Quenched Lyapunov exponent for the parabolic Anderson model in a dynamic random environment

J. Gärtner, F. den Hollander and G. Maillard

Abstract We continue our study of the parabolic Anderson equation $\partial u/\partial t = \kappa\Delta u + \gamma\xi u$ for the space-time field $u\colon\, \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$, where $\kappa \in [0,\infty)$ is the diffusion constant, Δ is the discrete Laplacian, $\gamma \in (0,\infty)$ is the coupling constant, and $\xi\colon\, \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$ is a space-time random environment that drives the equation. The solution of this equation describes the evolution of a "reactant" u under the influence of a "catalyst" ξ, both living on $\mathbb{Z}^d$.

In earlier work we considered three choices for ξ: independent simple random walks, the symmetric exclusion process, and the symmetric voter model, all in equilibrium at a given density. We analyzed the *annealed* Lyapunov exponents, i.e., the exponential growth rates of the successive moments of u w.r.t. ξ, and showed that these exponents display an interesting dependence on the diffusion constant κ, with qualitatively different behavior in different dimensions d. In the present paper we focus on the *quenched* Lyapunov exponent, i.e., the exponential growth rate of u conditional on ξ.

We first prove existence and derive some qualitative properties of the quenched Lyapunov exponent for a general ξ that is stationary and ergodic w.r.t. translations in $\mathbb{Z}^d$ and satisfies certain noisiness conditions. After that we focus on the three particular choices for ξ mentioned above and derive some more detailed properties. We close by formulating a number of open problems.

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1 Introduction

Section 1.1 defines the parabolic Anderson model, Section 1.2 introduces the quenched Lyapunov exponent, Section 1.3 summarizes what is known in the literature, Section 1.4 contains our main results, while Section 1.5 provides a discussion of these results and lists open problems.

1.1 Parabolic Anderson model

The parabolic Anderson model (PAM) is the partial differential equation

$$\frac{\partial}{\partial t}u(x,t) = \kappa\Delta u(x,t) + [\gamma\xi(x,t) - \delta]u(x,t), \qquad x \in \mathbb{Z}^d, t \geq 0. \tag{1}$$

Here, the u-field is $\mathbb{R}$-valued, $\kappa \in [0,\infty)$ is the diffusion constant, Δ is the discrete Laplacian acting on u as

$$\Delta u(x,t) = \sum_{\substack{y\in\mathbb{Z}^d \\ \|y-x\|=1}} [u(y,t) - u(x,t)] \tag{2}$$

($\|\cdot\|$ is the Euclidian norm), $\gamma \in [0,\infty)$ is the coupling constant, $\delta \in [0,\infty)$ is the killing constant, while

$$\xi = (\xi_t)_{t\geq 0} \text{ with } \xi_t = \{\xi(x,t)\colon\ x \in \mathbb{Z}^d\} \tag{3}$$

is an $\mathbb{R}$-valued random field that evolves with time and that drives the equation. The ξ-field provides a dynamic random environment defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. As initial condition for (1) we take

$$u(x,0) = \delta_0(x), \qquad x \in \mathbb{Z}^d. \tag{4}$$

One interpretation of (1) and (4) comes from population dynamics. Consider a system of two types of particles, A (catalyst) and B (reactant), subject to:

- A-particles evolve autonomously according to a prescribed stationary ergodic dynamics with $\xi(x,t)$ denoting the number of A-particles at site x at time t;
- B-particles perform independent random walks at rate $2d\kappa$ and split into two at a rate that is equal to γ times the number of A-particles present at the same location;
- B-particles die at rate δ;
- the initial configuration of B-particles is one particle at site 0 and no particle elsewhere.

Then

$$\begin{aligned} u(x,t) = \; &\text{the average number of } B\text{-particles at site } x \text{ at time } t \\ &\text{conditioned on the evolution of the } A\text{-particles.} \end{aligned} \tag{5}$$

It is possible to remove δ via the trivial transformation $u(x,t) \to u(x,t)e^{-\delta t}$. In what follows we will therefore put $\delta = 0$.

We will assume that ξ is *stationary* and *ergodic* w.r.t. translations in $\mathbb{Z}^d$, is *not constant*, and is such that

$$\forall\, \kappa,\gamma \in [0,\infty)\ \exists\, c = c(\kappa,\gamma) < \infty\colon\ \mathbb{E}\big(\log u(0,t)\big) \leq ct \quad \forall\, t \geq 0. \tag{6}$$

Three choices of ξ will receive special attention, namely ($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$):

(1) *Independent Simple Random Walks* (ISRW), where $\xi_t \in \Omega = \mathbb{N}_0^{\mathbb{Z}^d}$ and $\xi(x,t)$ represents the number of particles at site x at time t. Under the ISRW-dynamics particles move around independently as simple random walks stepping at rate 1. We draw ξ_0 according to the Poisson product measure ν_ρ with density $\rho \in (0,\infty)$. For this choice, ξ is stationary, ergodic and reversible in time (see Kipnis and Landim [22], Chapter 1).

(2) *Symmetric Exclusion Process* (SEP), where $\xi_t \in \Omega = \{0,1\}^{\mathbb{Z}^d}$ and $\xi(x,t)$ represents the presence ($\xi(x,t) = 1$) or absence ($\xi(x,t) = 0$) of a particle at site x at time t. Under the SEP-dynamics particles move around independently according to a symmetric random walk transition kernel at rate 1, but subject to the restriction that no two particles can occupy the same site. We draw ξ_0 according to the Bernoulli product measure ν_ρ with density $\rho \in (0,1)$. For this choice, the ξ-field is stationary, ergodic and reversible in time (see Liggett [23], Chapter VIII).

(3) *Symmetric Voter Model* (SVM), where $\xi_t \in \Omega = \{0,1\}^{\mathbb{Z}^d}$ and $\xi(x,t)$ represents two possible opinions or, alternatively, the presence ($\xi(x,t) = 1$) or absence ($\xi(x,t) = 0$) of a particle at site x at time t. Under the SVM-dynamics each site imposes its state on another site according to a symmetric random walk transition kernel at rate 1. We draw ξ_0 according to the equilibrium distribution ν_ρ with density $\rho \in (0,1)$, which is not a product measure. The ergodic properties of the SVM are qualitatively different in low and high dimensions, namely, when $d = 1,2$ all equilibria are trivial, i.e., $\nu_\rho = (1-\rho)\delta_0 + \rho\delta_1$, while when $d \geq 3$ there are also non-trivial equilibria, i.e., ergodic ν_ρ parametrized by the density ρ (see Liggett [23], Chapter V).

Contrary to ISRW and SEP, the dynamics of SVM is non-conservative and non-reversible: opinions are not preserved and the law of ξ is not invariant under time reversal. For each of these examples we study the quenched Lyapunov exponents as a function of d, κ, γ and ρ. Because ξ is dependent in space and time, these examples require techniques different from those developed in the case of a white noise potential ξ (see Carmona and Molchanov [6], Greven and den Hollander [18]).

Throughout the sequel, we write $\mathbb{P}_\eta$ for the law of ξ starting from $\eta \in \Omega$, and $\mathbb{P} = \int_\Omega \nu_\rho(d\eta)\,\mathbb{P}_\eta$ for the law of ξ in equilibrium ν_ρ at density ρ.

1.2 Lyapunov exponents

Our focus will be on the *quenched* Lyapunov exponent, i.e., the exponential growth rate of u conditional on ξ:

$$\lambda_0 = \lim_{t\to\infty} \frac{1}{t} \log u(0,t) \qquad \xi\text{-a.s.} \tag{7}$$

We will be interested in comparing λ_0 with the *annealed* Lyapunov exponents, defined by

$$\lambda_p = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\big([u(0,t)]^p\big)^{1/p}, \qquad p \in \mathbb{N}, \tag{8}$$

which were analyzed in detail in our earlier work (see Section 1.3). In (7–8) we pick $x = 0$ as the reference site to monitor the growth of u. However, it is easy to show that the Lyapunov exponents are the same at other sites.

By the Feynman-Kac formula, the solution of (1) reads

$$u(x,t) = \mathrm{E}_x\left(\exp\left[\gamma\int_0^t \xi\,(X^\kappa(s),t-s)\,ds\right] u\big(X^\kappa(t),0\big)\right), \tag{9}$$

where $X^\kappa = (X^\kappa(t))_{t\geq 0}$ is simple random walk on $\mathbb{Z}^d$ with step rate $2d\kappa$ and E_x denotes expectation with respect to X^κ given $X^\kappa(0) = x$. In particular, for stationary ξ and $t > 0$ we have

$$\begin{aligned} u(0,t) &= \sum_{y\in\mathbb{Z}^d} u(y,0)\,\mathrm{E}_0\bigg(\exp\left[\gamma\int_0^t \xi\big(X^\kappa(s),t-s\big)\,ds\right]\delta_y\big(X^\kappa(t)\big)\bigg)\\ &= \sum_{y\in\mathbb{Z}^d} u(y,0)\,\mathrm{E}_y\bigg(\exp\left[\gamma\int_0^t \xi\big(X^\kappa(s),s\big)\,ds\right]\delta_0\big(X^\kappa(t)\big)\bigg)\\ &\overset{\mathbb{P}}{=} \sum_{y\in\mathbb{Z}^d} u(y,0)\,\mathrm{E}_0\bigg(\exp\left[\gamma\int_0^t \xi\big(X^\kappa(s),s\big)\,ds\right]\delta_{-y}\big(X^\kappa(t)\big)\bigg), \end{aligned} \tag{10}$$

where in the second line we reverse time and use that X^κ is a reversible dynamics, while in the third line we use the stationarity of ξ to get equality in $\mathbb{P}$-distribution. Therefore, for any initial condition $u(\cdot,0)$ satisfying $u(x,0) = u(-x,0)$ for all $x \in \mathbb{Z}^d$, which is the case for our choice in (4), we can define

$$\Lambda_0(t) = \frac{1}{t}\log u(0,t) \overset{\mathbb{P}}{=} \frac{1}{t}\log \mathrm{E}_0\bigg(\exp\left[\gamma\int_0^t \xi\big(X^\kappa(s),s\big)\,ds\right] u\big(X^\kappa(t),0\big)\bigg). \tag{11}$$

If the last quantity admits a limit as $t \to \infty$, then

$$\lambda_0 = \lim_{t\to\infty} \Lambda_0(t) \qquad \xi\text{-a.s.}, \tag{12}$$

where the limit is expected to be ξ-a.s. constant. Condition (6) implies that $\lambda_0 \leq c$.

