THE PARABOLIC ANDERSON MODEL

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Abstract: This is a survey on the intermittent behavior of the parabolic Anderson model, which is the Cauchy problem for the heat equation with random potential on the lattice $\mathbb{Z}^d$. We first introduce the model and give heuristic explanations of the long-time behavior of the solution, both in the annealed and the quenched setting for time-independent potentials. We thereby consider examples of potentials studied in the literature. In the particularly important case of an i.i.d. potential with double-exponential tails we formulate the asymptotic results in detail. Furthermore, we explain that, under mild regularity assumptions, there are only four different universality classes of asymptotic behaviors. Finally, we study the moment Lyapunov exponents for space-time homogeneous catalytic potentials generated by a Poisson field of random walks.

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1. Introduction and heuristics

1.1 Evolution of spatially distributed systems in random media

One of the often adequate and frequently used methods for studying the evolution of spatially distributed systems under the influence of a random medium is *homogenization*. After rescaling, the system, modeled by partial differential equations with random coefficients, is approached by a system with 'properly averaged' deterministic coefficients, see e.g. [ZKO94]. But there are simple and important situations when random systems exhibit effects which cannot be recovered by such deterministic approximations and related fluctuation corrections. This concerns, in particular, *localization* effects for non-reversible random walks in random environment [S82] and for the electron transport in disordered media [And58].

Another such effect is that of *intermittency*. Roughly speaking, intermittency means that the solution of the system develops pronounced spatial structures on islands located far from each other that, in one or another sense, deliver the main output to the system. One of the sources of interest is magnetohydrodynamics and, in particular, the investigation of the induction equation with incompressible random velocity fields [Z84], [ZMRS87]. Another source are simple mathematical models such as the random Fisher-Eigen equation that have been used to derive caricatures of Darwinian evolution principles [EEEF84].

One of the simplest and most basic models exhibiting the effect of intermittency is the Cauchy problem for the spatially discrete heat equation with a random potential:

$$\begin{aligned} \partial_t u(t,x) &= \kappa\Delta u(t,x) + \xi(t,x)u(t,x), && (t,x)\in(0,\infty)\times\mathbb{Z}^d, \\ u(0,x) &= u_0(x), && x\in\mathbb{Z}^d. \end{aligned} \tag{1.1}$$

Here $\kappa > 0$ is a diffusion constant, Δ denotes the discrete Laplacian,

$$\Delta f(x) = \sum_{y\colon |y-x|=1} [f(y) - f(x)],$$

ξ is a space-time homogeneous ergodic random potential, and u_0 is a nonnegative initial function. Problem (1.1) is often called *parabolic problem for the Anderson model* or *parabolic Anderson model* (abbreviated *PAM*). As simplest localized initial datum one may take $u_0 = \delta_0$, and as non-local initial datum $u_0 = \mathbb{1}$. In the latter case, the solution $u(t,\cdot)$ is spatially homogeneous and ergodic for each t. Let us remark that the solution $u(t,x)$ of (1.1) allows the interpretation as average number of particles at site x at time t for branching random walks in random media given a realization of the medium ξ, cf. [CM94] and the remarks in the next subsection.

1.2 The PAM with time-independent potential

In the particular case when the potential $\xi(t,x) = \xi(x)$ is time-independent, the large-time behavior of the solution u to the PAM (1.1) is determined by the spectral properties of the Anderson Hamiltonian

$$\mathcal{H} = \kappa\Delta + \xi \tag{1.2}$$

and therefore closely related to the mentioned localization of the electron transport. Namely, since (under natural assumptions on ξ) the upper part of the spectrum of $\mathcal{H}$ in $\ell^2(\mathbb{Z}^d)$ is a pure point spectrum [FMSS85], [AM93], the solution u admits the spectral representation

$$u(t,\cdot) = \sum_i e^{\lambda_i t}\,(v_i, u_0)\,v_i(\cdot) \tag{1.3}$$

with respect to the random eigenvalues λ_i and the corresponding exponentially localized random eigenfunctions v_i. (For simplicity we ignore the possible occurrence of a continuous central part of the spectrum.) As t increases unboundedly, only summands with larger and larger eigenvalues will contribute to (1.3), and the corresponding eigenfunctions are expected to be localized more and more far from each other. Hence, for large t, the solution $u(t,\cdot)$ indeed looks like a weighted superposition of high peaks concentrated on distant islands.

A mathematically rigorous understanding of the nature of the spectrum of the Anderson Hamiltonian and the random Schrödinger operator is still far from being complete. For an overview about some recent developments we refer to the surveys [BKS04] and [LMW04] in this proceedings volume. The spectral results obtained so far do not yet seem directly applicable to answer the crucial questions about intermittency. A direct spectral approach clearly fails for space-time dependent potentials.

In this survey we present a part of the results about intermittency for the PAM which have been obtained by use of more intrinsic probabilistic methods. In the next subsections we stick to the PAM with time-independent potential and localized initial datum:

$$\begin{aligned} \partial_t u(t,x) &= \kappa\Delta u(t,x) + \xi(x)u(t,x), && (t,x)\in(0,\infty)\times\mathbb{Z}^d,\\ u(0,x) &= \delta_0(x), && x\in\mathbb{Z}^d. \end{aligned} \tag{1.4}$$

We assume throughout that $\xi = (\xi(x))_{x\in\mathbb{Z}^d}$ is a field of i.i.d. random variables with finite positive exponential moments. Under these basic assumptions, $u(t,x)$ has moments of all orders.

The solution u to (1.4) describes a random particle flow in $\mathbb{Z}^d$ in the presence of random sources (lattice sites x with $\xi(x) > 0$) and random sinks (sites x with $\xi(x) < 0$).[4] Two competing effects are present: the diffusion mechanism governed by the Laplacian, and the local growth governed by the potential. The diffusion tends to make the random field $u(t,\cdot)$ flat, whereas the random potential ξ has a tendency to make it irregular.

The solution u to (1.4) also admits a branching particle dynamics interpretation. Imagine that initially, at time $t = 0$, there is a single particle at the origin, and all other sites are vacant. This particle moves according to a continuous-time symmetric random walk with generator $\kappa\Delta$. When present at site x, the particle is split into two particles with rate $\xi_+(x)$ and is killed with rate $\xi_-(x)$, where $\xi_+ = (\xi_+(x))_{x\in\mathbb{Z}^d}$ and $\xi_- = (\xi_-(x))_{x\in\mathbb{Z}^d}$ are independent random i.i.d. fields ($\xi_-(x)$ may attain the value ∞). Every particle continues from its birth site in the same way as the parent particle, and their movements are independent. Put $\xi(x) = \xi_+(x) - \xi_-(x)$. Then, given ξ_- and ξ_+, the expected number of particles present at the site x at time t is equal to $u(t,x)$. Here the expectation is taken over the particle motion and over the splitting resp. killing mechanism, but not over the random medium (ξ_-,ξ_+).

A very useful standard tool for the probabilistic investigation of (1.4) is the well-known *Feynman-Kac formula* for the solution u, which (after time-reversal) reads

$$u(t,x) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X(s))\,ds\Big\}\delta_x(X(t))\Big], \qquad (t,x)\in[0,\infty)\times\mathbb{Z}^d, \tag{1.5}$$

where $(X(s))_{s\in[0,\infty)}$ is continuous-time random walk on $\mathbb{Z}^d$ with generator $\kappa\Delta$ starting at $x\in\mathbb{Z}^d$ under $\mathbb{E}_x$.

Our main interest concerns the large-time behavior of the random field $u(t,\cdot)$. In particular, we consider the total mass, i.e., the random variable

$$U(t) = \sum_{x\in\mathbb{Z}^d} u(t,x) = \mathbb{E}_0\Big[\exp\Big\{\int_0^t \xi(X(s))\,ds\Big\}\Big], \qquad t>0. \tag{1.6}$$

[4] Sites x with $\xi(x) = -\infty$ may be allowed and interpreted as ('hard') traps or obstacles, sites with $\xi(x)\in(-\infty,0)$ are sometimes called 'soft' traps.

Note that $U(t)$ coincides with the value $\hat{u}(t,0)$ of the solution $\hat{u}$ to the parabolic equation (1.4) with initial datum $\hat{u}(0,\cdot) = \mathbb{1}$ instead of $u(0,\cdot) = \delta_0$. One should have in mind that, because of this, our considerations below also concern the large-time asymptotics of $\hat{u}$.

We ask the following questions:

(1) What is the asymptotic behavior of $U(t)$ as $t \to \infty$?
(2) Where does the main mass of $u(t,\cdot)$ stem from? What are the regions that contribute most to $U(t)$? What are these regions determined by? How many of them are there and how far away are they from each other?
(3) What do the typical shapes of the potential $\xi(\cdot)$ and of the solution $u(t,\cdot)$ look like in these regions?

We call the regions that contribute the overwhelming part to the total mass $U(t)$ *relevant islands* or *relevant regions*. The notion of *intermittency* states that there does exist a small number of relevant islands which are far away from each other and carry asymptotically almost all the total mass $U(t)$ of $u(t,\cdot)$. See Section 1.3 for details.

This effect may also be studied from the point of view of *typical paths* $X(s)$, $s \in [0,t]$, giving the main contribution to the expectation in the Feynman-Kac formula (1.6). On the one hand, the random walker X should move quickly and as far as possible through the potential landscape to reach a region of exceptionally high potential and then stay there up to time t. This will make the integral in the exponent on the right of (1.6) large. On the other hand, the probability to reach such a distant potential peak up to t may be rather small. Hence, the first order contribution to $U(t)$ comes from paths that find a good compromise between the high potential values and the far distance. This contribution is given by the height of the peak. The second order contribution to $U(t)$ is determined by the precise manner in which the optimal walker moves within the potential peak, and this depends on the geometric properties of the potential in that peak.

It is part of our study to understand the effect of intermittency for the parabolic Anderson model in great detail. We distinguish between the so-called *quenched* setting, where we consider $u(t,\cdot)$ almost surely with respect to the medium ξ, and the *annealed* one, where we average with respect to ξ. It is clear that the quantitative details of the answers to the above questions strongly depend on the distribution of the field ξ (more precisely, on the upper tail of the distribution of the random variable $\xi(0)$), and that different phenomena occur in the quenched and the annealed settings.

