually L must be considered not as a set of functions but as a set of equivalence classes of functions coinciding $\mu(u,dx) \otimes du$—almost everywhere. But this is inessential, since $l_{s,t}(f) = l_{s,t}(g)$ if $\nabla_u f(x,u)$ and $\nabla_u g(x,u)$ belong to the same equivalence class which is immediate from (4.26) and (4.28).) Because of (4.26) and (4.28), the functional $l_{s,t}$ is bounded. Hence, by the Riesz Representation Theorem, there exists some $h_{s,t} \in L^2_\nabla(s,t)$ such that

$$l_{s,t}(f) = \int_s^t \langle \mu(u), (h_{s,t}(u), \nabla_u f(u))_u \rangle \, du, \qquad f \in C_k^{2,1}.$$

Now

$$\begin{aligned} l_{0,t}(f) &= l_{0,T}(f) - l_{t,T}(f) \\ &= \int_0^t \langle \mu(u), (h_{0,T}(u), \nabla_u f(u))_u \rangle \, du \\ &\quad + \int_t^T \langle \mu(u), (h_{0,T}(u) - h_{t,T}(u), \nabla_u f(u))_u \rangle \, du. \end{aligned}$$

Since $l_{0,t}(f)$ does not depend upon the values of f on the time interval $(t,T]$, the last integral on the right-hand side vanishes identically. So we finally get the representation

$$l_{0,t}(f) = \int_0^t \langle \mu(u), (h(u), \nabla_u f(u))_u \rangle \, du, \tag{4.29}$$

$t \in [0, T]$, $f \in C_k^{2,1}$, where $h = h_{0,T}$. Since $h \in L_\nabla^2(0, T)$,

$$\inf_{f \in C_k^{2,1}} \int_0^T \langle \mu(t), |h(t) - \nabla_t f(t)|_t^2 \rangle \, dt = 0. \tag{4.30}$$

4) Substituting (4.29) in (4.24) (for $s = 0$ and $t = T$) and taking into account (4.30), we obtain

$$\sup_{f \in C_k^{2,1}} I(\mu(\cdot); f) = \tfrac{1}{2} \int_0^T \langle \mu(t), |h(t)|_t^2 \rangle \, dt. \tag{4.31}$$

Therefore, in order to finish the proof of (4.21), it suffices to show that $S_\nu(\mu(\cdot))$ coincides with the integral expression on the right of (4.31). Comparing (4.23) with (4.29), we find that

$$\langle \mu(t), f \rangle - \langle \mu(s), f \rangle = \int_s^t \langle \mu(u), \mathscr{L}_u f + (h(u), \nabla_u f)_u \rangle \, du \tag{4.32}$$

for $0 \leqq s < t \leqq T$ and all $f \in \mathscr{D}$. This implies that $\mu(\cdot)$ is absolutely continuous as a map from $[0, T]$ to $\mathscr{D}'$. Hence, using Lemma 4.2, we conclude from (4.32) that for each $f \in \mathscr{D}$,

$$\langle \dot{\mu}(t), f \rangle = \langle \mu(t), \mathscr{L}_t f + (h(t), \nabla_t f)_t \rangle \tag{4.33}$$

for almost all $t \in [0, T]$. Since $\mathscr{D}$ is separable and the linear functionals on both sides of (4.33) belong to $\mathscr{D}'$ for almost all $t \in [0, T]$, there even exists a null set $N \subset [0, T]$ such that (4.33) holds for all $f \in \mathscr{D}$ and all $t \in [0, T] \backslash N$ simultaneously. Together with (4.9), (4.8), and (4.30) this finally yields

$$\begin{aligned}
S_\nu(\mu(\cdot)) &= \int_0^T \sup_{f \in \mathscr{D}} \left[\langle \dot{\mu}(t) - \mathscr{L}_t^* \mu(t), f \rangle - \tfrac{1}{2} \langle \mu(t), |\nabla_t f|_t^2 \rangle \right] dt \\
&= \int_0^T \sup_{f \in \mathscr{D}} \langle \mu(t), (h(t), \nabla_t f)_t - \tfrac{1}{2} |\nabla_t f|_t^2 \rangle \, dt \\
&= \tfrac{1}{2} \int_0^T \langle \mu(t), |h(t)|_t^2 \rangle \, dt - \tfrac{1}{2} \int_0^T \inf_{f \in \mathscr{D}} \langle \mu(t), |h(t) - \nabla_t f|_t^2 \rangle \, dt \\
&= \tfrac{1}{2} \int_0^T \langle \mu(t), |h(t)|_t^2 \rangle \, dt. \qquad \square
\end{aligned}$$

LEMMA 4.9 $S_\nu^{(1)} \geqq S_\nu$ for all $\nu \in \mathscr{M}$.

Proof Given $\nu \in \mathscr{M}$ and $\mu(\cdot) \in \mathscr{C}$, assume that $S_\nu^{(1)}(\mu(\cdot)) < \infty$. Then there exists a measure $P_{\min} \in \mathscr{M}(C)$ with one-dimensional distributions $\mu(t)$, $t \in [0, T]$, for which the minimum on the right of (4.10) is attained. Together with (4.11) this gives

$$\infty > S_\nu^{(1)}(\mu(\cdot)) \geqq \langle P_{\min}, F \rangle - \langle \nu, \log E_\cdot e^F \rangle \tag{4.34}$$

for all $F \in C_b(C)$. Putting in (4.34) $F(x(\cdot)) = f(x(0))$ and varying over all $f \in C_b(\mathbb{R}^d)$, one readily checks that $\mu(0) = \nu$. Now choose $f \in C_k^{2,1}$ arbitrarily. Since $\{P_{x,s}; (x, s) \in \mathbb{R}^d \times [0, T]\}$ is the solution to the martingale problem for $\{\mathscr{L}_t; t \in [0, T]\}$,

$$M_t(x(\cdot)) = f(x(t), t) - f(x(0), 0) - \int_0^t \left(\frac{\partial}{\partial u} + \mathscr{L}_u \right) f(x(u), u)\, du,$$

$t \in [0, T]$, is a bounded continuous P_x-martingale for each $x \in \mathbb{R}^d$ with quadratic characteristic ([23, Proposition 13.42])

$$\langle\!\langle M \rangle\!\rangle_t(x(\cdot)) = \int_0^t |\nabla_u f(x(u), u)|_u^2\, du$$

not depending on x. Put

$$F = M_T - \tfrac{1}{2}\langle\!\langle M \rangle\!\rangle_T.$$

Then

$$\langle P_{\min}, F \rangle = I(\mu(\cdot); f),$$

where $I(\mu(\cdot); f)$ is defined by (4.22). Further, since $\exp(M - \frac{1}{2}\langle\!\langle M \rangle\!\rangle)$ is a P_x-martingale (see e.g. [22, Chap. 3, Theorem 5.3]),

$$E_x e^F = 1, \qquad x \in \mathbb{R}^d.$$

We therefore derive from (4.34) that

$$S_\nu^{(1)}(\mu(\cdot)) \geqq I(\mu(\cdot); f)$$

for all $f \in C_k^{2,1}$. Together with Lemma 4.8 this gives $S_\nu^{(1)}(\mu(\cdot)) \geqq S_\nu(\mu(\cdot))$. □

LEMMA 4.10 *$S_\nu^{(2)} \leqq S_\nu$ for all $\nu \in \mathscr{M}$.*

The rest of this section is devoted to the proof of Lemma 4.10. Given $\nu \in \mathscr{M}$ and $\mu(\cdot) \in \mathscr{C}$, suppose that $S_\nu(\mu(\cdot)) < \infty$. By (4.9), (4.8), (4.17), and (4.18), it suffices to show that

$$\langle \mu(t), f \rangle - \langle \mu(s), \log U_{s,t} e^f \rangle$$

$$\leqq \int_s^t \sup_{g \in \mathscr{D}} [\langle \dot{\mu}(u), g \rangle - \langle \mu(u), \mathscr{L}_u g + \tfrac{1}{2} |\nabla_u g|_u^2 \rangle] \, du \tag{4.35}$$

for $0 \leqq s < t \leqq T$ and all $f \in \mathscr{D}$.

On a formal level of rigour, (4.35) is readily checked. Indeed, at least formally the function

$$g(x, u) = (U_{u,t} e^f)(x) = E_{x,u} \exp(f(x(t))),$$

$(x, u) \in \mathbb{R}^d \in [0, t]$, satisfies the backward Kolmogorov equation

$$\dot{g}(u) + \mathscr{L}_u g(u) = 0, \qquad g(t) = e^f.$$

Thus $h(x, u) = \log g(x, u)$ satisfies

$$\dot{h}(u) + \mathscr{L}_u h(u) + \tfrac{1}{2} |\nabla_u h(u)|_u^2 = 0, \qquad h(t) = f.$$

Together with a formal integration by parts this yields

$$\langle \mu(t), f \rangle - \langle \mu(s), \log U_{s,t} e^f \rangle$$

$$= \langle \mu(t), h(t) \rangle - \langle \mu(s), h(s) \rangle$$

$$= \int_s^t [\langle \dot{\mu}(u), h(u) \rangle + \langle \mu(u), \dot{h}(u) \rangle] \, du$$

$$= \int_s^t [\langle \dot{\mu}(u), h(u) \rangle - \langle \mu(u), \mathscr{L}_u h(u) + \tfrac{1}{2} |\nabla_u h(u)|_u^2 \rangle] \, du, \tag{4.36}$$

and we arrive at (4.35).

In general, the transformations in (4.36) do not make sense. The most striking reason for this is that the function h does not have compact support. One also has to keep in mind that we did not impose any restrictions on the growth at infinity of the drift or diffusion coefficients. To circumvent these problems, we modify the proof, passing from the semi-group $\{U_{s,t}\}$ to the semi-group $\{U_{s,t}^{(R)}\}$ associated to the diffusion process which is killed as soon as it leaves the open ball $B_R=\{x\in\mathbb{R}^d:|x|<R\}$. Let $\tau_R(s):C\to[0,\infty]$ be the stopping time of the first exit from B_R after time s. That is,

$$\tau_R(s)(x(\cdot))=\min\{t\in[s,T]:|x(t)|\geqq R\}$$

if the minimum exists and $\tau_R(s)(x(\cdot))=\infty$ otherwise. Then the semi-group $\{U_{s,t}^{(R)}\}$ acts on functions $f\in C_b(\mathbb{R}^d)$ according to

$$U_{s,t}^{(R)}f(x)=E_{x,s}f(x(t))\mathbb{1}_{\{\tau_R(s)>t\}}. \tag{4.37}$$

Here $\mathbb{1}_A$ is the indicator function of the event A. Let $\operatorname{supp} f$ denote the support of the function f. Instead of Lemma 4.10 (respectively (4.35)) we will check the following.