Clearly, λ_0 is a function of d, κ, γ and the parameters controlling ξ. In what follows, our main focus will be on the dependence on κ, and therefore we will often write $\lambda_0(\kappa)$.

1.3 Literature

1.3.1 White noise

The behavior of the Lyapunov exponents for the PAM in a *time-dependent* random environment has been the subject of several papers. Carmona and Molchanov [6] obtained a qualitative description of both the *quenched* and the *annealed* Lyapunov exponents when ξ is white noise, i.e.,

$$\xi(x,t) = \frac{\partial}{\partial t} W(x,t), \tag{13}$$

where $W = (W_t)_{t\geq 0}$ with $W_t = \{W(x,t)\colon\, x \in \mathbb{Z}^d\}$ is a space-time field of independent Brownian motions. They showed that if $u(\cdot,0)$ has compact support (e.g. $u(\cdot,0) = \delta_0(\cdot)$), then the quenched Lyapunov exponent $\lambda_0(\kappa)$ defined in (7) exists and is independent of $u(\cdot,0)$. Moreover, they found that the asymptotics of $\lambda_0(\kappa)$ as $\kappa \downarrow 0$ is singular, namely, there are constants $C_1, C_2 \in (0,\infty)$ and $\kappa_0 \in (0,\infty)$ such that

$$C_1 \frac{1}{\log(1/\kappa)} \leq \lambda_0(\kappa) \leq C_2 \frac{\log\log(1/\kappa)}{\log(1/\kappa)} \qquad \forall\, 0 < \kappa \leq \kappa_0. \tag{14}$$

Subsequently, Carmona, Molchanov and Viens [7], Carmona, Koralov and Molchanov [5], and Cranston, Mountford and Shiga [9], proved the existence of λ_0 when $u(\cdot,0)$ has non-compact support (e.g. $u(\cdot,0) \equiv 1$), showed that there is a constant $C \in (0,\infty)$ such that

$$\lim_{\kappa\downarrow 0} \log(1/\kappa)\, \lambda_0(\kappa) = C, \tag{15}$$

and proved that

$$\lim_{p\downarrow 0} \lambda_p(\kappa) = \lambda_0(\kappa) \quad \forall\, \kappa \in [0,\infty). \tag{16}$$

(These results were later extended to Lévy white noise by Cranston, Mountford and Shiga [10], and to colored noise by Kim, Viens and Vizcarra [20].) Further refinements on the behavior of the Lyapunov exponents were conjectured in Carmona and Molchanov [6] and proved in Greven and den Hollander [18]. In particular, it was shown that $\lambda_1(\kappa) = \frac{1}{2}$ for all $\kappa \in [0,\infty)$, while for the other Lyapunov exponents the following dichotomy holds (see Figs. 1–2):

- $d = 1,2$: $\lambda_0(\kappa) < \frac{1}{2}$, $\lambda_p(\kappa) > \frac{1}{2}$ for $p \in \mathbb{N}\backslash\{1\}$, for $\kappa \in [0,\infty)$;
- $d \geq 3$: there exist $0 < \kappa_1 \leq \kappa_2 \leq \ldots$ such that

$$\lambda_0(\kappa) - \tfrac{1}{2} \begin{cases} < 0, \text{ for } \kappa \in [0, \kappa_1), \\ = 0, \text{ for } \kappa \in [\kappa_1, \infty), \end{cases} \tag{17}$$

and

$$\lambda_p(\kappa) - \tfrac{1}{2} \begin{cases} > 0, \text{ for } \kappa \in [0, \kappa_p), \\ = 0, \text{ for } \kappa \in [\kappa_p, \infty), \end{cases} \qquad p \in \mathbb{N}\backslash\{1\}. \tag{18}$$

Moreover, variational formulas for κ_p were derived, which in turn led to upper and lower bounds on κ_p, and to the identification of the asymptotics of κ_p for $p \to \infty$ (κ_p grows linearly with p). In addition, it was shown that for every $p \in \mathbb{N}\backslash\{1\}$ there exists a $d(p) < \infty$ such that $\kappa_p < \kappa_{p+1}$ for $d \geq d(p)$. Moreover, it was shown that $\kappa_1 < \kappa_2$ in Birkner, Greven and den Hollander [2] ($d \geq 5$), Birkner and Sun [3] ($d = 4$), Berger and Toninelli [1], Birkner and Sun [4] ($d = 3$). Note that, by Hölder's inequality, all curves in Figs. 1–2 are distinct whenever they are different from $\frac{1}{2}$.

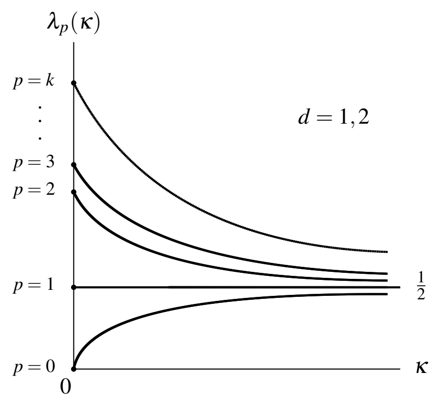

Fig. 1 Quenched and annealed Lyapunov exponents when $d = 1,2$ for white noise.

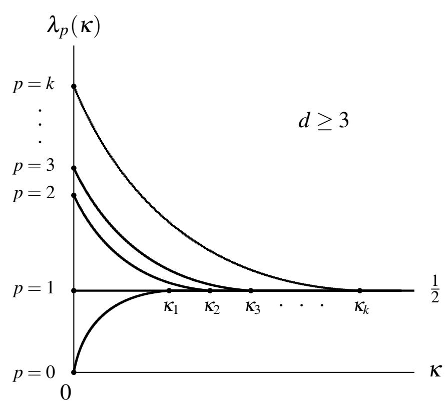

Fig. 2 Quenched and annealed Lyapunov exponents when $d \geq 3$ for white noise.

1.3.2 Interacting particle systems

Various models where ξ is *dependent in space and time* were looked at more recently. Kesten and Sidoravicius [19], and Gärtner and den Hollander [13], considered the case where ξ is a field of independent simple random walks in Poisson equilibrium (ISRW). The survival versus extinction pattern [19] and the annealed Lyapunov exponents [13] were analyzed, in particular, their dependence on d, κ, γ and ρ. The case where ξ is a single random walk was studied by Gärtner and Heydenreich [12]. Gärtner, den Hollander and Maillard [14], [16], [17] subsequently considered the cases where ξ is an exclusion process with a symmetric random walk transition kernel starting from a Bernoulli product measure (SEP), respectively, a voter model with a symmetric random walk transition kernel starting either from a Bernoulli product measure or from equilibrium (SVM). In each of these cases, a fairly complete picture of the behavior of the annealed Lyapunov exponents was obtained, including the presence or absence of *intermittency*, i.e., $\lambda_p(\kappa) > \lambda_{p-1}(\kappa)$ for some or all values of $p \in \mathbb{N}\backslash\{1\}$ and $\kappa \in [0,\infty)$. Several conjectures were formulated as well. In what follows we describe these results in some more detail. We refer the reader to Gärtner, den Hollander and Maillard [15] for an overview.

Let G_d be the Green function at the origin of simple random walk stepping at rate 1. It was shown in Gärtner and den Hollander [13], and Gärtner, den Hollander and Maillard [14], [16], [17] that for ISRW, SEP and SVM in equilibrium the function $\kappa \to \lambda_p(\kappa)$ satisfies:

- If $d \geq 1$ and $p \in \mathbb{N}$, then the limit in (8) exists for all $\kappa \in [0,\infty)$. Moreover, if $\lambda_p(0) < \infty$, then $\kappa \to \lambda_p(\kappa)$ is finite, continuous, strictly decreasing and convex on $[0,\infty)$.
- There are two regimes for the annealed Lyapunov exponents:
  - *Strongly catalytic regime* (see Fig. 3):
    - ISRW: $d = 1,2$, $p \in \mathbb{N}$ or $d \geq 3$, $p \geq 1/\gamma G_d$: $\lambda_p \equiv \infty$ on $[0,\infty)$.
    - SEP: $d = 1,2$, $p \in \mathbb{N}$: $\lambda_p \equiv \gamma$ on $[0,\infty)$.
    - SVM: $d = 1,2,3,4$, $p \in \mathbb{N}$: $\lambda_p \equiv \gamma$ on $[0,\infty)$.
  - *Weakly catalytic regime* (see Fig. 4–5):
    - ISRW: $d \geq 3$, $p < 1/\gamma G_d$: $\rho\gamma < \lambda_p < \infty$ on $[0,\infty)$.
    - SEP: $d \geq 3$, $p \in \mathbb{N}$: $\rho\gamma < \lambda_p < \gamma$ on $[0,\infty)$.
    - SVM: $d \geq 5$, $p \in \mathbb{N}$: $\rho\gamma < \lambda_p < \gamma$ on $[0,\infty)$.
- For all three dynamics, in the weakly catalytic regime $\lim_{\kappa\to\infty} \kappa[\lambda_p(\kappa) - \rho\gamma] = C_1 + C_2 p^2 1_{\{d=d_c\}}$ with $C_1, C_2 \in (0,\infty)$ and d_c a critical dimension: $d_c = 3$ for ISRW, SEP and $d_c = 5$ for SVM.
- Intermittent behavior:
  - In the strongly catalytic regime, there is no intermittency for all three dynamics.
  - In the weakly catalytic regime, there is full intermittency for:
    - all three dynamics when $0 \leq \kappa \ll 1$.
    - ISRW and SEP in $d = 3$ when $\kappa \gg 1$.