It will turn out that there is a universal picture present in the asymptotics of the parabolic Anderson model. Inside the relevant islands, after appropriate vertical shifting and spatial rescaling, the potential ξ will turn out to asymptotically approximate a universal, non-random shape, V, which is determined by a characteristic variational problem. The absolute height of the potential peaks and the diameter of the relevant islands are asymptotically determined by the upper tails of the random variable $\xi(0)$, while the number of the islands and their locations are random. Furthermore, after multiplication with an appropriate factor and rescaling, also the solution $u(t,\cdot)$ approaches a universal shape on these islands, namely the principal eigenfunction of the Hamiltonian $\kappa\Delta + V$ with V the above universal potential shape. Remarkably, there are only four universal classes of potential shapes for the PAM in (1.4), see Section 4 for details.

For a general discussion we refer to the monograph by Carmona and Molchanov [CM94], the lectures by Molchanov [M94], and also to the results by Sznitman about the important (spatially continuous) case of bounded from above Poisson-like potentials summarized in his monograph [S98]. A discussion from a physicist's and a chemist's point of view in the particular case of trapping problems (see also Section 2.2 below), including a survey on related mathematical models and a collection of open problems, is provided in [HW94]. A general mathematical background for the PAM is provided in [GM90].

1.3 Intermittency

As before, let $\hat{u}$ denote the solution to the equation in (1.4) with initial datum $\hat{u}(0,\cdot) = \mathbb{1}$, but now with a homogeneous ergodic potential $\xi = (\xi(x))_{x\in\mathbb{Z}^d}$. Assume that all positive exponential moments of $\xi(0)$ are finite. Let $\mathrm{Prob}\,(\cdot)$ and $\langle\cdot\rangle$ denote probability and expectation w.r.t. ξ.

A first, rough, mathematical approach to intermittency consists in a comparision of the growth of subsequent moments of the ergodic field $\hat{u}(t,\cdot)$ as $t\to\infty$. Define

$$\Lambda_p(t) = \log \langle \hat{u}(t,0)^p\rangle\,, \qquad p\in\mathbb{N},$$

and write $f \ll g$ if $\lim_{t\to\infty}[g(t)-f(t)] = \infty$.

Definition 1.1. For $p\in\mathbb{N}\setminus\{1\}$, the homogeneous ergodic field $\hat{u}(t,\cdot)$ is called *p-intermittent* as $t\to\infty$, if

$$\frac{\Lambda_{p-1}}{p-1} \ll \frac{\Lambda_p}{p}. \tag{1.7}$$

Note that, by Hölder's inequality, always $\Lambda_{p-1}/(p-1) \le \Lambda_p/p$. If the finite moment Lyapunov exponents

$$\lambda_p = \lim_{t\to\infty}\frac{1}{t}\Lambda_p(t), \qquad p\in\mathbb{N},$$

exist, then the strict inequality $\lambda_{p-1}/(p-1) < \lambda_p/p$ implies p-intermittency. Such a comparison of the moment Lyapunov exponents has first been used in the physics literature to study intermittency, cf. [ZMRS87], [ZMRS88]. We will use this approach in Section 5.

To explain the meaning of Definition 1.1, assume (1.7) for some $p\in\mathbb{N}\setminus\{1\}$ and choose a level function ℓ_p such that $\Lambda_{p-1}/(p-1) \ll \ell_p \ll \Lambda_p/p$. Then, by Chebyshev's inequality,

$$\mathrm{Prob}\Big(\hat{u}(t,0) > e^{\ell_p(t)}\Big) \le e^{-(p-1)\ell_p(t)}\left\langle \hat{u}(t,0)^{p-1}\right\rangle = \exp\{\Lambda_{p-1}(t) - (p-1)\ell_p(t)\}\,,$$

and the expression on the right converges to zero as $t\to\infty$. In other words, the density of the homogeneous point process

$$\Gamma(t) = \Big\{x\in\mathbb{Z}^d \colon \hat{u}(t,x) > e^{\ell_p(t)}\Big\}$$

vanishes asymptotically as $t\to\infty$. On the other hand,

$$\Big\langle \hat{u}(t,0)^p\,\mathbb{1}\Big\{\hat{u}(t,0)\le e^{\ell_p(t)}\Big\}\Big\rangle \le e^{p\ell_p(t)} = e^{p\ell_p(t)-\Lambda_p(t)}\left\langle \hat{u}(t,0)^p\right\rangle = o\left(\langle \hat{u}(t,0)^p\rangle\right)$$

and, consequently,

$$\langle \hat{u}(t,0)^p\rangle \;\sim\; \Big\langle \hat{u}(t,0)^p\,\mathbb{1}\Big\{\hat{u}(t,0) > e^{\ell_p(t)}\Big\}\Big\rangle$$

as $t\to\infty$. Hence, by Birkhoff's ergodic theorem, for large t and large centered boxes B in $\mathbb{Z}^d$,

$$|B|^{-1}\sum_{x\in B}\hat{u}(t,x)^p \;\approx\; |B|^{-1}\sum_{x\in B\cap\Gamma(t)}\hat{u}(t,x)^p.$$

This means that the p-th moment $\langle\hat{u}(t,0)^p\rangle$ is 'generated' by the high peaks of $\hat{u}(t,\cdot)$ on the 'thin' set $\Gamma(t)$ and therefore indicates the presence of intermittency in the above verbal sense. Unfortunately, this approach does not reflect the geometric structure of the set $\Gamma(t)$. This set might consist of islands or, e.g., have a net-like structure.

Theorem 1.2. *If $\xi = (\xi(x))_{x\in\mathbb{Z}^d}$ is a non-deterministic field of i.i.d. random variables with $\langle e^{t\xi(0)}\rangle < \infty$ for all $t>0$, then the solution $\hat{u}(t,\cdot)$ is p-intermittent for all $p\in\mathbb{N}\setminus\{1\}$.*

This is part of Theorem 3.2 in [GM90], where, for general homogeneous ergodic potentials ξ, necessary and sufficient conditions for p-intermittency of $\hat{u}(t,\cdot)$ have been given in spectral terms of the Hamiltonian (1.2).

1.4 Annealed second order asymptotics

Let us discuss, on a heuristic level, what the asymptotics of the moments of $U(t)$ are determined by, and how they can be described. For simplicity we restrict ourselves to the first moment.

The basic observation is that, as a consequence of the spectral representation (1.3),

$$U(t) \approx e^{t\lambda_t(\xi)} \tag{1.8}$$

(in the sense of logarithmic equivalence), where $\lambda_t(\varphi)$ denotes the principal (i.e., largest) eigenvalue of the operator $\kappa\Delta + \varphi$ with zero boundary condition in the 'macrobox' $B_t = [-t,t]^d \cap \mathbb{Z}^d$. Hence, we have to understand the large-time behavior of the exponential moments of the principal eigenvalue of the Anderson Hamiltonian $\mathcal{H}$ in a large, time-dependent box.

It turns out that the main contribution to $\langle e^{t\lambda_t(\xi)}\rangle$ comes from realizations of the potential ξ having high peaks on distant islands of some radius of order $\alpha(t)$ that is much smaller than t. But this implies that $\lambda_t(\xi)$ is close to the principal eigenvalue of $\mathcal{H}$ on one of these islands. Therefore, since the number of subboxes of B_t of radius of order $\alpha(t)$ grows only polynomial in t and ξ is spatially homogeneous, we may expect that

$$\Big\langle e^{t\lambda_t(\xi)}\Big\rangle \approx \Big\langle e^{t\lambda_{R\alpha(t)}(\xi)}\Big\rangle$$

for R large as $t \to \infty$.

The choice of the scale function $\alpha(t)$ depends on asymptotic 'stiffness' properties of the potential, more precisely of its tails at its essential supremum, and is determined by a large deviation principle, see (1.15) below. In Section 2 we shall see examples of potentials such that $\alpha(t)$ tends to 0, to ∞, or stays bounded and bounded away from zero as $t \to \infty$. In the present heuristics, we shall assume that $\alpha(t) \to \infty$, which implies the necessity of a spatial rescaling. In particular, after rescaling, the main quantities and objects will be described in terms of the continuous counterparts of the discrete objects we started with, i.e., instead of the discrete Laplacian, the continuous Laplace operator appears etc. The following heuristics can also be read in the case where $\alpha(t) \equiv 1$ by keeping the discrete versions for the limiting objects.

The optimal behavior of the field ξ in the 'microbox' $B_{R\alpha(t)}$ is to approximate a certain (deterministic) shape φ after appropriate spatial scaling and vertical shifting. It easily follows from the Feynman-Kac formula (1.6) that

$$e^{H(t)-2d\kappa t} \le \langle U(t)\rangle \le e^{H(t)},$$

where

$$H(t) = \log\big\langle e^{t\xi(0)}\big\rangle, \qquad t > 0, \tag{1.9}$$

denotes the *cumulant generating function* of $\xi(0)$ (often called *logarithmic moment generating function*). Hence, the peaks of $\xi(\cdot)$ mainly contributing to $\langle U(t)\rangle$ have height of order $H(t)/t$. Together with Brownian scaling this leads to the ansatz

$$\overline{\xi}_t(\cdot) = \alpha(t)^2\Big[\xi\big(\lfloor \cdot\,\alpha(t)\rfloor\big) - \frac{H(t)}{t}\Big], \tag{1.10}$$

for the spatially rescaled and vertically shifted potential in the cube $Q_R = (-R,R)^d$. Now the idea is that the main contribution to $\langle U(t)\rangle$ comes from fields that are shaped in such a way that $\overline{\xi}_t \approx \varphi$ in Q_R, for some $\varphi\colon Q_R \to \mathbb{R}$, which has to be chosen optimally. Observe that

$$\overline{\xi}_t \approx \varphi \quad \text{in } Q_R \qquad \Longleftrightarrow \qquad \xi(\cdot) \approx \frac{H(t)}{t} + \tfrac{1}{\alpha(t)^2}\varphi\big(\tfrac{\cdot}{\alpha(t)}\big) \quad \text{in } B_{R\alpha(t)}. \tag{1.11}$$