LEMMA 4.11 *Given* $\nu\in\mathscr{M}$ *and* $\mu(\cdot)\in\mathscr{C}$, *suppose that* $S_\nu(\mu(\cdot))<\infty$. *Then for each* $R>0$,

$$\langle\mu(t),f\rangle-\langle\mu(s),\log[1+U_{s,t}^{(R)}(e^f-1)]\rangle$$
$$\leqq\int_s^t\sup_{h\in\mathscr{D}}[\langle\dot\mu(u),h\rangle-\langle\mu(u),\mathscr{L}_u h+\tfrac{1}{2}|\nabla_u h|_u^2\rangle]\,du \tag{4.38}$$

for $0\leqq s<t\leqq T$ *and all* $f\in\mathscr{D}$ *with* $f\leqq 0$ *and* $\operatorname{supp} f\subset B_R$.

Inequality (4.38) implies (4.35). Indeed, letting $R\to\infty$ in (4.38), we find that (4.35) holds for all non-positive $f\in\mathscr{D}$. Now choose a sequence (h_n) in $\mathscr{D}$ so that $0\leqq h_n\uparrow 1$ pointwise. Given $f\in\mathscr{D}$ arbitrarily, set $f_n=h_n\cdot(f-\|f\|)$, where $\|f\|$ stands for the supremum norm of f. The functions f_n belong to $\mathscr{D}$ and are non-positive. Hence (4.35) holds for f_n instead of f. Letting $n\to\infty$, we conclude that (4.35) is also true for f. We have therefore reduced the proof of Lemma 4.10 to Lemma 4.11.

The function

$$g(x,s)=U_{s,t}^{(R)}(e^f-1)(x),\quad (x,s)\in\mathbb{R}^d\times[0,T], \tag{4.39}$$

appearing in (4.38) is the (unique) classical solution of the "initial" boundary value problem

$$\left(\frac{\partial}{\partial s}+\mathscr{L}_s\right)g(x,s)=0,\quad (x,s)\in B_R\times[0,t), \tag{4.40a}$$

$$g(x,t)=\exp(f(x))-1,\ x\in B_R, \tag{4.40b}$$

$$g(x,s)=0,\quad (x,s)\in\partial B_R\times[0,t). \tag{4.40c}$$

For, Assumption A.1 ensures that the problem (4.40) admits a unique classical solution g (see [24, Chap. 4, Theorem 5.1]). Since $\{P_{x,s}; (x,s)\in\mathbb{R}^d\times[0,T]\}$ solves the martingale problem for $\{\mathscr{L}_t; t\in[0,T]\}$, we conclude from this that $g(x(u\wedge\tau_R(s)), u\wedge\tau_R(s))$, $u\in[s,t]$, is a $P_{x,s}$-martingale after time s for each $(x,s)\in\bar{B}_R\times[0,t]$, ($u\wedge v$ denotes the minimum of u and v). Hence, taking into account (4.40b, c) and the definition of the stopping time $\tau_R(s)$, we obtain

$$\begin{aligned} g(x,s) &= E_{x,s}g(x(t\wedge\tau_R(s)), t\wedge\tau_R(s)) \\ &= E_{x,s}[\exp(f(x(t)))-1]\mathbb{1}_{\{\tau_R(s)>t\}} \end{aligned}$$

for $(x,s)\in\bar{B}_R\times[0,t]$. Together with (4.37) this yields (4.39).

The function (4.39) has compact support, but it is not smooth. It is even not differentiable on ∂B_R. To overcome this deficiency, we approach g by functions of the form

$$g_\varepsilon=k_\varepsilon * g,\qquad \varepsilon>0, \tag{4.41}$$

where $\{k_\varepsilon\}$ is a suitable family of smoothing kernels and $*$ denotes convolution on $\mathbb{R}^d$. To be concrete, we put $k_\varepsilon(x)=\varepsilon^{-d}k(\varepsilon^{-1}x)$, where k belongs to $\mathscr{D}$, is non-negative, and satisfies $\int k(x)\,dx=1$. The functions g_ε belong to $\mathscr{D}$. Of course, they do not satisfy Eq. (4.40a). However we do obtain the following bound.

LEMMA 4.12 *Given $R>0$, $t\in(0,T]$, and $f\in\mathscr{D}$ with* $\operatorname{supp} f\subset B_R$, *define g and g_ε by* (4.39) *and* (4.41), *respectively. Assume in addition that $f\leqq 0$. Then, for all sufficiently small $\varepsilon>0$, there exists a continuous function r_ε on $\mathbb{R}^d\times[0,t]$ vanishing outside of $B_{2R}\times[0,t]$*

such that

$$\left(\frac{\partial}{\partial s}+\mathscr{L}_s\right)g_\varepsilon \leqq r_\varepsilon \quad on \quad \mathbb{R}^d \times [0,t] \tag{4.42}$$

and

$$r_\varepsilon \to 0 \text{ uniformly on } \mathbb{R}^d \times [0,t] \text{ as } \varepsilon \to 0. \tag{4.43}$$

Proof Integrating by parts, we obtain for $(x,s) \in \mathbb{R}^d \in [0,t]$:

$$\begin{aligned}
&\left(\frac{\partial}{\partial s}+\mathscr{L}_s\right)g_\varepsilon(x,s) \\
&= \int_{\bar{B}_R} \left(\frac{\partial}{\partial s}+L_s^{(x)}\right)k_\varepsilon(x-y)g(y,s)\,dy = r_\varepsilon(x,s) \\
&\quad+\sum a^{ij}(x,s) \int_{\partial B_R} k_\varepsilon(x-y)\frac{\partial g}{\partial n}(y,s)n_i(y)n_j(y)\sigma(dy),
\end{aligned} \tag{4.44}$$

where

$$\begin{aligned}
r_\varepsilon(x,s) &= \int_{\bar{B}_R} k_\varepsilon(x-y)\frac{\partial g}{\partial s}(y,s)\,dy \\
&\quad+\tfrac{1}{2}\sum a^{ij}(x,s)\int_{\bar{B}_R} k_\varepsilon(x-y)\frac{\partial^2 g}{\partial y^i \partial y^j}(y,s)\,dy \\
&\quad+\sum b^i(x,s)\int_{\bar{B}_R} k_\varepsilon(x-y)\frac{\partial g}{\partial y^i}(y,s)\,dy.
\end{aligned} \tag{4.45}$$

The index x in $\mathscr{L}_s^{(x)}$ indicates that the operator $\mathscr{L}_s$ acts on the x-variable; σ denotes Lebesgue's surface measure on ∂B_R; $(\partial g/\partial n)(y,s)$ is the derivative of g with respect to the inner normal $n(y) = (n_1(y),\dots,n_d(y))$ to ∂B_R at point y. By assumption $f \leqq 0$. Hence $g \leqq 0$ on $\bar{B}_R \times [0,t]$ (see (4.37) and (4.39)). On the other hand, $g=0$ on $\partial B_R \times [0,t]$. Thus $\partial g/\partial n \leqq 0$. Consequently, the sum on the right of (4.44) is non-positive, and we arrive at (4.42). Clearly r_ε vanishes outside of $B_{2R}\times[0,t]$, provided that ε is sufficiently small. Because of (4.40a), we further conclude from (4.45) that

$$r_\varepsilon(x,s)=\int_{B_R} k_\varepsilon(x-y)\left\{\tfrac{1}{2}\sum[a^{ij}(x,s)-a^{ij}(y,s)]\frac{\partial^2 g}{\partial y^i\,\partial y^j}y,s)\right.$$

$$\left.+\sum[b^i(x,s)-b^i(y,s)]\frac{\partial g}{\partial y^i}(y,s)\right\}dy.$$

Letting $\varepsilon\to 0$, this yields (4.43), and the proof is complete. □

Proof of Lemma 4.11 The essence of this proof consists in a refinement of the formal transformations in (4.36). Fix $R>0$, $0\leqq s<t\leqq T$, and $f\in\mathscr{D}$ with $f\leqq 0$ and $\operatorname{supp} f\subset B_R$ arbitrarily. Define g and g_ε by (4.39) and (4.41), respectively. Let r_ε be as in Lemma 4.12. Set

$$h_\varepsilon=\log(1+g_\varepsilon). \tag{4.46}$$

Then, applying Lemma 4.3 and using (4.42), we obtain

$$\langle\mu(t),h_\varepsilon(t)\rangle-\langle\mu(s),h_\varepsilon(s)\rangle$$

$$=\int_s^t[\langle\dot\mu(u),h_\varepsilon(u)\rangle+\langle\mu(u),\dot h_\varepsilon(u)\rangle]\,du$$

$$=\int_s^t[\langle\dot\mu(u),h_\varepsilon(u)\rangle-\langle\mu(u),\mathscr{L}_u h_\varepsilon(u)+\tfrac{1}{2}|\nabla_u h_\varepsilon(u)|_u^2\rangle]\,du$$

$$+\int_s^t\left\langle\mu(u),\frac{\left(\dfrac{\partial}{\partial u}+\mathscr{L}_u\right)g_\varepsilon(u)}{1+g_\varepsilon(u)}\right\rangle du$$

$$\leqq\int_s^t\sup_{h\in\mathscr{D}}[\langle\dot\mu(u),h\rangle-\langle\mu(u),\mathscr{L}_u h+\tfrac{1}{2}|\nabla_u h|_u^2\rangle]\,du$$

$$+\int_s^t\left\langle\mu(u),\frac{r_\varepsilon(u)}{1+g_\varepsilon(u)}\right\rangle du. \tag{4.47}$$

From (4.39), (4.41), and (4.46) we conclude that $h_\varepsilon(s)\to\log(1+g(s))=\log[1+U_{s,t}^{(R)}(e^f-1)]$ and $h_\varepsilon(t)\to f$ uniformly as $\varepsilon\to 0$. We further know from Lemma 4.12 that $(1+g_\varepsilon)^{-1}r_\varepsilon$ vanishes outside of $B_{2R}\times[0,t]$ for small ε and converges to zero uniformly as $\varepsilon\to 0$. Thus, letting $\varepsilon\to 0$ in (4.47), we arrive at the desired estimate (4.38). □

4.5 Relaxation of Assumption A.1

The purpose of this section is to show that the assertion of Theorem 4.5 remains true, if we relax Assumption A.1, assuming that the drift coefficients b^i, $1\leqq i\leqq d$, are merely continuous instead of being locally Hölder continuous. Assumption A.1 has only been needed to prove Lemma 4.11. The non-degeneracy of the diffusion matrix a and the Hölder continuity of a and b on $\bar{B}_R\times[0,t]$ ensured that the function g defined by (4.39) is a classical solution of problem (4.40) and that Lemma 4.12 is applicable. It is therefore enough for us to show that inequality (4.38) remains true, if the drift coefficients are continuous but not locally Hölder continuous. This will be done by a limit procedure.