· SVM in $d = 5$ when $\kappa \gg 1$.

Note: For SVM the convexity of $\kappa \to \lambda_p(\kappa)$ and its scaling behavior for $\kappa \to \infty$ have not been proved, but have been argued on heuristic grounds.

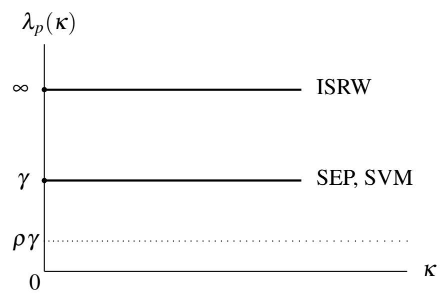

Fig. 3 Triviality of the annealed Lyapunov exponents for ISRW, SEP, SVM in the strongly catalytic regime.

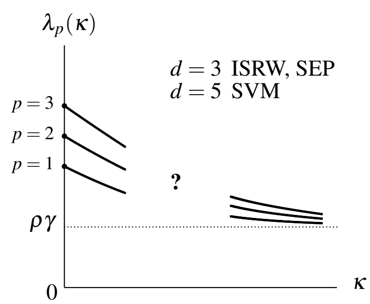

Fig. 4 Non-triviality of the annealed Lyapunov exponents for ISRW, SEP and SVM in the weakly catalytic regime at the critical dimension.

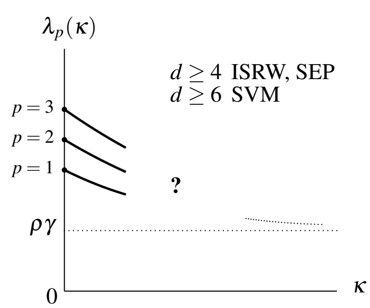

Fig. 5 Non-triviality of the annealed Lyapunov exponents for ISRW, SEP and SVM in the weakly catalytic regime above the critical dimension.

Recently, there has been further progress for the case where ξ consists of n independent random walks (Castell, Gün, Maillard [8]), for the trapping version of the PAM with $\gamma \in (-\infty, 0)$ (Drewitz, Gärtner, Ramírez and Sun [11]), and for the voter model (Maillard, Mountford and Schöpfer [24]). All these papers appear elsewhere in the present volume.

1.4 Main results

We have six theorems, all relating to the *quenched* Lyapunov exponent and extending the results on the annealed Lyapunov exponents listed in Section 1.3.2.

Our first three theorems will be proved in Section 2 and deal with a general ξ that is stationary and ergodic w.r.t. translations in $\mathbb{Z}^d$, and satisfies condition (6).

Theorem 1.1. *Fix $d \geq 1$, $\kappa \in [0, \infty)$ and $\gamma \in (0, \infty)$. If ξ is stationary and ergodic w.r.t. translations in $\mathbb{Z}^d$ and satisfies condition* (6)*, then the limit in* (7) *exists $\mathbb{P}$-a.s. and in $\mathbb{P}$-mean, and is finite.*

For our second theorem we need to make the additional assumption that

$$\liminf_{T\to\infty} \frac{\mathbb{E}(|I^\xi(0,T) - I^\xi(e,T)|)}{\log T} > 0, \tag{19}$$

where e is any nearest-neighbor site of 0, and

$$I^\xi(x,T) = \int_0^T [\xi(x,t) - \rho]\, dt, \qquad x \in \mathbb{Z}^d. \tag{20}$$

Theorem 1.2. *Fix $d \geq 1$ and $\gamma \in (0, \infty)$. If ξ is stationary and ergodic w.r.t. translations in $\mathbb{Z}^d$ and satisfies condition* (6)*, then (see Fig.* 6*):*
(i) $\kappa \mapsto \lambda_0(\kappa)$ is globally Lipschitz outside any neighborhood of 0.
(ii) $\kappa \mapsto \lambda_0(\kappa)$ is not Lipschitz at 0 *subject to condition* (19).
(iii) $\rho\gamma < \lambda_0(\kappa) < \infty$ for all $\kappa \in (0, \infty)$ with $\rho = \mathbb{E}(\xi(0,0))$.

Note that $\lambda_0(0) = \rho\gamma$, but that Theorem 1.2 does not include continuity of $\kappa \mapsto \lambda_0(\kappa)$ at 0. For our third theorem we need to make the additional assumption that

$$\limsup_{T\to\infty} \frac{1}{T} \log \left[\sup_{\eta\in\Omega} \mathbb{P}_\eta \left(\int_0^T \xi(0,s)\, ds > (\rho+\delta)T \right) \right] < 0 \qquad \forall\, \delta > 0. \tag{21}$$

Theorem 1.3. *Fix $d \geq 1$ and $\gamma \in (0, \infty)$. If ξ is stationary and ergodic w.r.t. translations in $\mathbb{Z}^d$, is bounded and satisfies condition* (21)*, then*

$$\limsup_{\kappa\downarrow 0} \frac{\log(1/\kappa)}{\log\log(1/\kappa)} [\lambda_0(\kappa) - \rho\gamma] < \infty. \tag{22}$$

For a discussion of conditions (19) and (21), see Section 1.5.

Our last three theorems deal with ISRW, SEP and SVM and will be proved in Section 3.

Theorem 1.4. *For ISRW, SEP and SVM in the weakly catalytic regime (see Fig.* 6*),* $\lim_{\kappa\to\infty}\lambda_0(\kappa)=\rho\gamma$.

Theorem 1.5. *For ISRW and SEP in the weakly catalytic regime (see Fig.* 6*),*

$$\liminf_{\kappa\downarrow 0}\,\log(1/\kappa)\,[\lambda_0(\kappa)-\rho\gamma]>0. \tag{23}$$

Theorem 1.6. *For ISRW in the strongly catalytic regime,* $\lambda_0(\kappa)<\lambda_1(\kappa)$ *for all* $\kappa\in[0,\infty)$ *(see Fig.* 7*).*

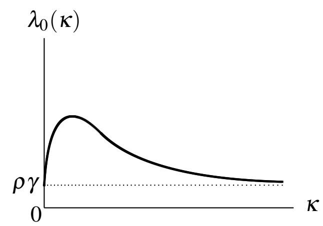

Fig. 6 The quenched Lyapunov exponent. Conjectured behavior in the weakly catalytic regime.

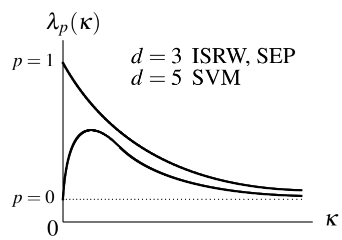

Fig. 7 Comparison between $\kappa\mapsto\lambda_0(\kappa)$ and $\kappa\mapsto\lambda_1(\kappa)$. Conjectured behavior for ISRW, SEP and SVM at the critical dimension.

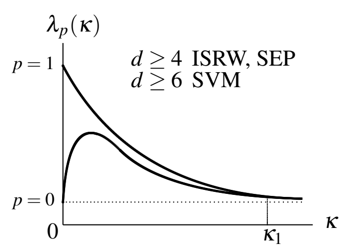

Fig. 8 Comparison between $\kappa\mapsto\lambda_0(\kappa)$ and $\kappa\mapsto\lambda_1(\kappa)$. Conjectured behavior for ISRW, SEP and SVM above the critical dimension.

1.5 Discussion and open problems

By (11–12), condition (6) is trivially satisfied for bounded ξ, which includes SEP and SVM. Condition (6) is a direct consequence of Theorem 2 in Kesten and Sidoravicius [19] when ξ is ISRW. Condition (19) is weak; we will see in Section 3.2 that it is satisfied for the three dynamics because the numerator of (19) grows polynomially rather than logarithmically. Condition (21) is strong; it fails for the three dynamics, but is satisfied e.g. for spin-flip dynamics in the so-called "$M < \varepsilon$ regime" (see Liggett [23], Section I.3).

The following problems remain open:

- Extend the existence of λ_0 to $u(\cdot,0) \equiv 1$, and prove that the limit is the same as for $u(\cdot,0) = \delta_0(\cdot)$ assumed in (4). It is straightforward to do the extension for $u(\cdot,0)$ symmetric with bounded support.
- Prove Theorem 1.4 for the three dynamics in the strongly catalytic regime, Theorem 1.5 for SVM in the weakly catalytic regime, and Theorem 1.6 for SEP and SVM in the strongly catalytic regime. The limits as $\kappa \downarrow 0$ and $\kappa \to \infty$ correspond to time ergodicity and space ergodicity, respectively, but are non-trivial because they require control on the large deviations of ξ.
- Derive an upper bound for $\lambda_0(\kappa) - \rho\gamma$ as $\kappa \downarrow 0$ that supplements the lower bound obtained in (23). The upper bound in Theorem 1.3 subject to (21) probably extends to ISRW, SEP and SVM. If so, then this would imply the continuity of $\kappa \mapsto \lambda_0(\kappa)$ at 0, which in turn would imply that there exists a $\kappa_1 > 0$ such that $\lambda_0(\kappa) < \lambda_1(\kappa)$ for all $\kappa \leq \kappa_1$ (see Fig. 8).
- In the weakly catalytic regime, find the asymptotics of $\lambda_0(\kappa)$ as $\kappa \to \infty$ and compare with the asymptotics of $\lambda_p(\kappa)$, $p \in \mathbb{N}$, as $\kappa \to \infty$ (see Figs. 4– 5).
- In the weakly catalytic regime, show that above the critical dimension there exists a $\kappa_1 < \infty$ such that $\lambda_0(\kappa) = \lambda_1(\kappa)$ for all $\kappa \geq \kappa_1$ (see Figs. 7–8)? For white noise dynamics such merging occurs for all $d \geq 3$ (see Figs. 1–2).
- Extend the existence of λ_p to all (non-integer) $p > 0$, and prove that $\lambda_p \downarrow \lambda_0$ as $p \downarrow 0$. For white noise this is achieved in (16).