Let us calculate the contribution to $\langle U(t)\rangle$ coming from such fields. Using (1.8), we obtain

$$\Big\langle U(t)\,\mathbb{1}\{\overline{\xi}_t \approx \varphi \text{ in } Q_R\}\Big\rangle \approx e^{H(t)}\exp\Big\{t\lambda_{R\alpha(t)}\big(\tfrac{1}{\alpha(t)^2}\varphi\big(\tfrac{\cdot}{\alpha(t)}\big)\big)\Big\}\,\mathrm{Prob}\,\big(\overline{\xi}_t \approx \varphi \text{ in } Q_R\big). \tag{1.12}$$

The asymptotic scaling properties of the discrete Laplacian, Δ, imply that

$$\lambda_{R\alpha(t)}\big(\tfrac{1}{\alpha(t)^2}\varphi\big(\tfrac{\cdot}{\alpha(t)}\big)\big) \approx \frac{1}{\alpha(t)^2}\lambda_R^{\rm c}(\varphi), \tag{1.13}$$

where $\lambda_R^{\rm c}(\varphi)$ denotes the principal eigenvalue of $\kappa\Delta^{\rm c}+\varphi$ in the cube Q_R with zero boundary condition, and $\Delta^{\rm c}$ is the usual 'continuous' Laplacian. This leads to

$$\big\langle U(t)\,1\!\!1\{\overline{\xi}_t\approx\varphi \text{ in } Q_R\}\big\rangle \approx e^{H(t)}\exp\Big\{\frac{t}{\alpha(t)^2}\lambda_R^{\rm c}(\varphi)\Big\}\,\mathrm{Prob}\,\big(\overline{\xi}_t\approx\varphi \text{ in } Q_R\big). \tag{1.14}$$

In order to achieve a balance between the second and the third factor on the right, it is necessary that the logarithmic decay rate of the considered probability is $t/\alpha(t)^2$. One expects to have a large deviation principle for the shifted, rescaled field, which reads

$$\mathrm{Prob}\,\big(\overline{\xi}_t\approx\varphi \text{ in } Q_R\big) \approx \exp\Big\{-\frac{t}{\alpha(t)^2}I_R(\varphi)\Big\}, \tag{1.15}$$

where the scale $\alpha(t)$ has to be determined in such a way that the rate function I_R is non-degenerate. Now substitute (1.15) into (1.14). Then the Laplace method tells us that the exponential asymptotics of $\langle U(t)\rangle$ is equal to the one of $\big\langle U(t)\,1\!\!1\{\overline{\xi}_t\approx\varphi \text{ in } Q_R\}$ with optimal φ. Hence, optimizing on φ and remembering that R is large, we arrive at

$$\big\langle U(t)\big\rangle \approx e^{H(t)}\exp\Big\{-\frac{t}{\alpha(t)^2}\chi\Big\}, \tag{1.16}$$

where the constant χ is given in terms of the characteristic variational problem

$$\chi = \lim_{R\to\infty}\inf_{\varphi\colon Q_R\to\mathbb{R}}\big[I_R(\varphi)-\lambda_R^{\rm c}(\varphi)\big]. \tag{1.17}$$

The first term on the right of (1.16) is determined by the absolute height of the typical realizations of the potential and the second contains information about the shape of the potential close to its maximum in spectral terms of the Anderson Hamiltonian $\mathcal{H}$ in this region. More precisely, those realizations of ξ with $\overline{\xi}_t\approx\varphi_*$ in Q_R for large R and φ_* a minimizer in the variational formula in (1.17) contribute most to $\langle U(t)\rangle$. In particular, the geometry of the relevant potential peaks is hidden via χ in the second asymptotic term of $\langle U(t)\rangle$.

1.5 Quenched second order asymptotics

Here we explain, again on a heuristic level, the almost sure asymptotics of $U(t)$ as $t\to\infty$. Because of (1.8), it suffices to study the asymptotics of the principal eigenvalue $\lambda_t(\xi)$.

Like for the annealed asymptotics, the main contribution to $\lambda_t(\xi)$ comes from islands whose radius is of a certain deterministic, time-depending order, which we denote $\widetilde{\alpha}(t)$. As $t\to\infty$, the scale function $\widetilde{\alpha}(t)$ tends to zero, one, or ∞, respectively, if the scale function $\alpha(t)$ for the moments tends to these respective values (see also (1.20) below). However, $\widetilde{\alpha}(t)$ is roughly of logarithmic order in $\alpha(t)$ if $\alpha(t)\to\infty$, hence it is *much* smaller than $\alpha(t)$.

The relevant islands ('microboxes') have radius $R\widetilde{\alpha}(t)$, where R is chosen large. Let $z\in B_t$ denote the (certainly random) center of one of these islands $\widetilde{B}=z+B_{R\widetilde{\alpha}(t)}$ meeting the two requirements (1) the potential ξ is very large in $\widetilde{B}$ and (2) ξ has an optimal shape within $\widetilde{B}$. This is further explained as follows. Let $h_t=\max_{B_t}\xi$ be the maximal potential value in the large box B_t. (Then h_t is a priori random, but well approximated by deterministic asymptotics, which can be deduced from asymptotics of $H(t)$.) Then $\xi-h_t$ is roughly of finite order within the relevant 'microbox' $\widetilde{B}$. Furthermore, $\xi-h_t$ should approximate a fixed deterministic shape in $\widetilde{B}$. Hence, we consider the shifted and rescaled field in the box $\widetilde{B}$,

$$\overline{\xi}_t(\cdot)=\widetilde{\alpha}(t)^2\Big[\xi\big(z+\cdot\,\widetilde{\alpha}(t)\big)-h_t\Big], \qquad \text{in } Q_R=(-R,R)^d. \tag{1.18}$$

Note that

$$\overline{\xi}_t \approx \varphi \quad \text{in } Q_R \qquad \Longleftrightarrow \qquad \xi(z+\cdot) \approx h_t + \tfrac{1}{\widetilde{\alpha}(t)^2}\varphi\big(\tfrac{\cdot}{\widetilde{\alpha}(t)}\big) \quad \text{in } \widetilde{B}-z. \tag{1.19}$$

A crucial Borel-Cantelli argument shows that, for a given shape φ, with probability one, for any t sufficiently large, there does exist at least one box $\widetilde{B}$ having radius $R\widetilde{\alpha}(t)$ such that the event $\{\overline{\xi}_t \approx \varphi \text{ in } Q_R\}$ occurs if $I_R(\varphi) < 1$, where I_R is the rate function of the large deviation principle in (1.15). If $I_R(\varphi) > 1$, then this happens with probability 0. For the Borel-Cantelli argument to work, one needs the scale function $\widetilde{\alpha}(t)$ to be defined in terms of the annealed scale function $\alpha(t)$ in the following way:

$$\frac{\widetilde{\alpha}(t)}{\alpha(\widetilde{\alpha}(t))^2} = d\log t, \tag{1.20}$$

i.e., $\widetilde{\alpha}(t)$ is the inverse of the map $t \mapsto t/\alpha(t)^2$, evaluated at $d\log t$. Note that the growth of $\widetilde{\alpha}(t)$ is roughly of logarithmic order of the growth of $\alpha(t)$, i.e., if the annealed relevant islands grow unboundedly, then the quenched relevant islands also grow unboundedly, but with much smaller velocity.

Hence, with probability one, for all large t, there is at least one box $\widetilde{B}$ in which the potential looks like the function on the right of (1.19). The contribution to $\lambda_t(\xi)$ coming from one of the boxes $\widetilde{B}$ is equal to the associated principal eigenvalue

$$\lambda_{\widetilde{B}-z}\big(h_t + \tfrac{1}{\widetilde{\alpha}(t)^2}\varphi\big(\tfrac{\cdot}{\widetilde{\alpha}(t)}\big)\big) \approx h_t + \frac{1}{\widetilde{\alpha}(t)^2}\lambda_R^{\mathrm{c}}(\varphi), \tag{1.21}$$

where we recall that $\lambda_R^{\mathrm{c}}(\varphi)$ is the principal Dirichlet eigenvalue of the operator $\kappa\Delta^c + \varphi$ in the 'continuous' cube Q_R. Obviously, $\lambda_t(\xi)$ is asymptotically not smaller than the expression on the right of (1.21). In terms of the Feynman-Kac formula in (1.5), this lower estimate is obtained by inserting the indicator on the event that the random path moves quickly to the box $\widetilde{B}$ and stays all the time until t in that box.

It is an important technical issue to show that, asymptotically as $t \to \infty$, $\lambda_t(\xi)$ is also estimated from *above* by the right hand side of (1.21), if φ is optimally chosen, i.e., if $\lambda_R^{\mathrm{c}}(\varphi)$ is optimized over all admissible φ and on R. This implies that the almost sure asymptotics of $U(t)$ are given as

$$\frac{1}{t}\log U(t) \approx \lambda_t(\xi) \approx h_t - \frac{1}{\widetilde{\alpha}(t)^2}\widetilde{\chi}, \qquad t \to \infty, \tag{1.22}$$

where $\widetilde{\chi}$ is given in terms of the characteristic variational problem

$$\widetilde{\chi} = \lim_{R\to\infty} \inf_{\varphi\colon Q_R \to \mathbb{R}, I_R(\varphi)<1} \left[-\lambda_R^{\mathrm{c}}(\varphi)\right]. \tag{1.23}$$

This ends the heuristic derivation of the almost sure asymptotics of $U(t)$. Like in the annealed case, there are two terms, which describe the absolute height of the potential in the 'macrobox' B_t, and the shape of the potential in the relevant 'microbox' $\widetilde{B}$, more precisely the spectral properties of $\kappa\Delta + \xi$ in that microbox. The interpretation is that, for R large and φ_* an approximate minimizer in (1.23), the main contribution to $U(t)$ comes from a small box $\widetilde{B}$ in B_t, with radius $R\widetilde{\alpha}(t)$, in which the shifted and rescaled potential $\overline{\xi}_t$ looks like φ_*. The condition $I_R(\varphi_*) < 1$ guarantees the existence of such a box, and $\lambda_R(\varphi_*)$ quantifies the contribution from that box.