To begin with, fix $\nu\in\mathcal{M}$ and $\mu(\cdot)\in\mathscr{C}$ with $S_\nu(\mu(\cdot))<\infty$, $R>0$, $0\leqq s<t\leqq T$, and $f\in\mathscr{D}$ with $f\leqq 0$ and $\operatorname{supp} f\subset B_R$ arbitrarily. Suppose that the drift vector $b\colon\mathbb{R}^d\times[0,T]\to\mathbb{R}^d$ is continuous but not locally Hölder continuous. Then one can find a sequence of continuous maps $b_n\colon\mathbb{R}^d\times[0,T]\to\mathbb{R}^d$ such that b_n is Hölder continuous on $\bar{B}_R\times[0,T]$, $b_n=b$ outside of $B_{2R}\times[0,T]$, and $b_n\to b$ uniformly. Denote by $\mathscr{L}_u^{(n)}$, $u\in[0,T]$, the diffusion operators with diffusion matrix a and drift vector b_n. Let $\{P_{x,u}^{(n)};\ (x,u)\in\mathbb{R}^d\times[0,T]\}$ and $\{U_{u,v}^{(R,n)};\ 0\leqq u<v\leqq T\}$ denote the solution of the associated martingale problem and the associated semi-group, respectively. That the martingale problem for $\{\mathscr{L}_u^{(n)}; u\in[0,T]\}$ is well posed follows from Assumption A.2 by an application of the Cameron–Martin–Girsanov formula (cf. [29, Section 6.4] where the case of bounded coefficients has been treated). Because of our assumptions on b_n, inequality (4.38) holds with $U_{s,t}^{(R)}$ and $\mathscr{L}_u$ replaced by $U_{s,t}^{(R,n)}$ and $\mathscr{L}_u^{(n)}$, respectively; i.e.

$$\langle\mu(t),f\rangle-\langle\mu(s),\log[1+U_{s,t}^{(R,n)}(e^f-1)]\rangle$$
$$\leqq\int_s^t\|\dot{\mu}(u)-(\mathscr{L}_u^{(n)})^*\mu(u)\|_{\mu(u),u}^2\,du. \tag{4.48}$$

Here we have also used (4.8). Letting $n\to\infty$ in (4.48), we arrive at (4.38). Indeed, from [29, Theorem 11.1.4 and Lemma 11.1.2], we know that

$$U_{s,t}^{(R,n)}(e^f-1)(x)=E_{x,s}^{(n)}[\exp(f(x(t)))-1]\mathbb{1}_{\{\tau_R(s)>t\}}$$

$$\to E_{x,s}[\exp(f(x(t)))-1]\mathbb{1}_{\{\tau_R(s)>t\}}=U_{s,t}^{(R)}(e^f-1)(x)$$

pointwise and boundedly as $n\to\infty$. Therefore the expression on the left of (4.48) converges to the corresponding expression in (4.38). Concerning the right-hand side of (4.48), it suffices to notice that

$$\|\dot\mu(u)-(\mathscr{L}_u^{(n)})^*\mu(u)\|_{\mu(u),u}$$

$$\leq\|\dot\mu(u)-\mathscr{L}_u^*\mu(u)\|_{\mu(u),u}+\|(\mathscr{L}_u^{(n)}-\mathscr{L}_u)^*\mu(u)\|_{\mu(u),u},$$

$$\|(\mathscr{L}_u^{(n)}-\mathscr{L}_u)^*\mu(u)\|_{\mu(u),u}^2$$

$$=\frac{1}{2}\sup_{\hbar\in\mathscr{D}_{\mu(u),u}}\frac{|\langle\mu(u),(b_n-b,\nabla_u\hbar)_u\rangle|^2}{\langle\mu(u),|\nabla_u\hbar|_u^2\rangle}$$

$$\leq\tfrac{1}{2}\langle\mu(u),|b_n-b|_u^2\rangle,$$

and $\langle\mu(u),|b_n-b|_u^2\rangle\to0$ as $n\to\infty$ uniformly in u. Here we have used the definitions of $\mathscr{L}_u^{(n)}$ and $\mathscr{L}_u$, (4.7), and the Cauchy–Schwarz inequality.

5. LARGE DEVIATIONS FOR WEAKLY INTERFACING DIFFUSIONS

5.1 The N-particle model

In this section we introduce the model of weakly interacting diffusions, describe the family of probability laws of the associated empirical process, and formulate the assumptions under which the large deviation result will be established.

We first define the state space $\mathscr{M}_\infty$ and the space of sample paths $\mathscr{C}_\infty=C([0,T];\mathscr{M}_\infty)$ for the empirical process. As in Section 4, let $\mathscr{M}$ denote the space of probability measures on $\mathbb{R}^d$ furnished with the weak topology, and let $C([0, T]; \mathcal{M})$ be the space of continuous maps $[0, T] \to \mathcal{M}$ endowed with the topology of uniform convergence. Let $\varphi: \mathbb{R}^d \to \mathbb{R}$ be a non-negative, two times continuously differentiable function with $\lim_{|x| \to \infty} \varphi(x) = \infty$. This function will serve as a Lyapunov function for our system of weakly interacting diffusions (cf. the Assumptions B.2 and B.3 below). Given $R > 0$, we denote by $\mathcal{M}_R$ the topological subspace of $\mathcal{M}$ consisting of all $\mu \in \mathcal{M}$ for which $\langle \mu, \varphi \rangle \leq R$. We set

$$\mathcal{M}_\infty = \bigcup_{R > 0} \mathcal{M}_R$$

and equip this space with an "inductive" topology. By definition, a set G is open in $\mathcal{M}_\infty$ if $G \cap \mathcal{M}_R$ is open in $\mathcal{M}_R$ for each $R > 0$. In the same way, furnishing $C([0, T]; \mathcal{M}_R)$, $R > 0$, with the subspace topology of $C([0, T]; \mathcal{M})$ and observing that

$$C([0, T]; \mathcal{M}_\infty) = \bigcup_{R > 0} C([0, T]; \mathcal{M}_R),$$

we define an "inductive" topology on $C([0, T]; \mathcal{M}_\infty)$. One easily checks that $\mu_n \to \mu$ in $\mathcal{M}_\infty$ if and only if $\mu_n \to \mu$ in $\mathcal{M}$ and $\sup_n \langle \mu_n, \varphi \rangle < \infty$. Ananlogously, $\mu_n(\cdot) \to \mu(\cdot)$ in $C([0, T]; \mathcal{M}_\infty)$ if and only if $\mu_n(\cdot) \to \mu(\cdot)$ in $C([0, T]; \mathcal{M})$ and $\sup_n \sup_{t \in [0, T]} \langle \mu_n(t), \varphi \rangle < \infty$. Although the topologies on $\mathcal{M}_\infty$ and $C([0, T]; \mathcal{M}_\infty)$ are not metrizable, this fact does not lead to serious trouble. In this context we mention that a real-valued function on $\mathbb{R}^d \times \mathcal{M}_\infty$ or on $C([0, T]; \mathcal{M}_\infty)$ is continuous if and only if it is sequentially continuous. For further details about these topologies we refer to the appendix in [20].

We will frequently use the abbreviations

$$\mathscr{C} = C([0, T]; \mathcal{M}), \quad \mathscr{C}_R = C([0, T]; \mathcal{M}_R), \quad \text{and} \quad \mathscr{C}_\infty = C([0, T]; \mathcal{M}_\infty).$$

Now suppose that we are given continuous maps $a: \mathbb{R}^d \to \mathbb{S}^d$ and $b: \mathbb{R}^d \times \mathcal{M}_\infty \to \mathbb{R}^d$. (As before, $\mathbb{S}^d$ denotes the space of symmetric non-negative definite $d \times d$ real matrices.) We want to consider a system of N diffusions on $\mathbb{R}^d$ with diffusion matrix $a = \{a^{ij}\}$ and drift vector $b = \{b^i\}$, interacting via their empirical measure which enters the drift vector. To this end, let us introduce diffusion operators $\mathscr{L}(\mu)$, $\mu \in \mathcal{M}_\infty$, and $\mathscr{L}^{(N)}$, $N = 1, 2, \ldots$, acting on functions on $\mathbb{R}^d$ and $(\mathbb{R}^d)^N$, respectively:

$$\mathscr{L}(\mu)f(x)=\frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(x)\frac{\partial^2 f(x)}{\partial x^i\, dx^j}+\sum_{i=1}^{d} b^i(x;\mu)\frac{\partial f(x)}{\partial x^i},$$

$$\mathscr{L}^{(N)}f(x_1,\dots,x_N)=\sum_{k=1}^{N}\mathscr{L}_k(\varepsilon_{\mathbf{x}})f(x_1,\dots,x_N).$$