2 Proof of Theorems 1.1–1.3

The proofs of Theorems 1.1–1.3 are given in Sections 2.1–2.3, respectively.

2.1 Proof of Theorem 1.1

Proof. Recall (4), (11) and (12), and abbreviate

$$\chi(s,t) = \mathrm{E}_0\left(\exp\left[\gamma\int_0^{t-s}\xi\left(X^\kappa(v),s+v\right)dv\right]\delta_0(X^\kappa(t-s))\right), \qquad 0\le s\le t<\infty. \tag{24}$$

Picking $u\in[s,t]$, inserting $\delta_0(X^\kappa(u-s))$ under the expectation in (24) and using the Markov property of X^κ at time $u-s$, we obtain

$$\chi(s,t)\ge\chi(s,u)\,\chi(u,t), \qquad 0\le s\le u\le t<\infty. \tag{25}$$

Thus, $(s,t)\mapsto\log\chi(s,t)$ is superadditive. Since ξ is stationary, ergodic, satisfies condition (6) and the law of $\{\chi(u+s,u+t)\colon 0\le s\le t<\infty\}$ is the same for all $u\ge 0$, the claim follows from the superadditive ergodic theorem (see Kingman [21]), i.e.,

$$\lambda_0=\lim_{t\to\infty}\frac{1}{t}\log\chi(0,t) \text{ exists } \mathbb{P}\text{-a.s. and in } \mathbb{P}\text{-mean}, \tag{26}$$

and the limit is finite. □

2.2 *Proof of Theorem 1.2(i)*

Proof. In Step 1 we give the proof for a general stationary and ergodic ξ that is bounded from above by 1. This proof is a copy of the proof in Gärtner, den Hollander and Maillard [17] of the Lipschitz continuity of the annealed Lyapunov exponents when ξ is SVM. In Step 2 we explain how to extend the proof to unbounded ξ subject to condition (6).

1. Pick $\kappa_1,\kappa_2\in(0,\infty)$ with $\kappa_1<\kappa_2$ arbitrarily. By Girsanov's formula,

$$\begin{aligned}
&\mathrm{E}_0\left(\exp\left[\gamma\int_0^t\xi(X^{\kappa_2}(s),s)\,ds\right]\delta_0(X^{\kappa_2}(t))\right)\\
&\quad=\mathrm{E}_0\Bigg(\exp\left[\gamma\int_0^t\xi(X^{\kappa_1}(s),s)\,ds\right]\delta_0(X^{\kappa_1}(t))\\
&\qquad\qquad\times\exp\Big[J(X^{\kappa_1};t)\log(\kappa_2/\kappa_1)-2d(\kappa_2-\kappa_1)t\Big]\Bigg)\\
&\quad=I+II,
\end{aligned} \tag{27}$$

where $J(X^{\kappa_1};t)$ is the number of jumps of X^{κ_1} up to time t, I and II are the contributions coming from the events $\{J(X^{\kappa_1};t)\le M2d\kappa_2 t\}$, respectively, $\{J(X^{\kappa_1};t)>M2d\kappa_2 t\}$, and $M>1$ is to be chosen. Clearly,

$$I\le\exp\Big[\Big(M2d\kappa_2\log(\kappa_2/\kappa_1)-2d(\kappa_2-\kappa_1)\Big)t\Big]\,\mathrm{E}_0\left(\exp\left[\gamma\int_0^t\xi(X^{\kappa_1}(s),s)\,ds\right]\right), \tag{28}$$

while

$$II\le e^{\gamma t}\,\mathrm{P}_0\Big(J(X^{\kappa_2};t)>M2d\kappa_2 t\Big) \tag{29}$$

because we may estimate

$$\int_0^t \xi(X^{\kappa_1}(s),s)\,ds \le t \tag{30}$$

and afterwards use Girsanov's formula in the reverse direction. Since $J(X^{\kappa_2};t) = J^*(2d\kappa_2 t)$ with $(J^*(t))_{t\ge 0}$ a rate-1 Poisson process, we have

$$\lim_{t\to\infty} \frac{1}{t} \log \mathrm{P}_0\Big(J(X^{\kappa_2};t) > M2d\kappa_2 t\Big) = -2d\kappa_2 \mathscr{I}(M) \tag{31}$$

with

$$\mathscr{I}(M) = \sup_{u\in\mathbb{R}} \big[Mu - \big(e^u - 1\big)\big] = M\log M - M + 1. \tag{32}$$

Recalling (11–12), we get from (27–31) that

$$\lambda_0(\kappa_2) \le \big[M2d\kappa_2 \log(\kappa_2/\kappa_1) - 2d(\kappa_2-\kappa_1) + \lambda_0(\kappa_1)\big] \vee \big[\gamma - 2d\kappa_2 \mathscr{I}(M)\big]. \tag{33}$$

On the other hand, estimating $J(X^{\kappa_1};t) \ge 0$ in (27), we have

$$\begin{aligned}
&\mathrm{E}_0\left(\exp\left[\gamma\int_0^t \xi(X^{\kappa_2}(s),s)\,ds\right]\delta_0(X^{\kappa_2}(t))\right)\\
&\qquad \ge \exp[-2d(\kappa_2-\kappa_1)t]\,\mathrm{E}_0\left(\exp\left[\gamma\int_0^t \xi(X^{\kappa_1}(s),s)\,ds\right]\delta_0(X^{\kappa_1}(t))\right),
\end{aligned} \tag{34}$$

which gives the lower bound

$$\lambda_0(\kappa_2) - \lambda_0(\kappa_1) \ge -2d(\kappa_2 - \kappa_1). \tag{35}$$

Next, for $\kappa \in (0,\infty)$, define

$$\begin{aligned}
D^+\lambda_0(\kappa) &= \limsup_{\delta\to 0} \delta^{-1}[\lambda_0(\kappa+\delta) - \lambda_0(\kappa)],\\
D^-\lambda_0(\kappa) &= \liminf_{\delta\to 0} \delta^{-1}[\lambda_0(\kappa+\delta) - \lambda_0(\kappa)].
\end{aligned} \tag{36}$$

Then, picking $\kappa_1 = \kappa$ and $\kappa_2 = \kappa+\delta$, respectively, $\kappa_1 = \kappa-\delta$ and $\kappa_2 = \kappa$ in (33) with $\delta > 0$ and letting $\delta \downarrow 0$, we get

$$D^+\lambda_0(\kappa) \le (M-1)2d \qquad \forall M>1:\ 2d\kappa\mathscr{I}(M) - (1-\rho)\gamma \ge 0 \tag{37}$$

(together with $\lambda_0(\kappa) \ge \rho\gamma$, the latter condition guarantees that the first term in the right-hand side of (33) is the maximum), while (35) gives

$$D^-\lambda_0(\kappa) \ge -2d. \tag{38}$$

We now pick

$$M = M(\kappa) = \mathscr{I}^{-1}\left(\frac{(1-\rho)\gamma}{2d\kappa}\right) \tag{39}$$

with $\mathscr{I}^{-1}$ the inverse of $\mathscr{I}\colon [1,\infty) \to \mathbb{R}$. Since $\mathscr{I}(M) = \frac{1}{2}(M-1)^2[1+o(1)]$ as $M \downarrow 1$, it follows that

$$[M(\kappa)-1]2d = 2d\sqrt{\gamma\frac{1-\rho}{d\kappa}}\,[1+o(1)] \quad \text{as } \kappa \to \infty. \tag{40}$$

By (37), the latter implies that $\kappa \mapsto D^+\lambda_0(\kappa)$ is bounded from above outside any neighborhood of 0. Since, by (38), $\kappa \mapsto D^-\lambda_0(\kappa)$ is bounded from below, the claim follows.

2. It remains to explain how to adapt the proof to the case where ξ is not bounded from above by 1. In that case (30) is no longer true, but by the Cauchy-Schwarz inequality we have

$$II \leq III \times IV \tag{41}$$

with

$$III = \left\{ \mathrm{E}_0\left(\exp\left[2\gamma \int_0^t \xi(X^{\kappa_1}(s),s)\,ds \right] \right) \right\}^{1/2} \tag{42}$$

and

$$\begin{aligned}
IV &= \Bigg\{ \mathrm{E}_0\Bigg(\exp\Big[2J(X^{\kappa_1};t)\log(\kappa_2/\kappa_1) - 4d(\kappa_2-\kappa_1)t \Big] \\
&\qquad\qquad \times 1\!\!1\{J(X^{\kappa_1};t) > M2d\kappa_2 t\} \Bigg) \Bigg\}^{1/2} \\
&= \exp\Big[\Big(d\kappa_1 - 2d\kappa_2 + d(\kappa_2^2/\kappa_1) \Big) t \Big] \\
&\qquad \times \Bigg\{ \mathrm{E}_0\Bigg(\exp\Bigg[J(X^{\kappa_1};t)\log\left(\frac{\kappa_2^2/\kappa_1}{\kappa_1}\right) - 2d\big(\kappa_2^2/\kappa_1 - \kappa_1\big)t \Bigg] \\
&\qquad\qquad \times 1\!\!1\{J(X^{\kappa_1};t) > M2d\kappa_2 t\} \Bigg) \Bigg\}^{1/2} \\
&= \exp\Big[\Big(d\kappa_1 - 2d\kappa_2 + d(\kappa_2^2/\kappa_1) \Big) t \Big] \left\{ \mathrm{P}_0\Big(J\Big(X^{\kappa_2^2/\kappa_1};t\Big) > M2d\kappa_2 t \Big) \right\}^{1/2},
\end{aligned} \tag{43}$$