Let us remark that the variational formulas in (1.23) and (1.17) are in close connection to each other. In particular, it can be shown that the minimizers of (1.23) are rescaled versions of the minimizers of (1.17). This means that, up to rescaling, the optimal potential shapes in the annealed and in the quenched setting are identical.

1.6 Geometric picture of intermittency

In this section we explain the geometric picture of intermittency, still on a heuristic level.

The heuristics for the total mass of $u(t,\cdot)$ in Section 1.5 makes use of only *one* of the relevant islands $\widetilde{B}$ in which the potential is optimally valued and shaped. In order to describe the entire function $u(t,\cdot)$, one has to take into account a certain (random) number of such islands. Let $n(t)$ denote their number, and let $z_1, z_2, \dots, z_{n(t)} \in B_t$ denote the centers of these relevant microboxes $B_1, B_2, \dots, B_{n(t)}$, whose radii are equal to $R\widetilde{\alpha}(t)$. Then, almost surely,

$$U(t) = \sum_{x\in\mathbb{Z}^d} u(t,x) \approx \sum_{i=1}^{n(t)} \sum_{x\in B_i} u(t,x), \qquad \text{as } t\to\infty, \tag{1.24}$$

i.e., asymptotically the total mass of the random field $u(t,\cdot)$ stems only from the unions of the relevant islands, $B_1,\dots,B_{n(t)}$. These islands are far away from each other. On each of them, the shifted and rescaled potential $\overline{\xi}_t$, see (1.18)), looks approximately like a minimizer φ_* of the variational problem in (1.23). In particular, it has an asymptotically *deterministic* shape. This is the universality in the potential landscape: the height and the (appropriately rescaled) shape of the potential on the relevant islands are deterministic, but their location and number are random.

The shape of the solution, $u(t,\cdot)$, on each of the relevant islands also approaches a universal deterministic shape, namely a time-dependent multiple of the principal eigenfunction of the operator $\kappa\Delta + \varphi_*$.

2. Examples of potentials

2.1 Double-exponential distributions

Consider a distribution which lies in the vicinity of the *double-exponential distribution*, i.e.,

$$\mathrm{Prob}\,(\xi(0) > r) \approx \exp\{-e^{r/\varrho}\}, \qquad r\to\infty, \tag{2.1}$$

with $\varrho\in(0,\infty)$ a parameter. It turns out [GM98] that this class of potentials constitutes a critical class in the sense that the radius of the relevant islands stays finite as $t\to\infty$. This is related to the characteristic property of the double-exponential distribution that

$$\mathrm{Prob}\,(\xi(x) > h) \approx \mathrm{Prob}\,(\xi(y) > h - \varrho\log 2,\, \xi(z) > h-\varrho\log 2),$$

meaning that single-site potential peaks of height $h \gg 1$ occur with the same frequency as two-site potential peaks with height of the same order. Hence, no spatial rescaling is necessary, and we put $\alpha(t) = 1$. In Sections 3.1–3.3 below we shall describe our results for this type of potentials more closely.

For the boundary cases $\varrho=\infty$ and $\varrho = 0$ ('beyond' and 'on this side of' the double-exponential distribution, respectively), [GM98] argued that the boundary cases $\alpha(t)\downarrow 0$ and $\alpha(t)\to\infty$ occur. In other words, the fields beyond the double-exponential (which includes, e.g., Gaussian fields) are simple in the sense that the main contribution comes from islands consisting of single lattice sites. Unbounded fields that are in the vicinity of the case $\varrho=0$ are called 'almost bounded' in [GM98].

2.2 Survival probabilities

The case when the field ξ assumes the values $-\infty$ and 0 only has a nice interpretation in terms of survival probabilities and is therefore of particular importance. The fundamental papers [DV75] and [DV79] by Donsker and Varadhan on the Wiener sausage contain apparently the first substantial annealed results on the asymptotics for the parabolic Anderson model. In the nineties, the thorough and deep work by Sznitman (see his monograph [S98]), pushed the rigorous understanding of the quenched situation much further.

2.2.1 Brownian motion in a Poisson field of traps. We consider the continuous case, i.e., the version of (1.4) with $\mathbb{Z}^d$ replaced by $\mathbb{R}^d$ and the lattice Laplacian replaced by the usual Laplace operator. The field ξ is given as follows. Let $(x_i)_{i\in I}$ be the points of a homogeneous Poisson point process in $\mathbb{R}^d$, and consider the union $\mathcal{O}$ of the balls $B_a(x_i)$ of radius a around the Poisson points x_i. We define a random potential by putting

$$\xi(x) = \begin{cases} 0 & \text{if } x \notin \mathcal{O}, \\ -\infty & \text{if } x \in \mathcal{O}. \end{cases} \tag{2.2}$$

(There are more general versions of this type of potentials, but for simplicity we keep with that.) The set $\mathcal{O}$ receives the meaning of the set of 'hard traps' or 'obstacles'. Let $T_{\mathcal{O}} = \inf\{t > 0 \colon X(t) \in \mathcal{O}\}$ denote the entrance time into $\mathcal{O}$ for a Brownian motion $(X(t))_{t\in[0,\infty)}$. Then we have the Feynman-Kac representation

$$u(t,x) = \mathbb{P}_0\left(T_{\mathcal{O}} > t,\, X(t) \in dx\right)/dx,$$

i.e., $u(t,x)$ is equal to the sub-probability density of $X(t)$ on survival in the Poisson field of traps by time t for Brownian motion starting from the origin. The total mass $U(t) = \mathbb{P}_0(T_{\mathcal{O}} > t)$ is the survival probability by time t. It is easily seen that the first moment of $U(t)$ coincides with a negative exponential moment of the volume of the Wiener sausage $\bigcup_{s\in[0,t]} B_a(X(s))$.

Donsker and Varadhan analyzed the leading asymptotics of $\langle U(t)\rangle$ by using their large deviation principle for Brownian occupation time measures. The relevant islands have radius of order $\alpha(t) = t^{1/(d+2)}$. To handle the quenched asymptotics of $U(t)$, Sznitman developed a coarse-graining scheme for Dirichlet eigenvalues on random subsets of $\mathbb{R}^d$, the so-called method of enlargement of obstacles (MEO). The MEO replaces the eigenvalues in certain complicated subsets of $\mathbb{R}^d$ by those in coarse-grained subsets belonging to a *discrete* class of much smaller combinatorial complexity such that control is kept on the relevant properties of the eigenvalue.

Qualitatively, the considered model falls into the class of bounded from above fields introduced in Section 2.3 with $\gamma = 0$.

Related potentials critically rescaled with time have been studied in particular by van den Berg et al. [BBH01] and by Merkl and Wüthrich [MW02].

2.2.2 Simple random walk among Bernoulli traps. This is the discrete version of Brownian motion among Poisson traps. Consider the i.i.d. field $\xi = (\xi(x))_{x\in\mathbb{Z}^d}$ where $\xi(x)$ takes the values 0 or $-\infty$ only. Again, $u(t,x)$ is the survival probability of continuous-time random walk paths from 0 to x among the set of traps $\mathcal{O} = \{y \in \mathbb{Z}^d \colon \xi(y) = -\infty\}$.

In their paper [DV79], Donsker and Varadhan also investigated the discrete case and described the logarithmic asymptotics of $\langle U(t)\rangle$ by proving and exploiting a large deviation principle for occupation times of random walks. Later Bolthausen [B94] carried out a deeper analysis of $\langle U(t)\rangle$ in the two-dimensional case using refined large deviation arguments. Antal [Ant94], [Ant95] developed a discrete variant of the MEO and demonstrated its value by proving limit theorems for the survival probability $U(t)$ and its moments.

2.3 General fields bounded from above

In [BK01a] and [BK01b], a large class of potentials with $\operatorname{esssup}\xi(0) < \infty$ is considered. Assume for simplicity that $\operatorname{esssup}\xi(0) = 0$ and that the tail of $\xi(0)$ at 0 is given by

$$\operatorname{Prob}\big(\xi(0) > -x\big) \approx \exp\Big\{-Dx^{-\frac{\gamma}{1-\gamma}}\Big\}, \qquad x \downarrow 0, \tag{2.3}$$

with $D > 0$ and $\gamma \in [0, 1)$ two parameters. The case $\gamma = 0$ contains simple random walk among Bernoulli traps as a particular case. The cumulant generating function is roughly $H(t) \approx -\text{const}\, t^\gamma$, and the annealed scale function is $\alpha(t) \approx t^\nu$ where $\nu = (1-\gamma)/(2+d-d\gamma)$. The power ν ranges from 0 to $1/(d+2)$ as the parameter γ ranges from 1 to 0.

It turns out in [BK01a] that the rate function I_R (see (1.15)) is given by

$$I_R(\varphi) = \text{const} \int_{Q_R} |\varphi(x)|^{-\frac{\gamma}{1-\gamma}}\, dx, \tag{2.4}$$

where in the case $\gamma = 0$ we interpret the integral as the Lebesgue measure of the support of φ. The characteristic variational formula for the annealed field shapes in (1.17) has been analyzed in great detail in the case $\gamma = 0$. In particular it was shown that the minimizer is unique and has compact support, and it was characterized in terms of Bessel functions. However, in the general case $\gamma \in (0, 1)$, an analysis of (1.17) has not yet been carried out.

2.4 Gaussian fields and Poisson shot noise

Two important particular cases in the *continuous* version of the parabolic Anderson model are considered in [GK00] and [GKM00] (see also [CM95] for first rough results). The continuous version of (1.4) replaces $\mathbb{Z}^d$ by $\mathbb{R}^d$ and the discrete lattice Laplacian by the usual Laplace operator. Unlike in the discrete case, where any distribution on $\mathbb{R}$ may be used for the definition of an i.i.d. potential, in the continuous case it is not easy to find examples of fields that can be expressed in easily manageable terms. Since a certain degree of regularity of the potential is required, the condition of spatial independence must be dropped.

In [GK00] and [GKM00], two types of fields are considered: a Gaussian field ξ whose covariance function B has a parabolic shape around zero with $B(0) = \sigma^2 > 0$, and a so-called Poisson shot-noise field, which is defined as the superposition of copies of parabolic-shaped *positive* clouds around the points of a homogeneous Poisson point process in $\mathbb{R}^d$ (in contrast to the trap case of Section 2.2). A certain (mild) assumption on the decay of the covariance function (respectively of the cloud) at infinity ensures sufficient independence between regions that are far apart.