The index k indicates that the operator $\mathscr{L}(\varepsilon_{\mathbf{x}})$ acts on the variable x_k; $\varepsilon_{\mathbf{x}}=N^{-1}\sum_{k=1}^{N}\delta_{x_k}$ is the empirical measure of the particle configuration $\mathbf{x}=(x_1,\dots,x_N)\in(\mathbb{R}^d)^N$. The precise assumptions on the diffusion and drift coefficients will be given later. For the moment we assume only that for each N, the martingale problem for $\mathscr{L}^{(N)}$ is well posed. Let $\{P^{(N)}_{\mathbf{x},s};(\mathbf{x},s)\in(\mathbb{R}^d)^N\times[0,T]\}$ be the associated family of solutions. $P^{(N)}_{\mathbf{x},s}$ is nothing else than the *joint probability law of N weakly interacting diffusions* (with diffusion matrix a and drift vector b) which starts at times s at $\mathbf{x}=(x_1,\dots,x_N)$. Further, given $N\geqq 1$ and $(\mathbf{x},s)\in(\mathbb{R}^d)^N\times[0,T]$, we will denote by $\mathscr{P}^{(N)}_{\varepsilon_{\mathbf{x}},s}$ the *probability law on $\mathscr{C}_\infty$ of the empirical process* $\varepsilon_{\mathbf{x}(\cdot)}$ under $P^{(N)}_{\mathbf{x},s}$. In other words, $\mathscr{P}^{(N)}_{\varepsilon_{\mathbf{x}},s}$ is the image of the measure $P^{(N)}_{\mathbf{x},s}$ with respect to the continuous map

$$C([0,T];(\mathbb{R}^d)^N)\ni\mathbf{x}(\cdot)\mapsto\varepsilon_{\mathbf{x}(\cdot)}\in C([0,T];\mathscr{M}_\infty).$$

Since the distribution of $\varepsilon_{\mathbf{x}(\cdot)}$ under $P^{(N)}_{\mathbf{x},s}$ is invariant with respect to permutations of the initial configuration $\mathbf{x}=(x_1,\dots,x_N)$, the probability measures $\mathscr{P}^{(N)}_{v,s}$, $(v,s)\in\mathscr{M}^{(N)}\times[0,T]$, are well defined. (As before, $\mathscr{M}^{(N)}$ denotes the space of N-particle empirical measures.) For each N, the measures $\mathscr{P}^{(N)}_{v,s}$ are concentrated on $C([0,T];\mathscr{M}^{(N)})$ and define a strong Markov–Feller process with state space $\mathscr{M}^{(N)}$. The last fact follows from the strong Markov property of the family $\{P^{(N)}_{\mathbf{x},s};(\mathbf{x},s)\in(\mathbb{R}^d)^N\times[0,T]\}$ (cf. [29, Theorem 10.1.1 and Corollary 11.1.5], and [11, Theorem 10.13 and Remark 1 to this theorem]). We will write $P^{(N)}_{\mathbf{x}}$ and $\mathscr{P}^{(N)}_{v}$ instead of $P^{(N)}_{x,0}$ and $\mathscr{P}^{(N)}_{v,0}$, respectively.

The joint diffusion matrix a of the operators $\mathscr{L}(\mu)$, $\mu\in\mathscr{M}_\infty$, induces a Riemannian structure on $\mathbb{R}^d$. We will denote by ∇ and $|\ |$ the Riemannian gradient and the Riemannian norm on the associated tangent spaces, respectively. They are defined in the same way as in Section 4.2, where the time-dependent case has been considered.

Given $N \geqq 1$ and $f \in C_k^{2,1}$, set $F(\mathbf{x}, t) = N^{-1} \sum_{k=1}^N f(x_k, t)$ for $\mathbf{x} = (x_1, \ldots, x_N) \in (\mathbb{R}^d)^N$ and $t \in [0, T]$. From the definition of $P_{\mathbf{x}}^{(N)}$ we know that

$$F(\mathbf{x}(t), t) - F(\mathbf{x}(0), 0) - \int_0^t \left(\frac{\partial}{\partial u} + \mathscr{L}^{(N)} \right) F(\mathbf{x}(u), u)\, du,$$

$t \in [0, T]$, is a $P_{\mathbf{x}}^{(N)}$-martingale for each $\mathbf{x} \in (\mathbb{R}^d)^N$. The quadratic characteristic of this martingale does not depend on $\mathbf{x}$ and has the form

$$\int_0^t \sum_{k=1}^N |\nabla_k F(\mathbf{x}(u), u)|^2\, du,$$

where the index k indicates that the operator ∇ acts on the kth component of the space variable (cf. [23, Proposition 13.42]). Clearly $F(\mathbf{x}, t) = \langle \varepsilon_{\mathbf{x}}, f(t) \rangle$, $\mathscr{L}^{(N)} F(\mathbf{x}, t) = \langle \varepsilon_{\mathbf{x}}, \mathscr{L}(\varepsilon_{\mathbf{x}}) f(t) \rangle$, and $\sum_{k=1}^N |\nabla_k F(\mathbf{x}, t)|^2 = N^{-1} \langle \varepsilon_{\mathbf{x}}, |\nabla f(t)|^2 \rangle$. Taking this into account and remembering the definition of the measures $\mathscr{P}_\nu^{(N)}$, we conclude from the above that

$$M_t^f(\mu(\cdot)) = \langle \mu(t), f(t) \rangle - \langle \mu(0), f(0) \rangle - \int_0^t \left\langle \mu(u), \left(\frac{\partial}{\partial u} + \mathscr{L}(\mu(u)) \right) f(u) \right\rangle du \tag{5.1}$$

is a $\mathscr{P}_\nu^{(N)}$-martingale with quadratic characteristic

$$\langle\!\langle M^f \rangle\!\rangle_t(\mu(\cdot)) = N^{-1} \int_0^t \langle \mu(u), |\nabla f(u)|^2 \rangle\, du \tag{5.2}$$

for all $N \geqq 1$, $f \in C_k^{2,1}$, and $\nu \in \mathscr{M}^{(N)}$.

Throughout the rest of this paper we will assume that the following hypotheses are satisfied, in which φ denotes the same function as in the definition of $\mathscr{M}_\infty$:

(*B.1*) The matrices $a(x)$, $x \in \mathbb{R}^d$, are strictly positive definite. The map $a\colon \mathbb{R}^d \to \mathbb{S}^d$ is locally Hölder continuous. The map $b\colon \mathbb{R}^d \times \mathscr{M}_\infty \to \mathbb{R}^d$ is continuous.

(*B.2*) There exists some constant $\lambda > 0$ such that

$$\langle \mu, \mathscr{L}(\mu)\varphi + \tfrac{1}{2}|\nabla\varphi|^2 \rangle \leqq \lambda \langle \mu, \varphi \rangle$$

for all probability measures μ on $\mathbb{R}^d$ with compact topological support.

(*B.3*) For each $\bar{\mu}(\cdot) \in \mathscr{C}_\infty$ there exists some constant $\bar{\lambda} = \bar{\lambda}(\bar{\mu}(\cdot)) > 0$ such that

$$\mathscr{L}(\bar{\mu}(t))\varphi + \tfrac{1}{2}|\nabla\varphi|^2 \leqq \bar{\lambda}\varphi$$

for all $t \in [0, T]$.

(*B.4*) For each $\bar{\mu}(\cdot) \in \mathscr{C}_\infty$ the function

$$\mathscr{C}_\infty \ni \mu(\cdot) \mapsto \int_0^T \langle \mu(t), |b(\cdot; \mu(t)) - b(\cdot; \bar{\mu}(t))|^2 \rangle \, dt \in [0, \infty]$$

is sequentially continuous at point $\mu(\cdot) = \bar{\mu}(\cdot)$.

For the Curie–Weiss model considered in the Introduction, we have

$$d = 1, \quad a(x) \equiv \sigma^2 > 0, \quad b(x; \mu) = -x^3 + (1-\theta)x - \theta \int y \mu(dy).$$

It is not difficult to check that in this case Assumptions (B.1)–(B.4) are satisfied for $\varphi(x) = 1 + x^2$ and also for $\varphi(x) = 1 + \gamma x^4$ with $0 < \gamma < (2\sigma^2)^{-1}$.

We close this section with some comments on Assumptions (B.1)–(B.4). First of all we remark that the local Hölder continuity of the diffusion matrix a in Assumption (B.1) will only be used to apply Theorem 4.5. But as it has been pointed out in Remark 4.4, this supposition can be relaxed, requiring only that a is continuous. Assumptions (B.1) and (B.2) ensure in particular that the martingale problem for $\mathscr{L}^{(N)}$ is well-posed for all N. Indeed, since the drift and diffusion coefficients of $\mathscr{L}^{(N)}$ are continuous and the diffusion matrix is non-degenerate, there exists at most one solution ([29, Theorem 10.1.3]). Applying (B.2) to the empirical measures $\mu = \varepsilon_{\mathbf{x}}$, $\mathbf{x} = (x_1, \ldots, x_N) \in (\mathbb{R}^d)^N$, we find that $\mathscr{L}^{(N)}\Phi \leqq \lambda\Phi$ for the function

$\Phi(\mathbf{x}) = \langle \varepsilon_{\mathbf{x}}, \varphi \rangle = N^{-1} \sum_{k=1}^{N} \varphi(x_k)$. Hence the non-explosion condition of [29, Theorem 10.2.1], is fulfilled, which guarantees the existence of a solution to the martingale problem. The term $|\nabla\varphi|^2$ in Assumption (B.2) will be needed to obtain the exponential bounds in Section 5.3. Assumption (B.3) is the analogue of Assumption (B.2) for independent identical diffusions with diffusion matrix $\bar{a}(x, t) = a(x)$ and drift vector $\bar{b}(x, t) = b(x; \bar{\mu}(t))$. Assumptions (B.1) and (B.3) will allow us to apply Theorem 4.5 for the diffusion matrix $\bar{a}$, the drift vector $\bar{b}$, and the associated diffusion operators $\mathscr{L}_t = \mathscr{L}(\bar{\mu}(t))$. Indeed, the relaxed form of Assumption (A.1) considered in Section 4.5 follows from (B.1); Assumption (A.2) is a consequence of (B.3) ([29, Theorem 10.2.1]). Finally, Assumption (B.4) will allow us to approach our empirical process locally (along a fixed path $\bar{\mu}(\cdot)$) by the empirical process of independent diffusions with diffusion matrix $\bar{a}$ and "frozen" drift vector $\bar{b}$.