where in the last line we use Girsanov's formula in the reverse direction. By Theorem 2 in Kesten and Sidoravicius [19], we have $III \leq e^{c_0 t}$ ξ-a.s. for t large enough and some $c_0 < \infty$. Therefore, combining (41–43), we get

$$II \leq \exp\Big[\Big(c_0 + d\kappa_1 - 2d\kappa_2 + d(\kappa_2^2/\kappa_1) \Big) t \Big] \left\{ \mathrm{P}_0\Big(J\Big(X^{\kappa_2^2/\kappa_1};t\Big) > M2d\kappa_2 t \Big) \right\}^{1/2} \tag{44}$$

instead of (29). The rest of the proof goes along the same lines as in (31–40), with $M > 1$ chosen such that

$$\frac{d(\kappa+\delta)^2}{\kappa}\mathscr{I}\left(\frac{M\kappa}{\kappa+\delta}\right)+\rho\gamma-c_0+d\kappa-2d(\kappa+\delta)+\frac{d(\kappa+\delta)^2}{\kappa}\geq 0 \tag{45}$$

instead of (29). □

2.3 *Proof of Theorem 1.2(ii)*

Proof. The proof of Theorem 1.2(ii) is based on the following lemma providing a lower bound for $\lambda_0(\kappa)-\rho\gamma$ when κ is small enough. Recall (20), and abbreviate

$$E_1(T)=\mathbb{E}\big(|I^\xi(0,T)-I^\xi(e,T)|\big), \qquad T>0. \tag{46}$$

Lemma 2.1. *For $\kappa\downarrow 0$ and $T\to\infty$ such that $\kappa T\downarrow 0$,*

$$\lambda_0(\kappa)-\rho\gamma\geq -d\kappa+\tfrac{1}{T}\left[\tfrac{\gamma}{2}E_1(\tfrac{1}{2}T)-\log(1/\kappa T)\right][1+o(1)]. \tag{47}$$

Proof. Recall (4), (11) and (12) and write

$$\lambda_0(\kappa)-\rho\gamma=\lim_{n\to\infty}\frac{1}{nT}\log \mathrm{E}_0\left(\exp\left[\gamma\int_0^{nT}\left[\xi(X^\kappa(s),s)-\rho\right]ds\right]\delta_0(X^\kappa(nT))\right). \tag{48}$$

1. Split the time interval $[0,nT]$ into $2n$ pieces of length $\frac{1}{2}T$,

$$B_i=\left[(i-1)T,(i-1)T+\tfrac{1}{2}T\right], \quad C_i=\left((i-1)T+\tfrac{1}{2}T,iT\right), \quad 1\leq i\leq n, \tag{49}$$

and define

$$I_i^\xi(x,T)=\int_{C_i}\left[\xi(x,s)-\rho\right]ds. \tag{50}$$

To obtain a lower bound for (48), let

$$Z_i^\xi=\operatorname{argmax}\left\{I_i^\xi(0,T),I_i^\xi(e,T)\right\} \tag{51}$$

and consider the event

$$A^\xi=\left[\bigcap_{i=1}^n\left\{X^\kappa(t)=Z_i^\xi\ \forall t\in C_i\right\}\right]\cap\{X^\kappa(nT)=0\}. \tag{52}$$

Then we get

$$
\begin{aligned}
&\mathrm{E}_0\left(\exp\left[\gamma\int_0^{nT}\big[\xi(X^\kappa(s),s)-\rho\big]\,ds\right]\delta_0(X^\kappa(nT))\right)\\
&\qquad\geq \mathrm{E}_0\left(\exp\left[\gamma\int_0^{nT}\big[\xi(X^\kappa(s),s)-\rho\big]\,ds\right]1\!\!1_{A^\xi}\right)\\
&\qquad\geq \mathrm{P}_0\big(A^\xi\big)\,\exp\left(\gamma\sum_{i=1}^n\max\big\{I_i^\xi(0,T),I_i^\xi(e,T)\big\}\right).
\end{aligned}
$$

By the ergodic theorem applied to ξ (which is stationary and ergodic w.r.t. translations in $\mathbb{Z}^d$), we have

$$
\sum_{i=1}^n\max\big\{I_i^\xi(0,T),I_i^\xi(e,T)\big\}=n[1+o(1)]\,\mathbb{E}\big(\max\big\{I_1^\xi(0,T),I_1^\xi(e,T)\big\}\big). \tag{53}
$$

Moreover, writing $p_t(x)=\mathrm{P}_0(X^1(t)=x)$, $x\in\mathbb{Z}^d$, $t\geq 0$, we have

$$
\mathrm{P}_0\big(A^\xi\big)\geq\Big(\min\big\{p_{\kappa T/2}(0),p_{\kappa T/2}(e)\big\}\Big)^{n+1}e^{-nd\kappa T}=\big(p_{\kappa T/2}(e)\big)^{n+1}e^{-nd\kappa T}, \tag{54}
$$

where in the right-hand side the first term is a lower bound for the probability that X^κ moves from 0 to e or vice-versa in time $\frac12 T$ in each of the time intervals B_i, while the second term is the probability that X^κ makes no jumps in each of the time intervals C_i.

2. Combining (48) and (53–54), and using that $p_{\kappa T/2}(e)=(\frac12\kappa T)[1+o(1)]$ as $\kappa T\downarrow 0$, we obtain

$$
\begin{aligned}
&\lambda_0(\kappa)-\rho\gamma\\
&\qquad\geq -d\kappa+[1+o(1)]\,\frac{1}{T}\left[\gamma\mathbb{E}\big(\max\big\{I_1^\xi(0,T),I_1^\xi(e,T)\big\}\big)+\log\big(\tfrac12\kappa T\big)\right].
\end{aligned} \tag{55}
$$

Because $I_1^\xi(0,T)$ and $I_1^\xi(e,T)$ have the same distribution under $\mathbb{P}$, and this distribution is continuous and has zero mean, we have

$$
\mathbb{E}\big(\max\big\{I_1^\xi(0,T),I_1^\xi(e,T)\big\}\big)=\tfrac12\,\mathbb{E}\big(\big|I_1^\xi(0,T)-I_1^\xi(e,T)\big|\big). \tag{56}
$$

The expectation in the right-hand side equals $E_1(\frac12 T)$ because $|C_1|=\frac12 T$, and so we get the claim. $\square$

Using Lemma 2.1, we can now complete the proof of Theorem 1.2(ii). By condition (19), there exists a $c>0$ such that $E_1(T)\geq c\log T$ for large enough T. Therefore, picking $T=T(\kappa)=\kappa^{-3/(3+c\gamma)}$ in (47) and letting $\kappa\downarrow 0$, we obtain

$$
\lambda_0(\kappa)-\rho\gamma\geq[1+o(1)]\,\tfrac{c\gamma}{2(3+c\gamma)}\,\kappa^{3/(3+c\gamma)}\log(1/\kappa). \tag{57}
$$

Since $\lambda_0(0)=\rho\gamma$, (57) implies that $\kappa\mapsto\lambda_0(\kappa)$ is not Lipschitz at 0. $\square$

2.4 Proof of Theorem 1.2(iii)

Proof. The upper bound is a direct consequence of condition (6).

1. To prove the lower bound, fix $T>0$ and consider the expression

$$\lambda_0 = \lim_{n\to\infty} \frac{1}{nT}\,\mathbb{E}\big(\log u(0,nT)\big), \tag{58}$$

where we recall that $\mathbb{E}$ denotes expectation w.r.t. ξ. By splitting the time-interval $[0,nT]$ into n pieces of length T and using the Markov property of X^κ at the end of each piece, we obtain

$$\begin{aligned} &u(0,nT)\\ &= \mathrm{E}_0\left(\exp\left[\gamma\int_0^{nT} \xi\big(X^\kappa(s),s\big)\,ds\right]\delta_0(X^\kappa(nT))\right)\\ &= \sum_{x_1,\dots,x_{n-1}\in\mathbb{Z}^d}\prod_{i=1}^n \mathrm{E}_{x_{i-1}}\left(\exp\left[\gamma\int_0^T \xi\big(X^\kappa(s),(i-1)T+s\big)\,ds\right]\delta_{x_i}(X^\kappa(T))\right) \end{aligned} \tag{59}$$

with $x_0=x_n=0$. Next, let $\mathrm{E}^{(T)}_{x,y}$ denote the conditional expectation X^κ given that $X^\kappa(0)=x$ and $X^\kappa(T)=y$, and abbreviate, for $1\le i\le n$,

$$\mathrm{E}^{(T)}_{x,y}(i) = \mathrm{E}^{(T)}_{x,y}\left(\exp\left[\gamma\int_0^T \xi\big(X^\kappa(s),(i-1)T+s\big)\,ds\right]\right). \tag{60}$$

Then we can write

$$\mathrm{E}_{x_{i-1}}\left(\exp\left[\gamma\int_0^T \xi\big(X^\kappa(s),(i-1)T+s\big)\,ds\right]\delta_{x_i}(X^\kappa(T))\right) = p_T(x_{i-1},x_i)\,\mathrm{E}^{(T)}_{x_{i-1},x_i}(i), \tag{61}$$

which, combined with (59), gives

$$\begin{aligned} u(0,nT) &= \sum_{x_1,\cdots,x_{n-1}\in\mathbb{Z}^d}\left(\prod_{i=1}^n p_T(x_{i-1},x_i)\right)\left(\prod_{i=1}^n \mathrm{E}^{(T)}_{x_{i-1},x_i}(i)\right)\\ &= p_{nT}(0,0)\,\mathrm{E}^{(nT)}_{0,0}\left(\prod_{i=1}^n \mathrm{E}^{(T)}_{X_{(i-1)T},X_{iT}}(i)\right). \end{aligned} \tag{62}$$