Both fields easily develop very high peaks on small islands (the Poisson shot noise field is large where many Poisson points are close together). The annealed scale function is $\alpha(t) = t^{-1/4}$ for the Gaussian field and $\alpha(t) = t^{d/8} e^{-\sigma^2 t/4}$ for the Poisson field [GK00].

3. Results for the double-exponential case

In this section, we formulate our results on the large-time asymptotics of the parabolic Anderson model in the particularly important case of a double-exponentially distributed random potential, see Section 2.1. We handle the annealed asymptotics of the total mass $U(t)$ in Section 3.1, the quenched ones in Section 3.2, and the geometric picture of intermittency in Section 3.3. The material of the first two subsections is taken from [GM98], that of the last subsection from [GKM04].

3.1 Annealed asymptotics

As before, we assume that $\xi = (\xi(x))_{x\in\mathbb{Z}^d}$ is a field of i.i.d. random variables. We impose the following assumption on the cumulant generating function of $\xi(0)$ defined by (1.9).

Assumption (H). *The function $H(t)$ is finite for all $t > 0$. There exists $\varrho \in [0, \infty]$ such that*

$$\lim_{t\to\infty} \frac{H(ct) - cH(t)}{t} = \varrho c \log c \qquad \textit{for all } c \in (0, 1).$$

Note that the vicinity of the double-exponential distribution (2.1) corresponds to $\varrho \in (0,\infty)$. If $\varrho = \infty$, then the upper tail of the distribution of $\xi(0)$ is heavier than in the double exponential case, whereas for $\varrho = 0$ it is thinner.

Let $\mathcal{P}(\mathbb{Z}^d)$ denote the space of probability measures on $\mathbb{Z}^d$. We introduce the Donsker-Varadhan functional S_d and the entropy functional I_d on $\mathcal{P}(\mathbb{Z}^d)$ by

$$S_d(\mu) = \sum_{\substack{\{x,y\}\subset\mathbb{Z}^d\\ |x-y|=1}} \left(\sqrt{\mu(x)} - \sqrt{\mu(y)}\right)^2 \quad \text{and} \quad I_d(\mu) = -\sum_{x\in\mathbb{Z}^d} \mu(x)\log\mu(x),$$

respectively, and set

$$\chi_d = \inf_{\mu\in\mathcal{P}(\mathbb{Z}^d)} \left[\kappa S_d(\mu) + \varrho I_d(\mu)\right], \qquad \varrho\in[0,\infty]. \tag{3.1}$$

As before, let $U(t)$ denote the total mass of the solution $u(t,\cdot)$ to the PAM (1.4).

Theorem 3.1. *Let Assumption (H) be satisfied. Then, for any $p \in \mathbb{N}$,*

$$\langle U(t)^p\rangle = \exp\{H(pt) - \chi_d pt + o(t)\} \qquad \text{as } t\to\infty.$$

It turns out that $\chi_d = 2d\kappa\chi_0(\varrho/\kappa)$, where $\chi_0\colon [0,\infty)\to[0,1)$ is strictly increasing and concave, $\chi_0(0) = 0$, and $\chi_0(\varrho)\to 1$ as $\varrho\to\infty$. Moreover, for $\varrho\in(0,\infty)$, each minimizer μ of the variational problem (3.1) has the form $\mu = \text{const}\, v^2$, where $v = v_1\otimes\cdots\otimes v_d$ and each of the factor $v_1,\dots,v_d$ is a positive solution of the equation

$$\kappa\Delta v + 2\varrho v\log v = 0 \qquad \text{on } \mathbb{Z}$$

with *minimal* ℓ^2-norm. Uniqueness of v modulo shifts holds for large ϱ/κ but is open for small values of this quantity.

Recall that $U(t) = \hat u(t,0)$, where $\hat u$ is the solution to (1.4), but with non-localized initial datum $\mathbb{1}$ instead of δ_0. A much deeper question is the computation of the asymptotics of the 'correlation'

$$c(t,x) = \frac{\langle \hat u(t,0)\hat u(t,x)\rangle}{\langle \hat u(t,0)^2\rangle}$$

of the spatially homogeneous solution $\hat u$ of the PAM. Assuming additional regularity of the cumulant generating function H and uniqueness modulo spatial shifts of the minimizer μ of the variational problem (3.1), it was shown in [GH99] that

$$\lim_{t\to\infty} c(t,x) = \frac{\sum_z v(z)v(z+x)}{\sum_z v(z)^2}.$$

This indicates that the second moment (considered as $\lim_{R\to\infty}|B_R|^{-1}\sum_{x\in B_R}\hat u^2(t,x)$) is generated by rare high peaks of the solution $\hat u(t,\cdot)$ *with shape* $v(\cdot)$.

3.2 Quenched asymptotics

Here we again consider i.i.d. random potentials $(\xi(x))_{x\in\mathbb{Z}^d}$ in the vicinity of the double-exponential distribution (2.1) but formulate our assumptions in a different manner.

To be precise, let F denote the distribution function of $\xi(0)$. Provided that F is continuous and $F(r) < \infty$ for all $r\in\mathbb{R}$ (i.e., ξ is unbounded from above), we may introduce the non-decreasing function

$$\varphi(r) = \log\frac{1}{1-F(r)}, \qquad r\in\mathbb{R}. \tag{3.2}$$

Its left-continuous inverse ψ is given by

$$\psi(s) = \min\{r \in \mathbb{R} \colon \varphi(r) \geq s\}, \qquad s > 0. \tag{3.3}$$

Note that ψ is strictly increasing with $\varphi(\psi(s)) = s$ for all $s > 0$. The relevance of ψ comes from the observation that ξ has the same distribution as $\psi \circ \eta$, where $\eta = (\eta(x))_{x \in \mathbb{Z}^d}$ is an i.i.d. field of exponentially distributed random variables with mean one.

We now formulate our main assumption.

Assumption (F). *The distribution function F is continuous, $F(r) < 1$ for all $r \in \mathbb{R}$, and, in dimension $d = 1$, $\int_{-\infty}^{-1} \log|r|\, F(\mathrm{d}r) < \infty$. There exists $\varrho \in (0, \infty]$ such that*

$$\lim_{s \to \infty} [\psi(cs) - \psi(s)] = \varrho \log c, \qquad c \in (0,1). \tag{3.4}$$

If $\varrho = \infty$, then ψ satisfies in addition

$$\lim_{s \to \infty} [\psi(s + \log s) - \psi(s)] = 0. \tag{3.5}$$

The crucial supposition (3.4) specifies that the upper tail of the distribution of $\xi(0)$ is close to the double-exponential distribution (2.1) for $\varrho \in (0, \infty)$ and is heavier for $\varrho = \infty$. Assumption (3.5) excludes too heavy tails. Note that (3.5) is fulfilled for Gaussian but not for exponential tails.

The reader easily checks that (3.4) implies that $\psi(t) \sim \varrho \log t$ as $t \to \infty$. Let

$$h_t = \max_{x \in B_t} \xi(x), \qquad t > 0, \tag{3.6}$$

be the height of the potential ξ in $B_t = [-t, t]^d \cap \mathbb{Z}^d$. It can be easily seen that, under Assumption (F), almost surely,

$$h_t = \psi(d \log t) + o(1) \qquad \text{as } t \to \infty. \tag{3.7}$$

Let us remark that it is condition (3.5) which ensures that the almost sure asymptotics of h_t in (3.7) is non-random up to order $o(1)$.

One of the main results in [GM98], Theorem 2.2, is the second order asymptotics of the total mass $U(t)$ defined in (1.6).

Theorem 3.2. *Under Assumption (F), with probability one,*

$$\log U(t) = t\,[h_t - \widetilde{\chi}_d + o(1)] \qquad \textit{as } t \to \infty. \tag{3.8}$$

Here $0 \leq \widetilde{\chi}_d \leq 2d\kappa$. An analytic description of $\widetilde{\chi}_d$ is as follows. Define $I \colon [-\infty, 0]^{\mathbb{Z}^d} \to [0, \infty]$ by

$$I(V) = \begin{cases} \sum_{x \in \mathbb{Z}^d} e^{V(x)/\varrho}, & \text{if } \varrho \in (0, \infty), \\ |\{x \in \mathbb{Z}^d \colon V(x) > -\infty\}|, & \text{if } \varrho = \infty. \end{cases} \tag{3.9}$$

One should regard I as *large deviation rate function* for the fields $\xi - h_t$ (recall (1.15) and note that $\alpha(t) = 1$ here). Indeed, if the distribution of ξ is exactly given by (2.1), then we have

$$\text{Prob}\left(\xi(\cdot) - h > V(\cdot) \text{ in } \mathbb{Z}^d\right) = \exp\left\{-e^{h/\varrho} I(V)\right\}$$

for any $V \colon \mathbb{Z}^d \to [-\infty, 0]$ and any $h \in (0, \infty)$. For $V \in [-\infty, 0]^{\mathbb{Z}^d}$, let $\lambda(V) \in [-\infty, 0]$ be the top of the spectrum of the self-adjoint operator $\kappa\Delta + V$ in the domain $\{V > -\infty\}$ with zero boundary condition. In terms of the Rayleigh-Ritz formula,

$$\lambda(V) = \sup_{f \in \ell^2(\mathbb{Z}^d) \colon \|f\|_2 = 1} \big\langle (\kappa\Delta + V) f, f \big\rangle, \tag{3.10}$$

where $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_2$ denote the inner product and the norm in $\ell^2(\mathbb{Z}^d)$, respectively. Then

$$-\widetilde{\chi}_d = \sup\Big\{\lambda(V)\colon V\in[-\infty,0]^{\mathbb{Z}^d},\, I(V)\le 1\Big\}. \tag{3.11}$$

This variational problem is 'dual' to the variational problem (3.1) and, in particular, $\widetilde{\chi}_d=\chi_d$.

3.3 Geometry of intermittency

In this section we give a precise formulation of the geometric picture of intermittency, which was heuristically explained in Section 1.6.