5.2 The main result

Given $\mu \in \mathscr{M}_\infty$ and $\vartheta \in \mathscr{D}'$, define

$$\|\vartheta\|_\mu^2 = \frac{1}{2} \sup_{f \in \mathscr{D}_\mu} \frac{|\langle \vartheta, f \rangle|^2}{\langle \mu, |\nabla f|^2 \rangle},$$

where $\mathscr{D}_\mu = \{f \in \mathscr{D} : \langle \mu, |\nabla f|^2 \rangle \neq 0\}$. One easily checks that $\| \ \|_\mu$ is a norm on the linear space $T_\mu = \{\vartheta \in \mathscr{D}' : \|\vartheta\|_\mu < \infty\}$. This allows the formal geometric interpretation that $\mathscr{M}_\infty$ is a Riemannian manifold with tangent spaces T_μ and Riemannian norm $\| \ \|_\mu$. We introduce a functional $S : \mathscr{C}_\infty \to [0, \infty]$ by setting

$$S(\mu(\cdot)) = \int_0^T \|\dot{\mu}(t) - \mathscr{L}(\mu(t))^* \|_{\mu(t)}^2 \, dt,$$

if $\mu(\cdot) \in \mathscr{C}_\infty$ is absolutely continuous (in the sense of Definition 4.1) and $S(\mu(\cdot)) = \infty$ otherwise. $\mathscr{L}(\mu)^*$ is the formal adjoint of the operator $\mathscr{L}(\mu)$ acting on $\mathscr{D}'$. The functional S is closely related to the functionals S_ν defined in Section 4.2. Indeed, given $\bar{\mu}(\cdot) \in \mathscr{C}_\infty$, set $\mathscr{L}_t = \mathscr{L}(\bar{\mu}(t))$, $t \in [0, T]$. Then $S(\bar{\mu}(\cdot))$ coincides with $S_{\bar{\mu}(0)}(\bar{\mu}(\cdot))$ defined by (4.7) and (4.9), where the Riemannian structure on $\mathbb{R}^d$ has to be taken with respect to the diffusion matrix $a(\cdot, t) \equiv a(\cdot)$.

We are now in a position to formulate the main result of this paper which holds under Assumptions (B.1)–(B.4).

THEOREM 5.1 *(a) Given* $\nu_N \in \mathcal{M}^{(N)}$ *and* $\nu \in \mathcal{M}_\infty$, *suppose that* $\nu_N \to \nu$ *in* $\mathcal{M}_\infty$. *Then*

$$\liminf_{N\to\infty} N^{-1} \log \mathcal{P}^{(N)}_{\nu_N}(G) \geqq -\inf\{S(\mu(\cdot)): \mu(\cdot) \in G, \mu(0)=\nu\}$$

for all open sets G in $\mathcal{C}_\infty$.

b) Given $\nu_N \in \mathcal{M}^{(N)}$ *and* $\nu \in \mathcal{M}_\infty$, *suppose that* $\nu_N \to \nu$ *in* $\mathcal{M}_\infty$. *Then*

$$\limsup_{N\to\infty} N^{-1} \log \mathcal{P}^{(N)}_{\nu_N}(F) \leqq -\inf\{S(\mu(\cdot)): \mu(\cdot) \in F, \mu(0)=\nu\}$$

for all closed sets F in $\mathcal{C}_\infty$.

c) For each compact set K in $\mathcal{M}_\infty$ *and each* $s \geqq 0$, *the set*

$$\Phi_K(s) = \{\mu(\cdot) \in \mathcal{C}_\infty : S(\mu(\cdot)) \leqq s, \mu(0) \in K\}$$

is compact in $\mathcal{C}_\infty$.

The proof of Theorem 5.1 will be divided into two parts. In Section 5.4 we will derive the following "local" version of Theorem 5.1.

THEOREM 5.2 *Given* $\nu_N \in \mathcal{M}^{(N)}$ *and* $\nu \in \mathcal{M}_\infty$, *suppose that* $\nu_N \to \nu$ *in* $\mathcal{M}_\infty$. *Then the following assertions are valid for each* $\bar{\mu}(\cdot) \in \mathcal{C}_\infty$ *with* $\bar{\mu}(0) = \nu$.

a) For each open neighborhood V of $\bar{\mu}(\cdot)$ *in* $\mathcal{C}_\infty$,

$$\liminf_{N\to\infty} N^{-1} \log \mathcal{P}^{(N)}_{\nu_N}(V) \geqq -S(\bar{\mu}(\cdot)).$$

b) For each $\gamma > 0$ *there exists an open neighborhood V of* $\bar{\mu}(\cdot)$ *in* $\mathcal{C}_\infty$ *such that*

$$\limsup_{N\to\infty} N^{-1} \log \mathcal{P}^{(N)}_{\nu_N}(V) \leqq -S(\bar{\mu}(\cdot)) + \gamma \tag{5.3}$$

provided that $S(\bar{\mu}(\cdot)) < \infty$. *If* $S(\bar{\mu}(\cdot)) = \infty$, *then this assertion holds with the expression on the right of* (5.3) *replaced by* $-\gamma$.

In Section 5.3 we will derive the following exponential bound for $\mathscr{P}_\nu^{(N)}$ which will allow us to convert the "local" result into a "global" one.

THEOREM 5.3 *For all positive numbers r and s there exists a compact set $\mathscr{K}$ in $\mathscr{C}_\infty$ such that*

$$\limsup_{N\to\infty} N^{-1}\log \sup_{\nu\in\mathscr{M}_r\cap\mathscr{M}^{(N)}} \mathscr{P}_\nu^{(N)}(\mathscr{C}_\infty\setminus\mathscr{K}) \leqq -s.$$

We complete this section by showing how Theorem 5.1 can be derived from Theorem 5.2 and Theorem 5.3.

Proof of Theorem 5.1 Applying assertion (a) of Theorem 5.2 for $V=G$ and all $\bar{\mu}(\cdot)\in G$ with $\bar{\mu}(0)=\nu$, we immediately arrive at assertion (a) of Theorem 5.1.

We next prove assertion (b). Given $\nu_N\in\mathscr{M}^{(N)}$ and $\nu\in\mathscr{M}_r$, suppose that $\nu_N\to\nu$ in $\mathscr{M}_\infty$. Let F be an arbitrary non-empty closed subset of $\mathscr{C}_\infty$, We set

$$s=\inf\{S(\mu(\cdot)):\mu(\cdot)\in F, \mu(0)=\nu\}$$

and assume that $s<\infty$. The case $s=\infty$ can be handled analogously. From Theorem 5.3 we know that there exists a compact set $\mathscr{K}$ in $\mathscr{C}_\infty$ such that

$$\limsup_{N\to\infty} N^{-1}\log \mathscr{P}_{\nu_N}^{(N)}(\mathscr{C}_\infty\setminus\mathscr{K}) \leqq -s. \tag{5.4}$$

Now fix $\gamma>0$ arbitrarily. By assertion (b) of Theorem 5.2, we find for each $\mu(\cdot)\in F\cap\mathscr{K}$ an open neighborhood V of $\mu(\cdot)$ such that

$$\limsup_{N\to\infty} N^{-1}\log \mathscr{P}_{\nu_N}^{(N)}(V\cap\mathscr{K}) \leqq -s+\gamma.$$

Since $F\cap\mathscr{K}$ is covered by a finite number of such neighborhoods, this yields

$$\limsup_{N\to\infty} N^{-1}\log \mathscr{P}_{\nu_N}^{(N)}(F\cap\mathscr{K}) \leqq -s+\gamma. \tag{5.5}$$

Combining (5.4) and (5.5), we obtain

$$\limsup_{N\to\infty} N^{-1}\log \mathscr{P}_{\nu_N}^{(N)}(F) \leqq -s+\gamma.$$

Since $\gamma > 0$ may be chosen arbitrarily small, this yields the desired result.

To prove assertion (c), fix $s \geqq 0$ and a non-empty compact set K in $\mathcal{M}_\infty$ arbitrarily. Choose $r > 0$ so that $K \subseteqq \mathcal{M}_{r/2}$. By Theorem 5.3, there exists a compact set $\mathcal{K}$ in $\mathcal{C}_\infty$ such that

$$\limsup_{N\to\infty} N^{-1} \log \sup_{\nu \in \mathcal{M}_r \cap \mathcal{M}^{(N)}} \mathcal{P}_\nu^{(N)}(\mathcal{C}_\infty \backslash \mathcal{K}) < -s. \tag{5.6}$$

We claim that $\Phi_K(s) \subseteqq \mathcal{K}$. To prove this, we choose $\bar{\mu}(\cdot) \in \mathcal{C}_\infty \backslash \mathcal{K}$ with $\bar{\mu}(0) \in K$ arbitrarily and show that $\bar{\mu}(\cdot)$ does not belong to $\Phi_K(s)$. We can find measures $\nu_N \in \mathcal{M}_r \cap \mathcal{M}^{(N)}$ such that $\nu_N \to \bar{\mu}(0)$ in $\mathcal{M}_\infty$. Since $\bar{\mu}(0) \in \mathcal{M}_{r/2}$, this follows, for example, from the law of large numbers for the empirical measures of independent random vectors with distribution $\bar{\mu}(0)$. By assertion (a) of Theorem 5.2,

$$\liminf_{N\to\infty} N^{-1} \log \mathcal{P}_{\nu_N}^{(N)}(\mathcal{C}_\infty \backslash \mathcal{K}) \geqq -S(\bar{\mu}(\cdot)). \tag{5.7}$$

Combining (5.6) with (5.7), we conclude that $S(\bar{\mu}(\cdot)) > s$ or, what is the same, $\bar{\mu}(\cdot) \notin \Phi_K(s)$. Thus $\Phi_K(s) \subseteqq \mathcal{K}$. We have therefore shown that $\Phi_K(s)$ is relatively compact.

Given $\mu(\cdot) \in \mathcal{C}_\infty$ and $f \in C_k^{2,1}$, define

$$I(\mu(\cdot); f) = \langle \mu(T), f(T) \rangle - \langle \mu(0), f(0) \rangle$$
$$- \int_0^T \left\langle \mu(t), \left(\frac{\partial}{\partial t} + \mathcal{L}(\mu(t)) \right) f(t) + \tfrac{1}{2} |\nabla f(t)|^2 \right\rangle dt.$$

One easily checks that the functions $\mu(\cdot) \mapsto I(\mu(\cdot); f)$, $f \in C_k^{2,1}$, are sequentially continuous on $\mathcal{C}_\infty$. Hence, they are continuous, and the sets $\Phi_K^f(s) = \{\mu(\cdot) \in \mathcal{C}_\infty : I(\mu(\cdot); f) \leqq s, \mu(0) \in K\}$ are closed. But from Lemma 4.8 we know that

$$S(\mu(\cdot)) = \inf_{f \in C_k^{2,1}} I(\mu(\cdot); f).$$

Therefore

$$\Phi_K(s) = \bigcap_{f \in C_k^{2,1}} \Phi_K^f(s).$$

This clearly proves that $\Phi_K(s)$ is closed, and we are done. □

5.3 Some exponential bounds

The purpose of this section is to prove Theorem 5.3. We denote by $C_k(\mathbb{R}^d)$ the space of real-valued continuous functions on $\mathbb{R}^d$ having compact support and endow it with the supremum norm. We need the following compactness criterion.