2. To estimate the last expectation in (62), we apply Jensen's inequality to write, for $x,y\in\mathbb{Z}^d$ and $1\le i\le n$,

$$\mathrm{E}^{(T)}_{x,y}(i) = \exp\left[\gamma\int_0^T \mathrm{E}^{(T)}_{x,y}\Big(\xi\big(X^\kappa(s),(i-1)T+s\big)\Big)\,ds + C_{x,y}\big(\xi_{[(i-1)T,iT]},T\big)\right] \tag{63}$$

with, for $I\subset[0,\infty)$ finite, $\xi_I=(\xi_t)_{t\in I}$ and $T>0$,

$$C_{x,y}(\xi_I,T) > 0 \quad \xi\text{-a.s.}, \tag{64}$$

where the strict positivity is an immediate consequence of the fact that ξ is not constant and $u \mapsto e^u$ is strictly convex. Combining (62–63) and again using Jensen's inequality, this time w.r.t. $\mathrm{E}_{0,0}^{(nT)}$, we obtain

$$\begin{aligned}
&\mathbb{E}\Big(\log u(0,nT)\Big)\\
&\geq \log p_{nT}(0,0) + \mathbb{E}\left(\mathrm{E}_{0,0}^{(nT)}\left(\sum_{i=1}^{n}\mathrm{E}^{(T)}_{X_{(i-1)T},X_{iT}}\left(\gamma\int_0^T \xi\left(X^\kappa(s),(i-1)T+s\right)ds\right.\right.\right.\\
&\qquad\qquad\qquad\left.\left.\left. + C_{X_{(i-1)T},X_{iT}}\left(\xi_{[(i-1)T,iT]},T\right)\right)\right)\right)\\
&= \log p_{nT}(0,0) + n\rho\gamma T\\
&\qquad + \mathbb{E}\left(\mathrm{E}_{0,0}^{(nT)}\left(\sum_{i=1}^{n}\mathrm{E}^{(T)}_{X_{(i-1)T},X_{iT}}\left(C_{X_{(i-1)T},X_{iT}}\left(\xi_{[(i-1)T,iT]},T\right)\right)\right)\right),
\end{aligned}$$

where in the last line the middle term is obtained after computing the expectation w.r.t. ξ. By inserting the indicator of the event $\{X_{(i-1)T} = X_{iT}\}$ for $1 \leq i \leq n$ in the last expectation in (65), we get

$$\begin{aligned}
&\mathbb{E}\left(\mathrm{E}_{0,0}^{(nT)}\left(\sum_{i=1}^{n}\mathrm{E}^{(T)}_{X_{(i-1)T},X_{iT}}\left(C_{X_{(i-1)T},X_{iT}}\left(\xi_{[(i-1)T,iT]},T\right)\right)\right)\right)\\
&\geq \sum_{i=1}^{n}\sum_{z\in\mathbb{Z}^d}\frac{p_{(i-1)T}(0,z)\,p_T(z,z)\,p_{(n-i)T}(z,0)}{p_{nT}(0,0)}\mathbb{E}\Big(C_{z,z}\big(\xi_{[(i-1)T,iT]},T\big)\Big)\\
&\geq nC_T\,p_T(0,0),
\end{aligned} \tag{65}$$

where we abbreviate

$$C_T = \mathbb{E}\Big(C_{z,z}\big(\xi_{[(i-1)T,iT]},T\big)\Big) > 0. \tag{66}$$

Note that the latter does not depend on i or z. Therefore, combining (58) and (65–66), and using that

$$\lim_{n\to\infty}\frac{1}{nT}\log p_{nT}(0,0) = 0, \tag{67}$$

we arrive at $\lambda_0 \geq \rho\gamma + (C_T/T)p_T(0,0) > \rho\gamma$. $\square$

2.5 Proof of Theorem 1.3

The proof borrows from Carmona and Molchanov [6], Section IV.3.

Proof. Recall (4), (11) and (12), and write

$$\lambda_0(\kappa) \le \lim_{n\to\infty} \frac{1}{nT} \log \mathrm{E}_0\left(\exp\left[\gamma \int_0^{nT} \xi(X^\kappa(s),s)\,ds\right]\right), \tag{68}$$

where we pick

$$T = T(\kappa) = K\log(1/\kappa), \qquad K \in (0,\infty). \tag{69}$$

Split the time interval $[0,nT)$ into n disjoint time intervals $I_j = [(j-1)T, jT)$, $1 \le j \le n$. Define N_j, $1 \le j \le n$, to be the number of jumps of X^κ in the time interval I_j, and color I_j *black* when $N_j > 0$ and *white* when $N_j = 0$. Using Cauchy-Schwarz, we can split λ_0 into a white part and a black part, and estimate

$$\lambda_0(\kappa) \le \lambda_0^{(\mathrm{b})}(\kappa) + \lambda_0^{(\mathrm{w})}(\kappa), \tag{70}$$

where

$$\lambda_0^{(\mathrm{b})}(\kappa) = \limsup_{n\to\infty} \tfrac{1}{2nT} \log \mathrm{E}_0\left(\exp\left[2\gamma \sum_{\substack{j=1\\ N_j>0}}^{n} \int_{I_j} \xi(X^\kappa(s),s)\,ds\right]\right), \tag{71}$$

$$\lambda_0^{(\mathrm{w})}(\kappa) = \limsup_{n\to\infty} \tfrac{1}{2nT} \log \mathrm{E}_0\left(\exp\left[2\gamma \sum_{\substack{j=1\\ N_j=0}}^{n} \int_{I_j} \xi(X^\kappa(s),s)\,ds\right]\right). \tag{72}$$

Lemma 2.2. *If ξ is bounded, then*

$$\limsup_{\kappa\downarrow 0} \lambda_0^{(\mathrm{b})}(\kappa) \le 0. \tag{73}$$

Lemma 2.3. *If ξ satisfies condition* (21)*, then*

$$\limsup_{\kappa\downarrow 0} \frac{\log(1/\kappa)}{\log\log(1/\kappa)}\,[\lambda_0^{(\mathrm{w})}(\kappa) - \rho\gamma] < \infty. \tag{74}$$

Theorem 1.3 follows from (70) and Lemmas 2.2–2.3. $\square$

We first give the proof of Lemma 2.2.

Proof. Let $N^{(\mathrm{b})} = |\{1 \le j \le n\colon N_j > 0\}|$ be the number of black time intervals. Since ξ is bounded, say $\xi \le 1$, we have

$$
\begin{aligned}
&\frac{1}{2nT}\log \mathrm{E}_0\left(\exp\left[2\gamma\sum_{\substack{j=1\\N_j>0}}^n\int_{I_j}\xi(X^\kappa(s),s)\,ds\right]\right)\\
&\quad\le\frac{1}{2nT}\log \mathrm{E}_0\Big(\exp\big[2\gamma T N^{(\mathrm{b})}\big]\Big)\\
&\quad=\frac{1}{2T}\log\left[\left(1-e^{-2d\kappa T}\right)e^{2\gamma T}+e^{-2d\kappa T}\right]\\
&\quad\le\frac{1}{2T}\log\left[2d\kappa Te^{2\gamma T}+1\right]\\
&\quad\le\frac{1}{2T}2d\kappa Te^{2\gamma T}\\
&\quad=d\kappa^{1-2\gamma K},
\end{aligned}
\tag{75}
$$

where the first equality uses that the distribution of $N^{(\mathrm{b})}$ is $\mathrm{BIN}(n,1-e^{-2d\kappa T})$, and the second equality uses (69). It follows from (71) and (75) that $\lambda_0^{(\mathrm{b})}(\kappa)\le d\kappa^{1-2\gamma K}$. The claim therefore follows by picking $0<K<1/2\gamma$ and letting $\kappa\downarrow 0$. $\square$

We next give the proof of Lemma 2.3.

Proof. The proof comes in 4 steps.

1. We begin with some definitions. To each time interval I_j, we associate the set of increments of X^κ occuring on I_j by putting

$$
\Gamma_j=\begin{cases}\emptyset & \text{if } I_j \text{ is white},\\ \{\Delta_1,\dots,\Delta_{N_j}\} & \text{if } I_j \text{ is black}.\end{cases}
\tag{76}
$$

Here, $\{\Delta_i\colon\ 1\le i\le N_j\}$ is the increment sequence of X^κ on the (black) time interval I_j, i.e., Δ_i, $1\le i\le N_j$, are random variables taking values in $\mathbb{Z}^d$ and satisfying $|\Delta_i|=1$. Next, we define the set of T-skeletons by putting

$$
\Psi=\{\chi\colon\ \Gamma=\chi\}\qquad\text{with}\qquad \chi=(\chi_1,\dots,\chi_n),\,\Gamma=(\Gamma_1,\dots,\Gamma_n),
\tag{77}
$$

which corresponds to the set of increments of X^κ on the time interval $[0,nT]$. Since X^κ is stepping at rate $2d\kappa$, the T-skeleton random variable Γ has distribution

$$
\mathrm{P}_0(\Gamma=\chi)=e^{-2d\kappa nT}\prod_{j=1}^n\frac{(2d\kappa T)^{|\chi_j|}}{|\chi_j|!},\qquad \chi\in\Psi.
\tag{78}
$$