We keep the assumptions of the last subsection. For our deeper investigations, in addition to Assumption (F), we introduce an assumption about the *optimal potential shape.*

Assumption (M). *Up to spatial shifts, the variational problem in* (3.11) *possesses a unique maximizer, which has a unique maximum.*

By V_* we denote the unique maximizer of (3.11) which attains its unique maximum at the origin. We will call V_* *optimal potential shape.* Assumption (M) is satisfied at least for large ϱ/κ. This fact as well as further important properties of the variational problem (3.11) are stated in the next proposition.

Proposition 3.3. *(a) For any $\varrho\in(0,\infty]$, the supremum in* (3.11) *is attained.*

(b) *If ϱ/κ is sufficiently large, then the maximizer in* (3.11) *is unique modulo shifts and has a unique maximum.*

(c) *If Assumption (M) is satisfied, then the optimal potential shape has the following properties.*
   (i) *If $\varrho\in(0,\infty)$, then $V_*=f_*\otimes\cdots\otimes f_*$ for some $f_*\colon\mathbb{Z}\to(-\infty,0)$. If $\varrho=\infty$, then V_* is degenerate in the sense that $V_*(0)=0$ and $V_*(x)=-\infty$ for $x\neq 0$.*
   (ii) *The operator $\kappa\Delta+V_*$ has a unique nonnegative eigenfunction $w_*\in\ell^2(\mathbb{Z}^d)$ with $w_*(0)=1$ corresponding to the eigenvalue $\lambda(V_*)$. Moreover, $w_*\in\ell^1(\mathbb{Z}^d)$. If $\varrho\in(0,\infty)$, then w_* is positive on $\mathbb{Z}^d$, while $w_*=\delta_0$ for $\varrho=\infty$.*

We shall see that the main contribution to the total mass $U(t)$ comes from a neighborhood of the set of best local coincidences of $\xi-h_t$ with spatial shifts of V_*. These neighborhoods are widely separated from each other and hence not numerous. We may restrict ourselves further to those neighborhoods in which, in addition, $u(t,\cdot)$, properly normalized, is close to w_*.

Denote by $B_R(y)=y+B_R$ the closed box of radius R centered at $y\in\mathbb{Z}^d$ and write

$$B_R(A)=\bigcup_{y\in A}B_R(y) \tag{3.12}$$

for the 'R-box neighborhood' of a set $A\subset\mathbb{Z}^d$. In particular, $B_0(A)=A$.

For any $\varepsilon>0$ and any sufficiently large $\varrho\in(0,\infty]$, let $r(\varepsilon,\varrho)$ denote the smallest $r\in\mathbb{N}_0$ such that

$$\|w_*\|_2^2\sum_{x\in\mathbb{Z}^d\setminus B_r}w_*(x)<\varepsilon. \tag{3.13}$$

Note that $r(\varepsilon,\infty)=0$, due to the degeneracy of w_∞. Given $f\colon\mathbb{Z}^d\to\mathbb{R}$ and $R>0$, let $\|f\|_R=\sup_{x\in B_R}|f(x)|$.

The main result of [GKM04] is the following.

Theorem 3.4. *Let the Assumptions (F) and (M) be satisfied. Then there exists a random time-dependent subset* $\Gamma^* = \Gamma^*_{t\log^2 t}$ *of* $B_{t\log^2 t}$ *such that, almost surely,*

$$(i) \qquad \liminf_{t\to\infty} \frac{1}{U(t)} \sum_{x\in B_{r(\varepsilon,\varrho)}(\Gamma^*)} u(t,x) \geq 1-\varepsilon, \qquad \varepsilon\in(0,1); \tag{3.14}$$

$$(ii) \qquad |\Gamma^*| \leq t^{o(1)} \quad \text{and} \quad \min_{y,\widetilde{y}\in\Gamma^*\colon y\neq\widetilde{y}} |y-\widetilde{y}| \geq t^{1-o(1)} \qquad \text{as } t\to\infty; \tag{3.15}$$

$$(iii) \qquad \lim_{t\to\infty} \max_{y\in\Gamma^*} \big\|\xi(y+\cdot) - h_t - V_*(\cdot)\big\|_R = 0, \qquad R>0; \tag{3.16}$$

$$(iv) \qquad \lim_{t\to\infty} \max_{y\in\Gamma^*} \Big\|\frac{u(t,y+\cdot)}{u(t,y)} - w_*(\cdot)\Big\|_R = 0, \qquad R>0. \tag{3.17}$$

Theorem 3.4 states that, up to an arbitrarily small relative error ε, the islands with centers in Γ^* and radius $r(\varepsilon,\varrho)$ carry the whole mass of the solution $u(t,\cdot)$. Locally, in an arbitrarily fixed R-neighborhood of each of these centers, the shapes of the potential and the normalized solution resemble h_t+V_* and w_*, respectively. The number of these islands increases at most as an arbitrarily small power of t and their distance increases almost like t. Note that, for $\varrho=\infty$, the set $B_{r(\varepsilon,\varrho)}(\Gamma^*)$ in (3.14) is equal to Γ^* and, hence, the islands consist of single lattice sites.

It is an open problem under what assumptions on the potential ξ the number $|\Gamma^*|$ of relevant peaks stays bounded as $t\to\infty$; we have made no attempt to choose Γ^* as small as possible.

4. Universality

In this section we explain that, under some mild regularity assumptions on the tails of $\xi(0)$ at its essential supremum, there are only four universality classes of asymptotic behaviors of the parabolic Anderson model. Three of them have already been analyzed in the literature: the double-exponential distribution with $\varrho\in(0,\infty)$ respectively $\varrho=\infty$ [GM98], [GH99], [GK00], [GKM00], [GKM04]) (see Sections 2.1 and 3) and general fields bounded from above [BK01a], [BK01b], [S98], [Ant94], [Ant95] (see Section 2.3). A fourth and new universality class is currently under investigation, see [HKM04]. This class lies in the union of the boundary cases $\varrho=0$ of the double-exponential distribution ('almost bounded' fields) and $\gamma=1$ for the general bounded from above fields. Examples of distributions that fall into this class look somewhat odd, but it turns out that the optimal potential shape and the optimal shape of the solution are perfectly parabolic, respectively Gaussian, which makes this class rather appealing. In particular, the appearing variational formulas can be easily solved explicitly and uniquely.

We now summarize [HKM04]. Our basic assumption on the logarithmic moment generating function H in (1.9) is the following.

Assumption $(\hat{\mathbf{H}})$: *There are a function* $\widehat{H}\colon(0,\infty)\to\mathbb{R}$ *and a continuous auxiliary function* $\eta\colon(0,\infty)\to(0,\infty)$ *such that*

$$\lim_{t\uparrow\infty} \frac{H(ty)-yH(t)}{\eta(t)} = \widehat{H}(y) \neq 0 \qquad \text{for } y\neq 1. \tag{4.1}$$

The function $\widehat{H}$ extracts the asymptotic scaling properties of the cumulant generating function H. In the language of the theory of regular functions, the assumption is that the logarithmic moment generating function H is in the de Haan class, which does not leave many possibilities for $\widehat{H}$:

Proposition 4.1. *Suppose that Assumption $(\hat{H})$ holds.*

(i) *There is a* $\gamma \geq 0$ *such that* $\lim_{t\uparrow\infty} \eta(yt)/\eta(t) = y^\gamma$ *for any* $y > 0$*, i.e.,* η *is regularly varying of index* γ*. In particular,* $\eta(t) = t^{\gamma+o(1)}$ *as* $t \to \infty$*.*

(ii) *There exists a parameter* $\rho > 0$ *such that, for every* $y > 0$*,*

(a) $\widehat{H}(y) = \rho \dfrac{y - y^\gamma}{1-\gamma}$ *if* $\gamma \neq 1$*,*

(b) $\widehat{H}(y) = \rho y \log y$ *if* $\gamma = 1$*.*

Our second regularity assumption is a mild supposition on the auxiliary function η. This assumption is necessary only in the case $\gamma = 1$ (which will turn out to be the critical case).

Assumption (K): *The limit* $\eta_* = \lim_{t\to\infty} \eta(t)/t \in [0,\infty]$ *exists.*

We now introduce a scale function $\alpha\colon (0,\infty) \to (0,\infty)$, by

$$\frac{\eta\big(t\alpha(t)^{-d}\big)}{t\alpha(t)^{-d}} = \frac{1}{\alpha(t)^2}\,. \tag{4.2}$$

The function $\alpha(t)$ turns out to be the annealed scale function for the radius of the relevant islands in the parabolic Anderson model. We can easily say something about the asymptotics of $\alpha(t)$:

Lemma 4.2. *Suppose that Assumptions* ($\hat{H}$) *and (K) hold. If* $\gamma \leq 1$ *and* $\eta_* < \infty$*, then there exists a unique solution* $\alpha\colon (0,\infty) \to (0,\infty)$ *to* (4.2)*, and it satisfies* $\lim_{t\to\infty} t\alpha(t)^{-d} = \infty$*. Moreover,*

(i) *If* $\gamma = 1$ *and* $0 < \eta_* < \infty$*, then* $\lim_{t\to\infty} \alpha(t) = 1/\sqrt{\eta_*} \in (0,\infty)$*.*

(ii) *If* $\gamma = 1$ *and* $\eta_* = 0$*, then* $\alpha(t) = t^{\nu+o(1)}$ *as* $t \to \infty$*, where* $\nu = (1-\gamma)/(d+2-d\gamma)$*.*

Now, under Assumptions ($\hat{H}$) and (K), we can formulate a complete distinction of the PAM into four cases:

(1) $\eta_* = \infty$ (in particular, $\gamma \geq 1$).
This is the boundary case $\varrho = \infty$ of the double-exponential case. We have $\alpha(t) \to 0$ as $t \to \infty$, as is seen from (4.2), i.e., the relevant islands consist of single lattice sites.

(2) $\eta_* \in (0,\infty)$ (in particular, $\gamma = 1$).
This is the case of the double-exponential distribution in Section 2.1. By rescaling, one can achieve that $\eta_* = 1$. The parameter ϱ of Proposition 4.1(ii)(b) is identical to the one in Assumption (H) of Section 3.1.