LEMMA 5.4 *Let* $\{f_n; n=1,2,\ldots\}$ *be a countable dense subset of* $C_k(\mathbb{R}^d)$. *A set* $\mathscr{K}$ *is relatively compact in* $\mathscr{C}_\infty$ *if and only if it is contained in a set of the form*

$$\hat{\mathscr{K}} = \mathscr{C}_k \cap \bigcap_n \mathscr{K}_n. \tag{5.8}$$

where $R>0$,

$$\mathscr{K}_n = \{\mu(\cdot) \in \mathscr{C}_\infty : \langle \mu(\cdot), f_n \rangle \in K_n\}, \tag{5.9}$$

and K_n *are compact subsets of* $C([0, T]; \mathbb{R})$.

Proof First of all we remark that a set $\mathscr{K}$ is relatively compact in $\mathscr{C}_\infty$ if and only if it is relatively compact in $\mathscr{C}$ and entirely contained in $\mathscr{C}_R$ for some $R>0$ (cf. the Appendix of [20]). It will therefore be enough to show that a set $\mathscr{K}'$ is relatively compact in $\mathscr{C}$ if and only if it is contained in a set of the form

$$\hat{\mathscr{K}}' = \mathscr{K}'_{\mathscr{M}} \cap \bigcap_n \mathscr{K}'_n,$$

where

$$\mathscr{K}'_{\mathscr{M}} = \{\mu(\cdot) \in \mathscr{C} : \mu(t) \in K'_{\mathscr{M}} \quad \text{for all } t \in [0, T]\}$$

and

$$\mathscr{K}'_n = \{\mu(\cdot) \in \mathscr{C} : \langle \mu(\cdot), f_n \rangle \in K'_n\}.$$

Here $K'_{\mathscr{M}}$ and K'_n denote compact subsets of $\mathscr{M}$ and $C([0, T]; \mathbb{R})$, respectively. Since $\mathscr{C}$ is metrizable, the last assertion is readily checked by proving sequential compactness instead of compactness. For details the reader is referred to [20]. □

LEMMA 5.5 *For any positive numbers r, R and all N.*

$$\sup_{\nu \in \mathcal{M}_r \cap \mathcal{M}^{(N)}} \mathcal{P}_\nu^{(N)}(\mathcal{C}_\infty \setminus \mathcal{C}_R) \leqq \exp(-NR_T),$$

where $R_T = R\exp(-\lambda T) - r$ *and* λ *is taken from Assumption (B.2).*

Proof Let $\sigma_N(\mu(\cdot)) = \inf\{t \in [0, T]: \mu(t) \notin \mathcal{M}^{(N)}\}$, where, by convention, $\inf \phi = \infty$. Note that σ_N is a stopping time with respect to the right-continuous extension of the canonical filtration on $\mathcal{C}_\infty$. Define

$$M_t(\mu(\cdot)) = \exp(-\lambda(t \wedge \sigma_N))\langle \mu(t \wedge \sigma_N), \varphi\rangle - \langle \mu(0), \varphi\rangle$$

$$- \int_0^{t \wedge \sigma_N} \exp(-\lambda u)\langle \mu(u), (\mathcal{L}(\mu(u)) - \lambda)\varphi\rangle\, du,$$

$t \in [0, T]$, where λ and φ are taken from Assumption (B.2) and $t \wedge \sigma_N$ denotes the minimum of t and σ_N. The stopping time σ_N ensures that the expression on the right makes sense for all $\mu(\cdot) \in \mathcal{C}_\infty$. From (5.1) and (5.2) it follows that, for each N and each $\nu \in \mathcal{M}^{(N)}$, M is a continuous local $\mathcal{P}_\nu^{(N)}$-martingale with quadratic characteristic

$$\langle\langle M\rangle\rangle_t(\mu(\cdot)) = N^{-1} \int_0^{t \wedge \sigma_N} \exp(-2\lambda u)\langle \mu(u), |\nabla\varphi|^2\rangle\, du$$

(with respect to the right-continuous extension of the canonical filtration). Using Assumption (B.2), we obtain for all N and all $\nu \in \mathcal{M}_r \cap \mathcal{M}^{(N)}$:

$$\exp(-\lambda t)\langle \mu(t), \varphi\rangle \leqq r + M_t - \frac{N}{2}\langle\langle M\rangle\rangle_t, \quad t \in [0, T], \quad \mathcal{P}_\nu^{(N)}\text{-a.s.}$$

Hence, for all N and all $\nu \in \mathcal{M}_r \cap \mathcal{M}^{(N)}$,

$$\mathcal{P}_\nu^{(N)}(\mathcal{C}_\infty \setminus \mathcal{C}_R) = \mathcal{P}_\nu^{(N)}\left(\sup_{t \in [0,T]} \langle \mu(t), \varphi\rangle > R\right)$$

$$\leqq \mathcal{P}_\nu^{(N)}\left(\sup_{t \in [0,T]} \exp\left(NM_t - \frac{N^2}{2}\langle\langle M\rangle\rangle_t\right) > \exp(NR_T)\right).$$

Since $\exp[NM-(N^2/2)\langle\langle M\rangle\rangle]$ is a non-negative $\mathscr{P}_\nu^{(N)}$-supermartingale (see, for example, [22, Chap. 3, Theorem 5.2]), the assertion now follows by an application of Doob's supermartingale inequality. □

LEMMA 5.6 *For all* $R>0$, $s>0$, *and* $f\in\mathscr{D}$ *there exists a compact subset* K *of* $C([0,T];\mathbb{R})$ *such that*

$$\mathscr{P}_\nu^{(N)}(\mathscr{C}_R\backslash\mathscr{K}_f)\leqq\exp(-Ns)$$

for all N *and all* $\nu\in\mathscr{M}^{(N)}$, *where*

$$\mathscr{K}_f=\{\mu(\cdot)\in\mathscr{C}_\infty:\langle\mu(\cdot),f\rangle\in K\}. \tag{5.10}$$

Proof Fix $R>0$, $s>0$, and $f\in\mathscr{D}$ arbitrarily. Applying the Markov property, we derive for any $\delta\in(0,T/2]$ and $\rho>0$ the estimate

$$\begin{aligned}&\sup_{\nu\in\mathscr{M}^{(N)}}\mathscr{P}_\nu^{(N)}\Bigg(\sup_{\substack{0\leqq u<v\leqq T\\ v-u<\delta}}|\langle\mu(v),f\rangle-\langle\mu(u),f\rangle|>\rho,\mu(\cdot)\in\mathscr{C}_R\Bigg)\\
&\leqq\sum_{k=0}^{[T/\delta]-1}\sup_{\nu\in\mathscr{M}^{(N)}}\mathscr{P}_\nu^{(N)}\Bigg(\sup_{k\delta\leqq t<((k+2)\delta)\wedge T}\\
&\qquad\times|\langle\mu(t),f\rangle-\langle\mu(k\delta),f\rangle|>\frac{\rho}{2},\mu(\cdot)\in\mathscr{C}_R\Bigg)\\
&\leqq\frac{T}{\delta}\sup_{\nu\in\mathscr{M}^{(N)}}\mathscr{P}_\nu^{(N)}\Bigg(\sup_{t<2\delta}|\langle\mu(t),f\rangle-\langle\mu(0),f\rangle|>\frac{\rho}{2},\mu(\cdot)\in\mathscr{C}_{R,2\delta}\Bigg).\end{aligned} \tag{5.11}$$

Here $\mathscr{C}_{R,2\delta}=C([0,2\delta];\mathscr{M}_R)$; $[T/\delta]$ denotes the integer part of T/δ. Clearly

$$\sup_{\mu\in\mathscr{M}_R}\langle\mu,|\mathscr{L}(\mu)f|+\tfrac{1}{2}|\nabla f|^2\rangle\leqq\kappa<\infty$$

for some constant κ. It therefore follows from (5.1) and (5.2) that

$$\langle\mu(t),f\rangle-\langle\mu(0),f\rangle\leqq(1+\gamma)\kappa t+M_t^f-\frac{N\gamma}{2}\langle\langle M^f\rangle\rangle_t$$

for all $\gamma \geqq 0$, $\mu(\cdot) \in \mathscr{C}_{R,2\delta}$, and $t \in [0, 2\delta]$. Using this, we obtain for all N and all $\nu \in \mathscr{M}^{(N)}$ the estimate

$$\mathscr{P}_\nu^{(N)}\left(\sup_{t<2\delta}\left[\langle \mu(t), f\rangle - \langle \mu(0), f\rangle\right] > \frac{\rho}{2}, \mu(\cdot) \in \mathscr{C}_{R,2\delta}\right)$$

$$= \mathscr{P}_\nu^{(N)}\left(\sup_{t<2\delta}\left[M_t^f - \frac{N\gamma}{2}\langle\!\langle M^f \rangle\!\rangle_t\right] > \frac{\rho}{2} - 2(1+\gamma)\kappa\delta\right)$$

$$\leqq \exp\left(-N\gamma\left(\frac{\rho}{2} - 2(1+\gamma)\kappa\delta\right)\right). \tag{5.12}$$

In the last step we have applied Doob's inequality to the exponential $\mathscr{P}_\nu^{(N)}$-martingale $\exp[N\gamma M^f - (N^2\gamma^2/2)\langle\!\langle M^f \rangle\!\rangle]$ (cf. [22, Chap. 3, Theorem 5.3]). Minimizing the expression on the right of (5.12) with respect to $\gamma \geqq 0$, we get for $\rho > 4\kappa\delta$:

$$\mathscr{P}_\nu^{(N)}\left(\sup_{t<2\delta}\left[\langle \mu(t), f\rangle - \langle \mu(0), f\rangle > \frac{\rho}{2}, \mu(\cdot) \in \mathscr{C}_{R,2\delta}\right)\right.$$

$$\leqq \exp\left(-N\frac{(\rho - 4\kappa\delta)^2}{32\kappa\delta}\right). \tag{5.13}$$

In exactly the same way we obtain

$$\mathscr{P}_\nu^{(N)}\left(\sup_{t<2\delta}\left[\langle \mu(0), f\rangle - \langle \mu(t), f\rangle\right] > \frac{\rho}{2}, \mu(\cdot) \in \mathscr{C}_{R,2\delta}\right)$$

$$\leqq \exp\left(-N\frac{(\rho - 4\kappa\delta)^2}{32\kappa\delta}\right). \tag{5.14}$$

Combining (5.11) with (5.13) and (5.14), we finally find that

$$\sup_{\nu \in \mathscr{M}^{(N)}} \mathscr{P}_\nu^{(N)}\left(\sup_{\substack{0 \leqq u < v \leqq T \\ v-u<\delta}} \left|\langle \mu(v), f\rangle - \langle \mu(u), f\rangle\right| > \rho, \mu(\cdot) \in \mathscr{C}_R\right)$$

$$\leqq \frac{2T}{\delta}\exp\left(-N\frac{(\rho - 4\kappa\delta)^2}{32\kappa\delta}\right) \tag{5.15}$$

for all $\delta \in (0, T/2]$ and $\rho > 4\kappa\delta$.