For a given realization $\{n_j\colon\ 1\le j\le n\}$ of $\{N_j\colon 1\le j\le n\}$, we define the event

$$
A^{(n)}(\chi;\lambda)=\left\{\sum_{\substack{j=1\\n_j=0}}^n\int_{I_j}\big[\xi(x_j,s)-\rho\big]\,ds\ge\lambda\right\},\qquad \chi\in\Psi,\ \lambda>0,
\tag{79}
$$

where, for $\chi_j=\{x_{j,1},\dots,x_{j,n_j}\}$ with $1\le j\le n$ and $n_j>0$,

$$x_j = \sum_{i=1}^{j-1} \sum_{k=1}^{n_j} x_{i,k} \tag{80}$$

is the starting point of the T-skeleton χ in the time interval I_j. Finally, we abbreviate

$$f_\delta(T) = \sup_{\eta \in \Omega} \mathbb{P}_\eta \left(\int_0^T \xi(0,s)\, ds > (\rho + \delta)T \right), \qquad \delta > 0. \tag{81}$$

2. With the above definitions, we can now start our proof. Fix $\chi \in \Psi$, and let $k_0(\chi) = |\{1 \le j \le n\colon\ |\chi_j| = 0\}|$ be the number of white intervals associated to χ. Then

$$\sum_{\substack{j=1 \\ n_j=0}}^{n} \int_{I_j} \big[\xi(x_j, s) - \rho\big]\, ds \preceq LT + (k_0(\chi) - L)\delta T, \qquad \delta > 0, \tag{82}$$

where $\preceq$ means "stochastically dominated", and L is the random variable with distribution $\mathrm{BIN}(k_0(\chi), f_\delta(T))$. By (79), (82) and the exponential Chebychev inequality, we have

$$\begin{aligned} \mathbb{P}\big(A^{(n)}(\chi;\lambda)\big) &\le \mathrm{P}\big(LT + (k_0(\chi) - L)\delta T \ge \lambda\big) \\ &\le \inf_{c>0} e^{-c\lambda}\, \mathbb{E}\Big(e^{c[LT + (k_0(\chi) - L)\delta T]}\Big) \\ &= \inf_{c>0} e^{-c\lambda} \left\{ f_\delta(T)\, e^{cT} + [1 - f_\delta(T)]\, e^{c\delta T} \right\}^{k_0(\chi)}. \end{aligned} \tag{83}$$

Using condition (21), which implies that there exists a $C = C(\delta) \in (0,\infty)$ such that $f_\delta(T) \le e^{-CT}$ (see Liggett [23], Section I.3), and choosing $c = C\lambda/2k_0(\chi)T$, we obtain from (83) that

$$\mathbb{P}\big(A^{(n)}(\chi;\lambda)\big) \le \exp\left[-\frac{C\lambda^2}{2k_0(\chi)T}\right] \left\{ \exp\left[\frac{C\lambda}{2k_0(\chi)} - CT\right] + \exp\left[\frac{C\lambda\delta}{2k_0(\chi)}\right] \right\}^{k_0(\chi)}. \tag{84}$$

3. Our next step is to choose λ. Recall (69), and put

$$\lambda = \sum_{l=0}^{\infty} a_l k_l(\chi) \tag{85}$$

with

$$a_0 = K' \log\log(1/\kappa), \quad K' \in (0,\infty), \qquad a_l = lT, \quad l \ge 1, \tag{86}$$

and

$$k_l(\chi) = |\{1 \le j \le n\colon\ |\chi_j| = l\}|, \qquad l \ge 0. \tag{87}$$

Then, using that $\lambda > k_0(\chi)T$ and choosing $0 < \delta \ll \lambda/2k_0(\chi)T$, we obtain

$$\text{r.h.s. } (84) \le \exp\left[-\frac{C\lambda^2}{4k_0(\chi)T} + \frac{C\lambda\delta}{2}\right] \le \exp\left[-\frac{C'\lambda^2}{2k_0(\chi)T}\right] \tag{88}$$

for some $C' = C'(\delta) \in (0,\infty)$. Recalling (79), combining (82), (84) and (88), and using (85) and (87), we get

$$
\begin{aligned}
\sum_{\chi\in\Psi} \mathrm{P}\Big(A^{(n)}(\chi;\lambda)\Big) &\leq \sum_{\chi\in\Psi} \exp\left[-\frac{C'}{2k_0(\chi)T}\left(\sum_{l=0}^{\infty} a_l k_l(\chi)\right)^2\right] \\
&\leq \sum_{\chi\in\Psi} \exp\left[-\frac{C'}{2T}a_0^2 k_0(\chi) - \frac{C'a_0}{T}\sum_{l=1}^{\infty} a_l k_l(\chi)\right] \\
&\leq \left(e^{-\frac{C'}{2T}a_0^2} + \sum_{l=1}^{\infty}(2d)^l e^{-\frac{C'}{T}a_0 a_l}\right)^n,
\end{aligned}
\tag{89}
$$

where in the last line we perform the sum over $\chi \in \Psi$ as a sum over all integers $k_l(\chi)$, $l \geq 0$, that sum up to n, and we take into account that there are $(2d)^l$ different χ_j with $|\chi_j| = l$, $1 \leq j \leq n$. By (69) and (86), $a_0 \to \infty$ and $a_0^2/T \downarrow 0$ as $\kappa \downarrow 0$. Hence, picking $K' > 1/C'$ and κ small enough, we have

$$
\begin{aligned}
e^{-\frac{C'}{2T}a_0^2} &+ \sum_{l=1}^{\infty}(2d)^l e^{-\frac{C'}{T}a_0 a_l} = e^{-\frac{C'}{2T}a_0^2} + \sum_{l=1}^{\infty}\Big(2de^{-C'a_0}\Big)^l \\
&= e^{-\frac{C'}{2T}a_0^2} + \frac{2de^{-C'a_0}}{1-2de^{-C'a_0}} \leq \Big(1 - \frac{C'}{4T}a_0^2\Big) + 4de^{-C'a_0} \\
&= \Big(1 - \frac{C'[K'\log\log(1/\kappa)]^2}{4K\log(1/\kappa)}\Big) + 4d[\log(1/\kappa)]^{-C'K'} < 1.
\end{aligned}
\tag{90}
$$

It follows from (89–90) and the Borel-Cantelli lemma that $\mathbb{P}$-a.s. there exists an $n_0 = n_0(\xi) \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$
\sum_{\substack{j=1 \\ n_j=0}}^{n} \int_{I_j} \big[\xi(x_j,s) - \rho\big]\,ds \leq \sum_{l=0}^{\infty} a_l k_l(\chi). \tag{91}
$$

4. The estimate in (91) allows us to proceed as follows. Combining (78) and (91), we obtain, for $n \geq n_0$,

$$
\begin{aligned}
&\mathrm{E}_0\left(\exp\left[2\gamma\sum_{\substack{j=1 \\ N_j=0}}^{n}\int_{I_j}\big[\xi(X^\kappa(s),s)-\rho\big]\,ds\right]\right) \\
&\qquad \leq e^{-2d\kappa nT}\sum_{\chi\in\Psi}\prod_{j=1}^{n}\frac{(2d\kappa T)^{|\chi_j|}}{|\chi_j|!}\exp\left[2\gamma\sum_{l=0}^{\infty}a_l k_l(\chi)\right].
\end{aligned}
\tag{92}
$$

Now, for any sequence $\{n_j\colon\ 1 \leq j \leq n\}$ such that $\sum_{j=1}^{n} n_j = n$, the number of T-skeletons χ such that $k_j(\chi) = n_{j+1}$, $0 \leq j \leq n-1$, equals $n!/\prod_{j=1}^{n} n_j!$. Hence, for any $\chi \in \Psi$,

$$\prod_{j=1}^{n} \frac{(2d\kappa T)^{|\chi_j|}}{|\chi_j|!} = \frac{n!}{\prod_{l=0}^{\infty} k_l(\chi)!} \prod_{l=0}^{\infty} \left(\frac{(2d\kappa T)^l)}{l!} \right)^{k_l(\chi)}. \tag{93}$$

Combining (92–93), we get

$$\begin{aligned} &\mathrm{E}_0\left(\exp\left[2\gamma \sum_{\substack{j=1 \\ n_j=0}}^{n} \int_{I_j} \left[\xi(X^\kappa(s),s) - \rho \right] ds \right] \right) \\ &\leq e^{-2d\kappa nT} \sum_{\chi \in \Psi} \frac{n!}{\prod_{l=0}^{\infty} k_l(\chi)!} \prod_{l=0}^{\infty} \left(\frac{(2d\kappa T)^l}{l!} e^{2\gamma a_l} \right)^{k_l(\chi)} \\ &\leq e^{-2d\kappa nT} \left(\sum_{l=0}^{\infty} \frac{(4d^2\kappa T)^l}{l!} e^{2\gamma a_l} \right)^n, \end{aligned} \tag{94}$$

where in the last line we do the same computation as in the last line of (89). Using (69) and (86), we have

$$\begin{aligned} &\frac{1}{2nT} \log \mathrm{E}_0\left(\exp\left[2\gamma \sum_{\substack{j=1 \\ N_j=0}}^{n} \int_{I_j} \left[\xi(X^\kappa(s),s) - \rho \right] ds \right] \right) \\ &\leq -2d\kappa + \frac{1}{T} \log\left(\sum_{l=0}^{\infty} \frac{(4d^2\kappa T)^l}{l!} e^{2\gamma a_l} \right). \end{aligned} \tag{95}$$

Note that the r.h.s. of (95) does not depend on n. Therefore, letting $n \to \infty$ and recalling (75), we get

$$\lambda_0(\kappa) \leq -2d\kappa + \frac{1}{T} \log\left(\sum_{l=0}^{\infty} \frac{(4d^2\kappa T)^l}{l!} e^{2\gamma a_l} \right). \tag{96}$$

Finally, by (69) and (86), if $0 < K < 1/2\gamma$, then

$$\begin{aligned} \sum_{l=0}^{\infty} \frac{(4d^2\kappa T)^l}{l!} e^{2\gamma a_l} &= [\log(1/\kappa)]^{2\gamma K'} + \sum_{l=1}^{\infty} \frac{(4d^2\kappa^{1-2\gamma K})^l}{l!} \\ &= [\log(1/\kappa)]^{2\gamma K'} + o(1), \qquad \kappa \downarrow 0, \end{aligned} \tag{97}$$

and hence

$$\lambda_0(\kappa) \leq [1+o(1)] \frac{2\gamma K' \log\log(1/\kappa)}{K \log(1/\kappa)}, \qquad \kappa \downarrow 0, \tag{98}$$

which proves the claim. $\square$

3 Proof of Theorems 1.4–1.6

The proofs of Theorems 1.4–1.6 are given in Sections 3.1–3.3, respectively.