(3) $\eta_* = 0$ and $\gamma = 1$.
This is the case of islands of slowly growing size, i.e., $\alpha(t) \to \infty$ as $t \to \infty$ slower than any power of t. This case comprises 'almost bounded' and bounded from above potentials. This class is the subject of [HKM04], see also below.

(4) $\gamma < 1$ (in particular, $\eta_* = 0$)
This is the case of islands of rapidly growing size, i.e., $\alpha(t) \to \infty$ as $t \to \infty$ at least as fast as some power of t. Here the potential ξ is necessarily bounded from above. This case was treated in [BK01a]; see Section 2.3.

Let us comment on the class (3), which appears to be new in the literature and is under investigation in [HKM04]. One obtains examples of potentials (unbounded from above) that fall into this class by replacing ϱ in (2.1) by a sufficiently regular function $\varrho(r)$ that tends to 0 as $r \to \infty$, and other examples (bounded from above) by replacing γ in (2.3) by a sufficiently regular function $\gamma(x)$ tending to 1 as $x \downarrow 0$. According to Lemma 4.2, the scale function $\alpha(t)$ defined in (4.2) tends to infinity, but is slowly varying. The annealed rate function for the rescaled potential shape, I_R, introduced in (1.15) turns out to be

$$I_R(\varphi) = \text{const} \int_{Q_R} e^{\varphi(x)/\varrho}\,dx. \tag{4.3}$$

The characteristic variational problem for the annealed potential shape in (1.17), χ, turns out to be uniquely minimized by a parabolic function $\varphi_*(x) = \mathrm{const} - \varrho\|x\|_2^2$, and the principal eigenfunction v_* of the operator $\kappa\Delta + \varphi_*$ is the Gaussian density $v_*(x) = \mathrm{const}\, e^{-\varrho\|x\|_2^2}$.

5. Time-dependent random potentials

In this section we study the intermittent behavior of the parabolic Anderson model (PAM) with a space-time homogeneous ergodic random potential ξ:

$$\begin{aligned} \partial_t u(t,x) &= \kappa\Delta u(t,x) + [\xi(t,x) - \langle \xi(t,x)\rangle]\, u(t,x), && (t,x) \in (0,\infty)\times\mathbb{Z}^d,\\ u(0,x) &= 1, && x \in \mathbb{Z}^d. \end{aligned} \tag{5.1}$$

Note that for time-dependent potentials the direct connection to the spectral representation of the Anderson Hamiltonian (1.2) is lost. Our focus will be on the situation when the potential ξ is given by a field $\{Y_k(t); k \in \mathbb{N}\}$ of independent random walks on $\mathbb{Z}^d$ with diffusion constant ϱ in Poisson equilibrium with density ν:

$$\xi(t,x) = \gamma \sum_{k\in\mathbb{N}} \delta_{Y_k(t)}(x), \tag{5.2}$$

where γ denotes a positive coupling constant. Clearly

$$\langle \xi(t,x)\rangle = \nu\gamma.$$

This deterministic correction to the potential in (5.1) has been added for convenience to eliminate non-random terms.

The form (5.2) of the potential is motivated by the following particle model. Consider a system of two types of independent particles, A and B, performing independent continuous-time simple random walks with diffusion constants κ and ϱ and Poisson initial distribution with densities ν and 1, respectively. Assume that the B-particles split into two at a rate that is γ times the number of A-particles present at the same location and die at rate $\nu\gamma$. Hence, the A- and B-particles may be regarded as catalysts and reactants in a simple catalytic reaction model. Then the (spatially homogeneous and ergodic) solution $u(t,x)$ of the PAM (5.1) is nothing but the average number of reactants at site x at time t given a realization of the catalytic dynamics (5.2). Such a particle model (with arbitrary death rate) has been considered by Kesten and Sidoravicius [KS03]. We will come back to the results in [KS03] at the end of this section. We further refer to the overview papers by Dawson and Fleischmann [DF00] and by Klenke [Kl00] for continuum models with singular catalysts in a measure-valued context where questions different from ours have been addressed.

Our aim is to study the moment Lyapunov exponents

$$\lambda_p = \lim_{t\to\infty} \frac{1}{t} \log \langle u(t,0)^p\rangle$$

as well as the quantities

$$\lambda_p^* = \lim_{t\to\infty} \frac{1}{t} \log\log \langle u(t,0)^p\rangle$$

($p = 1, 2, \dots$) as functions of the model parameters. The phase diagram will turn out to be different in dimensions $d = 1, 2$, $d = 3$, and $d \geq 4$.

Definition 5.1. a) For $p \in \mathbb{N}$, we will say that the PAM (5.1)–(5.2) is *strongly p-catalytic* if $\lambda_p^* > 0$. Otherwise the PAM will be called *weakly p-catalytic*.

b) For $p \in \mathbb{N}\setminus\{1\}$ and $\lambda_p < \infty$, we will say that the PAM is *p-intermittent* if $\lambda_p/p > \lambda_{p-1}/(p-1)$.

We believe that strongly catalytic behavior is related to heavy tails of the Poisson distribution of the catalytic point process $\{Y_k(t); k \in \mathbb{N}\}$ and may occur if the main contribution to the p-th moment comes from realizations with a huge number of catalysts at the same lattice site. Recall that p-intermittency means that, for large t, the p-th moment, considered as $\lim_{R\to\infty} |B_R|^{-1} \sum_{x\in B_R} u(t,x)^p$, is 'generated' by high peaks of the solution $u(t,\cdot)$ located far from each other.

For potentials of the form
$$\xi(t,x) = \gamma \dot{W}_x(t)$$
with $(W_x(t))_{x\in\mathbb{Z}^d}$ being a field of i.i.d. (or correlated) Brownian motions and (5.1) understood as a system of Itô equations, the moment Lyapunov exponents have been shown by Carmona and Molchanov [CM94] to exhibit the following behavior. In dimensions $d = 1, 2$ there is p-intermittency for all $p \in \mathbb{N} \setminus \{1\}$ and all choices of the model parameters κ and γ. If $d \geq 3$, then there exist critical points $0 < c_2 < c_3 < \cdots$ such that p-intermittency holds if and only if $\kappa/\gamma < c_p$. For this model, the asymptotics of the almost sure ('quenched') Lyapunov exponent as $\kappa \to 0$ has been investigated in [CM94], [CMV96], [CKM01], and [CMS02].

In the following we present the results for catalytic potentials of the form (5.2) obtained in [GH04].

Our analysis of the moment Lyapunov exponents is based on the following probabilistic representation of the p-th moment which is easily derived from the Feynman-Kac formula for $u(t,0)$:
$$\langle u(t,0)^p \rangle = \mathbb{E}_0^{(p)} \exp\left\{ \nu\gamma \int_0^t \sum_{i=1}^p w(s, X_i(s))\, ds \right\}, \tag{5.3}$$
where $\mathbb{E}_0^{(p)}$ denotes expectation with respect to p independent random walks $X_1, \dots, X_p$ on $\mathbb{Z}^d$ with generator $\kappa\Delta$ starting at the origin, and w is the solution of the random initial value problem
$$\begin{aligned} \partial_t\, w(t,x) &= \varrho\Delta w(t,x) + \gamma \textstyle\sum_{i=1}^p \delta_{X_i(t)}(x)\,(1 + w(t,x))\,, && (t,x) \in (0,\infty)\times\mathbb{Z}^d,\\ w(0,x) &= 0, && x \in \mathbb{Z}^d. \end{aligned}$$
One of the main difficulties in analyzing (5.3) is related to the observation that $w(t, X_i(t))$ depends in a nontrivial way *on the whole past* $\{X_j(s); 0 \leq s \leq t\}$, $j = 1, \dots, p$, of our random walks (although our notation does not reflect this).

For $r \geq 0$, let $\mu(r)$ denote the upper boundary of the spectrum of the operator $\Delta + r\delta_0$ in $\ell^2(\mathbb{Z}^d)$. It is well-known that, in dimensions $d = 1, 2$, $\mu(r) > 0$ for all r and, in dimensions $d \geq 3$, $\mu(r) = 0$ for $0 \leq r \leq r_d$ and $\mu(r) > 0$ for $r > r_d$, where
$$r_d = 1/G_d(0)$$
and G_d denotes the Green function associated with the discrete Laplacian Δ.

Theorem 5.2. *For any choice of the parameters of the PAM (5.1)–(5.2), the limit λ_p^* exists, and*
$$\lambda_p^* = \varrho\, \mu(p\gamma/\varrho), \qquad p \in \mathbb{N}.$$

Hence, for $d = 1, 2$, the PAM (5.1)–(5.2) is always strongly p-catalytic, whereas for $d \geq 3$ this is true only if $p\gamma/\varrho$ exceeds the critical threshold r_d. Note that λ_p^* does not depend on κ nor on ν.

We next study the behavior of the moment Lyapunov exponents $\lambda_p = \lambda_p(\kappa)$ as a function of the diffusion constant κ in the weakly catalytic regime $0 < p\gamma/\varrho < r_d$ for $d \geq 3$. We will mainly describe their behavior for small and for large values of κ.

Theorem 5.3. *Let $d \geq 3$, $p \in \mathbb{N}$, and $0 < p\gamma/\varrho < r_d$. Then the limit λ_p exists and is finite for all κ and ν. Moreover, $\kappa \mapsto \lambda_p(\kappa)$ is strictly decreasing and convex on $[0,\infty)$ and satisfies*
$$\lim_{\kappa\downarrow 0} \frac{\lambda_p(\kappa)}{p} = \frac{\lambda_p(0)}{p} = \nu\gamma \frac{p\gamma/\varrho}{r_d - p\gamma/\varrho}.$$

Hence, in any dimension $d \geq 3$, the PAM (5.1)–(5.2) is p-intermittent in the weakly catalytic regime for small values of the diffusion constant κ.