Let (δ_n) and (ρ_n) be arbitrary null sequences satisfying $\delta_n \in (0, T/2]$ and $\rho_n > 4\kappa\delta_n$ for all n. By the Arzelà–Ascoli theorem, the set

$$K = \{|x(0)| \leqq \sup_y |f(y)|\} \cap \bigcap_n \left\{ \sup_{\substack{0 \leqq u < v \leqq T \\ v-u<\delta_n}} |x(v) - x(u)| \leqq \rho_n \right\}$$

is compact in $C([0, T]; \mathbb{R})$. Define $\mathscr{K}_f$ by (5.10). Then (5.15) yields

$$\sup_{v \in \mathscr{M}^{(N)}} \mathscr{P}_v^{(N)}(\mathscr{C}_R \setminus \mathscr{K}_f) \leqq \sum_n \frac{2T}{\delta_n} \exp\left(-N \frac{(\rho_n - 4\kappa\delta_n)^2}{32\kappa\delta_n} \right). \tag{5.16}$$

To finish the proof, let us assume without loss of generality that $s > 1$ and $\kappa > T^{-1}$. It then suffices to choose the null sequences (δ_n) and (ρ_n) in such a way that, for each N, the sum on the right of (5.16) does not exceed $\exp(-Ns)$. To see that such a choice is possible, we pick

$$\delta_n = \frac{T}{2} n^{-2} \quad \text{and} \quad \rho_n = 10T\kappa s^{1/2} n^{-1/2}.$$

Then $\delta_n \in (0, T/2]$ and $\rho_n > 4\kappa\delta_n$ for all N. Moreover, we get for all N:

$$\sum_{n=1}^{\infty} \frac{2T}{\delta_n} \left(-N \frac{(\rho_n - 4\kappa\delta_n)^2}{32\kappa\delta_n} \right) \leqq \sum_{n=1}^{\infty} 4n^2 \exp(-4Nsn)$$

$$\leqq \sum_{n=1}^{\infty} \exp(-2Nsn) = \frac{\exp(-2Ns)}{1 - \exp(-2Ns)} \leqq \frac{e^{-s}}{1 - e^{-s}} \exp(-Ns)$$

$$\leqq \exp(-Ns).$$

This completes the proof of Lemma 5.6. □

Now, combining the Lemmas 5.4–5.6, we are at last in a position to prove Theorem 5.3.

Proof of Theorem 5.3 Let us fix arbitrary positive numbers r and s and let $\{f_n; n = 1, 2, \ldots\}$ be a countable dense subset of $\mathscr{D}$. For $\mathscr{K}$ we put the compact set $\hat{\mathscr{K}}$ defined by (5.8) and (5.9), where R and

K_n will be specified in the course of the proof. Then we get

$$\sup_{\nu \in \mathcal{M}_r \cap \mathcal{M}^{(N)}} \mathcal{P}_\nu^{(N)}(\mathcal{C}_\infty \backslash \mathcal{K})$$

$$\leqq \sup_{\nu \in \mathcal{M}_r \cap \mathcal{M}^{(N)}} \mathcal{P}_\nu^{(N)}(\mathcal{C}_\infty \backslash \mathcal{C}_R) + \sum_{n=1}^{\infty} \sup_{\nu \in \mathcal{M}^{(N)}} \mathcal{P}_\nu^{(N)}(\mathcal{C}_R \backslash \mathcal{K}_n). \tag{5.17}$$

We choose R so large that

$$\sup_{\nu \in \mathcal{M}_r \cap \mathcal{M}^{(N)}} \mathcal{P}_\nu^{(N)}(\mathcal{C}_\infty \backslash \mathcal{C}_R) \leqq \exp(-Ns) \tag{5.18}$$

for all N. This is possible because of Lemma 5.5. By Lemma 5.6, we can choose for each n the compact set $K_n \subset C([0, T]; \mathbb{R})$ so "large" that

$$\sup_{\nu \in \mathcal{M}^{(N)}} \mathcal{P}_\nu^{(N)}(\mathcal{C}_R \backslash \mathcal{K}_n) \leqq \exp(-nNs) \tag{5.19}$$

for all n and N, where $\mathcal{K}_n$ is defined by (5.9). Combining (5.17) with (5.18), we arrive at the desired result. □

5.4 Proof of the local result

The present section is devoted to the proof of Theorem 5.2. Locally, along a fixed path $\bar{\mu}(\cdot) \in \mathcal{C}_\infty$, the empirical process of our system of weakly interacting diffusions may be regarded as a small perturbation of the empirical process for independent diffusions with the same diffusion matrix $\bar{a}(x, t) = a(x)$ and the drift vector $\bar{b}(x, t) = b(x; \bar{\mu}(t))$ which is "frozen" along $\bar{\mu}(\cdot)$. This observation allows us to reduce the proof of Theorem 5.2 essentially to an application of Theorem 4.5 and an estimation of the deviation of the perturbed process from the unperturbed one. The deviation can be measured with the help of the Cameron–Martin–Girsanov formula. That it is small in a narrow vicinity of $\bar{\mu}(\cdot)$ will be shown to follow from Assumption (B.4).

Fix $\bar{\mu}(\cdot) \in \mathcal{C}_\infty$ arbitrarily and define

$$\mathcal{L}_t^{(N)} f(x_1, \ldots, x_N) = \sum_{k=1}^{N} \mathcal{L}_k(\bar{\mu}(t)) f(x_1, \ldots, x_N),$$

where the index k in $\mathscr{L}_k(\bar{\mu}(t))$ indicates that this operator acts on the variable x_k. It follows from Assumptions (B.1) and (B.3) that the martingale problem for $\{\bar{\mathscr{L}}_t^{(N)}; \in[0, T]\}$ is well posed for each N. This can be shown in the same way as it has been done for $\mathscr{L}^{(N)}$ in Section 5.1. Let $\{\bar{P}_{\mathbf{x},s}^{(N)}; (\mathbf{x}, s) \in (\mathbb{R}^d)^N \times [0, T]\}$ be the associated family of probability laws on $C([0, T]; (\mathbb{R}^d)^N) = C([0, T]; \mathbb{R}^d)^N$. It is not hard to check that

$$\bar{P}_{\mathbf{x},s}^{(N)} = \bar{P}_{x_1,s} \otimes \cdots \otimes \bar{P}_{x_N,s}$$

for all $N, \mathbf{x} = (x_1, \ldots, x_N) \in (\mathbb{R}^d)^N$, and $s \in [0, T]$, where $\bar{P}_{x,s} = \bar{P}_{x,s}^{(1)}$ is the solution of the martingale problem for $\{\mathscr{L}(\bar{\mu}(t)); t \in [0, T]\}$. Let $\{\bar{\mathscr{P}}_{v,s}^{(N)}; (v, s) \in \mathscr{M}^{(N)} \times [0, T]\}$ denote the family of probability laws on $\mathscr{C}$ of the empirical process $\varepsilon_{\mathbf{x}(\cdot)}$ induced by the measures $\bar{\mathscr{P}}_{\mathbf{x},s}^{(N)}$. Clearly Theorem 4.5 is applicable to the measures $\bar{\mathscr{P}}_v^{(N)} = \bar{\mathscr{P}}_{v,0}^{(N)}$.

The operators $\mathscr{L}^{(N)}$ and $\bar{\mathscr{L}}_t^{(N)}$ coincide up to different drift vectors, and the associated martingale problems are well-posed. This implies that, for each N and $\mathbf{x} \in (\mathbb{R}^d)^N$, the measure $P_{\mathbf{x}}^{(N)}$ is absolutely continuous with respect to $\bar{P}_{\mathbf{x}}^{(N)} = \bar{P}_{\mathbf{x},0}^{(N)}$. In the Riemannian metric induced by the joint diffusion matrix, the squared distance between the drift vectors of $\mathscr{L}^{(N)}$ and $\bar{\mathscr{L}}_t^{(N)}$ at point $(\mathbf{x}, t) \in (\mathbb{R}^d)^N \times [0, T]$ equals $N\langle \varepsilon_{\mathbf{x}}, |b(\cdot; \varepsilon_{\mathbf{x}}) - b(\cdot; \bar{\mu}(t))|^2 \rangle$. Hence the Cameron–Martin–Girsanov formula yields for the Radon–Nikodym derivative

$$\frac{dP_{\mathbf{x}}^{(N)}}{d\bar{P}_{\mathbf{x}}^{(N)}} = \exp(M_T^{(N)} - \tfrac{1}{2}\langle\!\langle M^{(N)} \rangle\!\rangle_T). \tag{5.20}$$

where $M^{(N)}$ is a continuous local $\bar{P}_{\mathbf{x}}^{(N)}$-martingale with $M_0^{(N)} = 0$. Its quadratic characteristic does not depend on $\mathbf{x}$ and has the form

$$\langle\!\langle M^{(N)} \rangle\!\rangle_t(\mathbf{x}(\cdot)) = N \int_0^t \langle \varepsilon_{\mathbf{x}(u)}, |b(\cdot; \varepsilon_{\mathbf{x}(u)}) - b(\cdot; \bar{\mu}(u))|^2 \rangle \, du \tag{5.21}$$

In the case of bounded drift and diffusion coefficients, these assertions can be found, e.g. in [29, Section 6.4]. The case of unbounded coefficients can be reduced to the previous one by spatial localization, using the second part of Theorem 10.1.1 in [29].