3.1 Proof of Theorem 1.4

Proof. For ISRW, SEP and SVM in the weakly catalytic regime, it is known that $\lim_{\kappa\to\infty}\lambda_1(\kappa)=\rho\gamma$ (recall Section 1.3.2). The claim therefore follows from the fact that $\rho\gamma\le\lambda_0(\kappa)\le\lambda_1(\kappa)$ for all $\kappa\in[0,\infty)$. □

3.2 Proof of Theorem 1.5

Proof. Recall (20) and define

$$\begin{aligned} E_k(T) &= \mathbb{E}\big(|I^\xi(0,T)-I^\xi(e,T)|^k\big),\\ \bar{E}_k(T) &= \mathbb{E}\big(|I^\xi(0,T)|^k\big), \end{aligned} \qquad T>0,\, k\in\mathbb{N}. \tag{99}$$

The proof is based on the following lemma.

Lemma 3.1. *For ISRW and SEP in the weakly catalytic regime,*

$$\liminf_{T\to\infty} T^{-1}E_2(T)>0, \qquad \limsup_{T\to\infty} T^{-2}\bar{E}_4(T)<\infty. \tag{100}$$

Before proving Lemma 3.1, we complete the proof of Theorem 1.5. Estimate, for $N>0$,

$$\begin{aligned} E_1(T) &= \mathbb{E}\big(|I^\xi(0,T)-I^\xi(e,T)|\big)\\ &\ge \tfrac{1}{2N}\,\mathbb{E}\Big(|I^\xi(0,T)-I^\xi(e,T)|^2\,1\!\!1_{\{|I^\xi(0,T)|\le N \text{ and } |I^\xi(e,T)|\le N\}}\Big)\\ &= \tfrac{1}{2N}\Big[E_2(T)-\mathbb{E}\Big(|I^\xi(0,T)-I^\xi(e,T)|^2\,1\!\!1_{\{|I^\xi(0,T)|>N \text{ or } |I^\xi(e,T)|>N\}}\Big)\Big]. \end{aligned} \tag{101}$$

By Cauchy-Schwarz,

$$\begin{aligned} &\mathbb{E}\Big(|I^\xi(0,T)-I^\xi(e,T)|^2\,1\!\!1_{\{|I^\xi(0,T)|>N \text{ or } |I^\xi(e,T)|>N\}}\Big)\\ &\qquad\le [E_4(T)]^{1/2}\Big[\mathbb{P}\Big(|I^\xi(0,T)|>N \text{ or } |I^\xi(e,T)|>N\Big)\Big]^{1/2}. \end{aligned} \tag{102}$$

Moreover, $E_4(T)\le 16\bar{E}_4(T)$ and

$$\mathbb{P}\Big(|I^\xi(0,T)|>N \text{ or } |I^\xi(e,T)|>N\Big)\le \tfrac{2}{N^2}\,\bar{E}_2(T)\le \tfrac{2}{N^2}\,[\bar{E}_4(T)]^{1/2}. \tag{103}$$

By (100), there exist an $a>0$ such that $E_2(T) \geq aT$ and a $b<\infty$ such that $\bar{E}_4(T) \leq bT^2$ for T large enough. Therefore, combining (101–103) and picking $N = cT^{1/2}$, we obtain

$$E_1(T) \geq A\,T^{1/2} \text{ with } A = \tfrac{1}{2c}\left(a - 2^{5/2} b^{3/4} \tfrac{1}{c}\right), \tag{104}$$

where we note that $A>0$ for c large enough. Inserting this bound into Lemma 2.1 and picking $T = T(\kappa) = B[\log(1/\kappa)]^2$, we find that

$$\lambda_0(\kappa) - \rho\gamma \geq C\,[\log(1/\kappa)]^{-1}\,[1+o(1)] \text{ with } C = \tfrac{1}{B}\left(\tfrac{\gamma A B^{1/2}}{2^{3/2}} - 1\right). \tag{105}$$

Since $C>0$ for $A>0$ and B large enough, this proves the claim. □

We finish by proving Lemma 3.1.

Proof. Let

$$C(x,t) = \mathbb{E}\big([\xi(0,0)-\rho][\xi(x,t)-\rho]\big), \qquad x \in \mathbb{Z}^d, t \geq 0, \tag{106}$$

denote the two-point correlation function of ξ. By the stationarity of ξ, we have

$$\begin{aligned} E_2(T) &= \int_0^T ds \int_0^T dt\ \mathbb{E}\big([\xi(0,s)-\xi(e,s)][\xi(0,t)-\xi(e,t)]\big) \\ &= 4\int_0^T ds \int_0^{T-s} dt\ [C(0,t)-C(e,t)]. \end{aligned} \tag{107}$$

Recall that $G_d = G_d(0,0)$ denotes the Green function at the origin of simple random walk stepping at rate 1.

Lemma 3.2. *For $x \in \mathbb{Z}^d$ and $t \geq 0$,*

$$C(x,t) = \begin{cases} \rho\, p_t(0,x), & d \geq 1 \textit{ ISRW}, \\ \rho(1-\rho) p_t(0,x), & d \geq 1 \textit{ SEP}, \\ [\rho(1-\rho)/G_d] \int_0^\infty p_{t+s}(0,x)\, ds, & d \geq 3 \textit{ SVM}. \end{cases} \tag{108}$$

Proof. For ISRW, we have

$$\xi(x,t) = \sum_{y \in \mathbb{Z}^d} \sum_{j=1}^{N_y} \delta_x\big(Y_j^y\big), \qquad x \in \mathbb{Z}^d, t \geq 0, \tag{109}$$

where $\{N_y\colon y \in \mathbb{Z}^d\}$ are i.i.d. Poisson random variables with mean $\rho \in (0,\infty)$, and $\{Y_j^y\colon y \in \mathbb{Z}^d, 1 \leq j \leq N_y\}$ is a collection of independent simple random walks with jump rate 1 (Y_j^y is the j-th random walk starting from $y \in \mathbb{Z}^d$ at time 0). Inserting (109) into (106), we get the first line in (108). For SEP and SVM, the claim follows via the graphical representation (see [14], Eq. (1.5.5) and [17], Lemma A.1, respectively). □

Combining (107) and Lemma 3.2, we obtain

$$\lim_{T\to\infty}\frac{1}{T}E_2(T)=\begin{cases}4\rho[G_d(0,0)-G_d(0,e)], & d\geq 3 \text{ ISRW},\\ 4\rho(1-\rho)[G_d(0,0)-G_d(0,e)], & d\geq 3 \text{ SEP},\\ 4\rho(1-\rho)[G_d^*(0,0)-G_d^*(0,e)]/G_d(0,0), & d\geq 5 \text{ SVM},\end{cases}\tag{110}$$

where $G_d^*(0,x)=\int_0^\infty t p_t(0,x)dt$. By using the strong Markov property at the first jump time of simple random walk stepping at rate 1, we get

$$\begin{aligned}G_d(0,0)-G_d(0,e)&=1,\\ G_d^*(0,0)-G_d^*(0,e)&=G_d(0,0).\end{aligned}\tag{111}$$

Hence (110) gives

$$\lim_{T\to\infty}\frac{1}{T}E_2(T)=\begin{cases}4\rho, & d\geq 3 \text{ ISRW},\\ 4\rho(1-\rho), & d\geq 3 \text{ SEP and } d\geq 5 \text{ SVM},\end{cases}\tag{112}$$

which proves the first part of (100).

Let

$$C(x,t;y,u;z,v)=\mathbb{E}\big([\xi(0,0)-\rho][\xi(x,t)-\rho][\xi(y,u)-\rho][\xi(z,v)-\rho]\big),\qquad x,y,z\in\mathbb{Z}^d,\ 0\leq t\leq u\leq v,\tag{113}$$

denotes the four-point correlation function of ξ. Then

$$\bar{E}_4(T)=4!\int_0^T ds\int_0^{T-s}dt\int_t^{T-s}du\int_u^{T-s}dv\ C(0,t;0,u;0,v).\tag{114}$$

To prove the second part of (100), we must estimate $C(0,t;0,u;0,v)$. For ISRW, this can be done by using (109). For SEP, the proof uses the Markov property and the graphical representation. In both cases the computations are long but straightforward, with leading terms of the form

$$C(\rho)p_a(0,0)p_b(0,0)\tag{115}$$

with a,b linear in t, u or v, and $C(\rho)$ a positive constant depending on ρ. Each of these leading terms, after being integrated as in (114), can be bounded from above by a term of order $O(T^2)$ in $d\geq 3$.

We expect the second part of (100) to hold also for SVM. However, the graphical representation, which is based on coalescing random walks, seems a bit too complicated to carry through the computations. □

3.3 Proof of Theorem 1.6

Proof. For ISRW in the strongly catalytic regime, we know that $\lambda_1(\kappa) = \infty$ for all $\kappa \in [0,\infty)$ (recall Fig. 3), while $\lambda_0(\kappa) < \infty$ for all $\kappa \in [0,\infty)$ (by Theorem 2 in Kesten and Sidoravicius [19]). $\square$

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