To formulate the behavior of the moment Lyapunov exponents for $\kappa \to \infty$, we introduce the variational expression $\mathcal{P}$ for the polaron problem analyzed in [L77], [DV83], and [BDS93]:

$$\mathcal{P} = \sup_{\substack{f \in C_c^\infty(\mathbb{R}^3) \\ \|f\|_2 = 1}} \left[\left\| (-\Delta)^{-1/2} f^2 \right\|_2^2 - \|\nabla f\|_2^2 \right].$$

Theorem 5.4. *Let $d \geq 3$, $p \in \mathbb{N}$, and $0 < p\gamma/\varrho < r_d$. Then*

$$\lim_{\kappa \to \infty} \kappa \frac{\lambda_p(\kappa)}{p} = \begin{cases} \frac{\nu\gamma^2}{r_3} + \sqrt{p}\sqrt{\frac{\nu\gamma^2}{\varrho}\mathcal{P}}, & \text{if } d = 3, \\ \frac{\nu\gamma^2}{r_d}, & \text{if } d \geq 4. \end{cases}$$

In other words, in dimensions $d = 3$ for p in the weakly catalytic regime, the PAM (5.1)–(5.2) is p-intermittent also for large values of κ. We conjecture intermittent behavior for all κ. In dimensions $d \geq 4$, the leading term in the asymptotics of $\lambda_p(\kappa)/p$ as $\kappa \to \infty$ is the same for all p in the weakly catalytic regime. We in fact conjecture that there is even no intermittency in high dimensions for large κ.

Let us remark that in [KS03] Kesten and Sidoravicius obtained the following results for the above mentioned catalytic particle model with arbitrary death rate δ instead of $\nu\gamma$. If $d = 1, 2$, then for all choices of the parameters, the average number of B-particles per site tends to infinity faster than exponential. If $d \geq 3$, γ small enough and δ large enough, then the average number of B-particles per site tends to zero exponentially fast. The first of these two results is covered by Theorem 5.2, since the death rate δ does not affect λ_1^*. The second result is covered by Theorem 5.3, since $0 < \lambda_1 < \infty$ in the weakly catalytic regime and the death rate δ shifts λ_1 by $\gamma\nu - \delta$. Kesten and Sidoravicius further show that in all dimensions for γ large enough, conditioned on the evolution of the A-particles, there is a phase transition. Namely, for small δ the B-particles locally survive, while for large δ they become locally extinct. In [GH04] there are no results for the quenched situation. The analysis in [KS03] does not lead to an identification of Lyapunov exponents, but is more robust under an adaption of the model than the above analysis based on the Feynman-Kac representation.

References

[And58] P.W. Anderson, Absence of diffusion in certain random lattices. *Phys. Rev.* **109**, 1492–1505, 1958.

[Ant94] P. Antal, *Trapping Problems for the Simple Random Walk,* Dissertation ETH Zürich, No. 10759, 1994.

[Ant95] P. Antal, Enlargement of obstacles for the simple random walk. *Ann. Probab.* **23**, 3, 1061–1101, 1995.

[AM93] M. Aizenman and S. Molchanov, Localization at large disorder and extreme energies: an elementary derivation. *Comm. Math. Phys.* **157**, 2, 245–278, 1993.

[BBH01] M. van den Berg, E. Bolthausen and F. den Hollander, Moderate deviations for the volume of the Wiener sausage. *Ann. of Math. (2)* **153**, 2, 355–406, 2001.

[BK01a] M. Biskup and W. König, Long-time tails in the parabolic Anderson model with bounded potential. *Ann. Probab.* **29**, 2, 636-682, 2001.

[BK01b] M. Biskup and W. König, Screening effect due to heavy lower tails in one-dimensional parabolic Anderson model. *J. Stat. Phys.* **102**, 5/6, 1253-1270, 2001.

[BKS04] S. Böcker, W. Kirsch and P. Stollmann, Spectral theory for nonstationary random potentials. In this proceedings volume.

[B94] E. Bolthausen, Localization of a two-dimensional random walk with an attractive path interaction. *Ann. Probab.* **22**, 875–918, 1994.

[BDS93] E. Bolthausen, J.-D. Deuschel and U. Schmock, Convergence of path measures arising from a mean field or polaron type interaction. *Probab. Theory Relat. Fields* **95**, 283–310, 1993.

[CKM01] R.A. Carmona, L. Koralov and S.A. Molchanov, Asymptotics for the almost-sure Lyapunov exponent for the solution of the parabolic Anderson problem. *Random Oper. Stochastic Equations* **9**, 77–86, 2001.

[CM94] R. Carmona, S.A. Molchanov, Parabolic Anderson problem and intermittency. *Memoirs of the AMS* **108**, nr. 518, 1994.

[CM95] R.A. Carmona and S.A. Molchanov, Stationary parabolic Anderson model and intermittency. *Probab. Theory Relat. Fields* **102**, 433–453, 1995.

[CMV96] R.A. Carmona, S.A. Molchanov and F. Viens, Sharp upper bound on the almost-sure exponential behavior of a stochastic partial differential equation. *Random Oper. Stochastic Equations* **4**, 43–49, 1996.

[CMS02] M. Cranston, T.S. Mountford and T. Shiga, Lyapunov exponents for the parabolic Anderson model. *Acta Math. Univ. Comeniane* **LXXI**, 163–188, 2002.

[DF00] D.A. Dawson and K. Fleischmann, Catalytic and mutually catalytic branching, *in:* Infinite dimensional stochastic analysis, 145–170, Ph. Clément, F. den Hollander, J. van Neerven and B. de Pagter, eds., R. Neth. Acad. Sci., Amsterdam 2000.

[DV75] M. Donsker and S.R.S. Varadhan, Asymptotics for the Wiener sausage. *Comm. Pure Appl. Math.* **28,** 525–565, 1975.

[DV79] M. Donsker and S.R.S. Varadhan, On the number of distinct sites visited by a random walk. *Comm. Pure Appl. Math.* **32**, 721–747, 1979.

[DV83] M. Donsker and S.R.S. Varadhan, Asymptotics for the polaron. *Comm. Pure Appl. Math.* **36**, 505–528, 1983.

[EEEF84] W. Ebeling, A. Engel, B. Esser and R. Feistel, Diffusion and reaction in random media and models of evolution processes. *J. Stat. Phys.* **37**, 369–385, 1984.

[FMSS85] J. Fröhlich, F. Martinelli, E. Scoppola and T. Spencer, Constructive proof of localization in the Anderson tight binding model. *Comm. Math. Phys.* **101**, 21–46, 1985.

[GH99] J. Gärtner and F. den Hollander, Correlation structure of intermittency in the parabolic Anderson model. *Probab. Theory Relat. Fields* **114**, 1–54, 1999.

[GH04] J. Gärtner and F. den Hollander, Intermittency in a catalytic random medium. In preparation.

[GK00] J. Gärtner and W. König, Moment asymptotics for the continuous parabolic Anderson model. *Ann. Appl. Probab.* **10**, 3, 192–217, 2000.

[GKM00] J. Gärtner, W. König and S. Molchanov, Almost sure asymptotics for the continuous parabolic Anderson model. *Probab. Theory Relat. Fields* **118**, 4, 547-573, 2000.

[GKM04] J. Gärtner, W. König and S. Molchanov, Geometric characterization of intermittency in the parabolic Anderson model. In preparation.

[GM90] J. Gärtner and S. Molchanov, Parabolic problems for the Anderson model. I. Intermittency and related topics. *Commun. Math. Phys.* **132**, 613–655, 1990.

[GM98] J. Gärtner and S. Molchanov, Parabolic problems for the Anderson model. II. Second-order asymptotics and structure of high peaks. *Probab. Theory Relat. Fields* **111**, 17–55, 1998.

[GM00] J. Gärtner and S. Molchanov, Moment asymptotics and Lifshitz tails for the parabolic Anderson model. Canadian Math. Soc. Conference Proceedings 26, Luis G. Gorostiza and B. Gail Ivanoff, eds., AMS, Providence, 141–157, 2000.

[HKM04] R. van der Hofstad, W. König and P. Mörters, Universality for the parabolic Anderson model. In preparation.

[HW94] F. den Hollander and G.H. Weiss, Aspects of trapping in transport processes, *in:* Contemporary problems in Statistical Physics, G.H. Weiss, ed., SIAM, Philadelphia, 1994.

[KS03] H. Kesten and V. Sidoravicius, Branching random walk with catalysts, *Electr. J. Prob.* **8**, Paper no. 5, 1–51, 2003.

[Kl00] A. Klenke, A review on spatial catalytic branching, *in:* Stochastic models, 245–263, CMS Conf. Proc. 26, Amer. Math. Soc. 2000.

[LMW04] H. Leschke, P. Müller and S. Warzel, A survey of rigorous results on random Schrödinger operators for amorphous solids. In this proceedings volume.

[L77] E.H. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation. *Studies in Applied Mathematics* **57**, 93–105, 1977.

[MW02] F. Merkl and M.V. Wüthrich, Infinite volume asymptotics for the ground state energy in a scaled Poissonian potential. *Ann. Inst. H. Poincaré Probab. Statist.* **38**, 3, 253–284, 2002.

[M94] S. Molchanov, Lectures on random media. In: D. Bakry, R.D. Gill, and S. Molchanov, *Lectures on Probability Theory*, Ecole d'Eté de Probabilités de Saint-Flour XXII-1992, LNM 1581, pp. 242–411. Berlin, Springer 1994.

[S82] Ya.G. Sinai, Limit behavior of one-dimensional random walks in random environment. *Theor. Probab. Appl.* **27**, 247–258, 1982.

[S98] A.-S. Sznitman, *Brownian motion, Obstacles and Random Media.* Springer Berlin 1998.

[Z84] Ya.B. Zel'dovich, *Selected Papers. Chemical Physics and Hydrodynamics* (in Russian), Nauka, Moscow 1984.

[ZKO94] V.V. Zhikov, S.M. Kozlov and O.A. Olejnik, *Homogenization of Differential Operators and Integral Functionals.* Springer, Berlin 1994.

[ZMRS87] Ya.B. Zel'dovich, S.A. Molchanov, S.A. Ruzmajkin and D.D. Sokolov, Intermittency in random media. *Sov. Phys. Uspekhi* **30**, 5, 353–369, 1987.

[ZMRS88] Ya.B. Zel'dovich, S.A. Molchanov, S.A. Ruzmajkin and D.D. Sokolov, Intermittency, diffusion and generation in a nonstationary random medium. *Sov. Sci. Rev. Sect. C, Math. Phys. Rev.* **7**, 1–110, 1988.