We now have all the ingredients necessary for the proof of Theorem 5.2. Given $\nu_N \in \mathscr{M}^{(N)}$, suppose that $\nu_N \to \bar{\mu}(0)$ in $\mathscr{M}_\infty$. Pick $\mathbf{x}_N \in (\mathbb{R}^d)^N$ so that $\varepsilon_{\mathbf{x}_N} = \nu_N$. Fix $\gamma > 0$ arbitrarily.

We first prove assertion (a). To this end, we assume without loss of generality that $S(\bar{\mu}(\cdot)) < \infty$ and consider an arbitrary open neighborhood V of $\bar{\mu}(\cdot)$. By the definition of the measures $\mathscr{P}_\nu^{(N)}$, it is certainly enough to show that

$$\liminf_{N \to \infty} N^{-1} \log P_{\mathbf{x}_N}^{(N)}(\varepsilon_{\mathbf{x}(\cdot)} \in V) \geqq -S(\bar{\mu}(\cdot)) - \gamma. \tag{5.22}$$

Assumption (B.3) guarantees that Lemma 5.5 is applicable to $\bar{\mathscr{P}}_\nu^{(N)}$ instead of $\mathscr{P}_\nu^{(N)}$ with λ replaced by $\bar{\lambda}$. Consequently, there exists some $R > 0$ such that $\bar{\mu}(\cdot) \in \mathscr{C}_R$ and

$$\limsup_{N \to \infty} N^{-1} \log \bar{P}_{\mathbf{x}_N}^{(N)}(\varepsilon_{\mathbf{x}(\cdot)} \notin \mathscr{C}_R) \leqq -S(\bar{\mu}(\cdot)) - \gamma. \tag{5.23}$$

We next choose $p, q > 1$ with $p^{-1} + q^{-1} = 1$ and $\delta > 0$ so that

$$\frac{1}{2}\left(1 + \frac{q}{p}\right)\delta + pS(\bar{\mu}(\cdot)) \leqq S(\bar{\mu}(\cdot)) + \gamma. \tag{5.24}$$

Because of Assumption (B.4) and (5.21) there exists a neighborhood W of $\bar{\mu}(\cdot)$ in $\mathscr{C}$ such that $W \cap \mathscr{C}_R \subseteqq V$ and $\langle\!\langle M^{(N)} \rangle\!\rangle_T < N\delta$ on $\{\varepsilon_{\mathbf{x}(\cdot)} \in W \cap \mathscr{C}_R\}$. Thus, applying (5.20) and Hölder's inequality, we obtain for all N:

$$\begin{aligned}
P_{\mathbf{x}_N}^{(N)}(\varepsilon_{\mathbf{x}(\cdot)} \in V) &\geqq P_{\mathbf{x}_N}^{(N)}(\varepsilon_{\mathbf{x}(\cdot)} \in W \cap \mathscr{C}_R) \\
&= \bar{E}_{\mathbf{x}_N}^{(N)} \exp(M_T^{(N)} - \tfrac{1}{2}\langle\!\langle M^{(N)} \rangle\!\rangle_T) \mathbb{1}_{\{\varepsilon_{\mathbf{x}(\cdot)} \in W \cap \mathscr{C}_R\}} \\
&\geqq \exp\left(-\frac{1}{2}\left(1 + \frac{q}{p}\right)\delta N\right) \\
&\quad \times \bar{E}_{\mathbf{x}_N}^{(N)} \exp\left(M_T^{(N)} + \frac{q}{2p}\langle\!\langle M^{(N)} \rangle\!\rangle_T\right) \mathbb{1}_{\{\varepsilon_{\mathbf{x}(\cdot)} \in W \cap \mathscr{C}_R\}} \\
&\geqq \exp\left(-\frac{1}{2}\left(1 + \frac{q}{p}\right)\delta N\right)
\end{aligned}$$

$$\times\left[\bar{E}^{(N)}_{\mathbf{x}_N}\exp\left(-\frac{q}{p}M^{(N)}_T-\frac{1}{2}\left\langle\!\!\left\langle\frac{q}{p}M^{(N)}\right\rangle\!\!\right\rangle_T\right)\right]^{-p/q}$$

$$\times[\bar{P}^{(N)}_{\mathbf{x}_N}(\varepsilon_{\mathbf{x}(\cdot)}\in W\cap\mathscr{C}_R)]^p.$$

Here $\bar{E}^{(N)}_{\mathbf{x}}$ denotes expectation with respect to $\bar{P}^{(N)}_{\mathbf{x}}$. Since

$$\exp\left(-\frac{q}{p}M^{(N)}-\frac{1}{2}\left\langle\!\!\left\langle\left(\frac{q}{p}\right)M^{(N)}\right\rangle\!\!\right\rangle\right)$$

is a $\bar{P}^{(N)}_{\mathbf{x}_N}$-supermartingale with expectation not exceeding one ([22, Chap. 3, Theorem 5.2]), we arrive at

$$P^{(N)}_{\mathbf{x}_N}(\varepsilon_{\mathbf{x}(\cdot)}\in V)$$

$$\geqq\exp\left(-\frac{1}{2}\left(1+\frac{q}{p}\right)\delta N\right)[\bar{P}^{(N)}_{\mathbf{x}_N}(\varepsilon_{\mathbf{x}(\cdot)}\in W\cap\mathscr{C}_R)]^p. \tag{5.25}$$

Now $(\mathscr{C},\bar{P}^{(N)}_{\nu_N},N)$ is a large deviation system by Theorem 4.5. The value of the corresponding action functional at $\bar{\mu}(\cdot)$ equals $S(\bar{\mu}(\cdot))$. From this and the definition of $\bar{\mathscr{P}}^{(N)}_{\nu}$ we conclude that

$$\liminf_{N\to\infty}N^{-1}\log\bar{P}^{(N)}_{\mathbf{x}_N}(\varepsilon_{\mathbf{x}(\cdot)}\in W)\geqq-S(\bar{\mu}(\cdot)). \tag{5.26}$$

Combining (5.25) with (5.26) and (5.23) and taking into account (5.24), we finally arrive at (5.22).

We now turn to the proof of assertion (b). Because of Lemma 5.5 and the definition of the measures $\mathscr{P}^{(N)}_{\nu}$, it suffices to show that, for each R for which $\bar{\mu}(\cdot)\mathscr{C}_R$ there exists an open neighborhood W of $\bar{\mu}(\cdot)$ in $\mathscr{C}$ such that

$$\limsup_{N\to\infty}N^{-1}\log P^{(N)}_{\mathbf{x}_N}(\varepsilon_{\mathbf{x}(\cdot)}\in W\cap\mathscr{C}_R)\leqq-S(\bar{\mu}(\cdot))+\gamma. \tag{5.27}$$

To this end, we pick $p,q>1$ with $p^{-1}+q^{-1}=1$ and $\delta>0$ so that

$$-\frac{p-1}{2}\delta+\frac{1}{q}\left(S(\bar{\mu}(\cdot))-\frac{\gamma}{2}\right)\geqq S(\bar{\mu}(\cdot))-\gamma. \tag{5.28}$$

We choose W so "small" that

$$\langle\!\langle M^{(N)} \rangle\!\rangle_T < \delta N \quad \text{on} \quad \{\varepsilon_{x(\cdot)} \in W \cap \mathscr{C}_R\} \tag{5.29}$$

for all N and

$$\limsup_{N\to\infty} N^{-1} \log \bar{P}^{(N)}_{x_N}(\varepsilon_{x(\cdot)} \in W) \leqq -S(\bar{\mu}(\cdot)) + \frac{\gamma}{2}. \tag{5.30}$$

Inequality (5.29) can be achieved for "small" neighborhoods W of $\bar{\mu}(\cdot)$ because of Assumption (B.4) and (5.21). That assertion (5.30) holds for "small" W follows from the definition of the measures $\bar{\mathscr{P}}^{(N)}_{\nu}$, the fact that $(\mathscr{C}, \bar{\mathscr{P}}^{(N)}_{\nu_N}, N)$ is a large deviation system, and the observation that the value of the corresponding action functional at $\bar{\mu}(\cdot)$ equals $S(\bar{\mu}(\cdot))$. Applying (5.20), (5.29), and Hölder's inequality, we obtain for all N:

$$\begin{aligned} P^{(N)}_{x_N}(\varepsilon_{x(\cdot)} \in W \cap \mathscr{C}_R) &= \bar{E}^{(N)}_{x_N} \exp(M^{(N)}_T - \tfrac{1}{2}\langle\!\langle M^{(N)} \rangle\!\rangle_T) \\ &\quad \times \mathbb{1}_{\{\varepsilon_{x(\cdot)} \in W \cap \mathscr{C}_R\}} \\ &\leqq \exp\left(\frac{p-1}{2}\delta N\right) \bar{E}^{(N)}_{x_N} \exp\left(M^{(N)}_T - \frac{p}{2}\langle\!\langle M^{(N)} \rangle\!\rangle_T\right) \\ &\quad \times \mathbb{1}_{\{\varepsilon_{x(\cdot)} \in W \cap \mathscr{C}_R\}} \\ &\leqq \exp\left(\frac{p-1}{2}\delta N\right) [\bar{E}^{(N)}_{x_N} \exp(pM^{(N)}_T - \tfrac{1}{2}\langle\!\langle pM^{(N)} \rangle\!\rangle_T)]^{1/p} \\ &\quad \times [\bar{P}^{(N)}_{x_N}(\varepsilon_{x(\cdot)} \in W)]^{1/q}. \end{aligned}$$

Since $\exp(pM^{(N)}_T - \tfrac{1}{2}\langle\!\langle pM^{(N)} \rangle\!\rangle_T)$ is a $\bar{P}^{(N)}_{x_N}$-supermartingale with expectation not exceeding one, we arrive at

$$P^{(N)}_{x_N}(\varepsilon_{x(\cdot)} \in W \cap \mathscr{C}_R) \leqq \exp\left(\frac{p-1}{2}\delta N\right) [\bar{P}^{(N)}_{x_N}(\varepsilon_{x(\cdot)} \in W)]^{1/q}.$$

A combination of this estimate with (5.30) and (5.28) finally yields (5.27).

The proof of Theorem 5.2 is now complete.

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