A NOTE ON LATTICE PACKINGS VIA LATTICE REFINEMENTS

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ABSTRACT. Rogers proved in a constructive way that every packing lattice Λ of a symmetric convex body K in $\mathbb{R}^n$ is contained in a packing lattice whose covering radius is less than 3. By a slight modification of Rogers' approach better bounds for l_p-balls are obtained. Together with Rogers' constructive proof this leads, for instance, to a simple $O(n^{n/2})$ running time algorithm that refines successively the packing lattice D^n (checkboard lattice) of the unit ball B^n and terminates with a packing lattice $\overline{\Lambda}$ with density $\delta(\overline{\Lambda}, B^n) > 2^{-1.197\,n}$. We have also implemented this algorithm and in small dimensions (≤ 25) and for certain simple structured start lattices like $\mathbb{Z}^n$ or D^n the algorithm often terminates with packing lattices achieving the best known lattice densities.

1. INTRODUCTION

Let $\mathcal{K}_o^n$ be the set of all convex bodies $K \subset \mathbb{R}^n$ in the n-dimensional Euclidean space $\mathbb{R}^n$, which are symmetric with respect to the origin $\mathbf{0}$, i.e., $K = -K$. The norm (or distance function) induced by K is given by $|\cdot|_K : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with $|\boldsymbol{x}|_K = \min\{\rho \geq 0 : \boldsymbol{x} \in \rho\, K\}$. In the case of the l_p-(unit) balls $B_p^n = \{\boldsymbol{x} \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^p \leq 1\} \in \mathcal{K}_o^n$, we write $|\boldsymbol{x}|_p$ instead of $|\boldsymbol{x}|_{B_p^n} = (\sum_{i=1}^n |x_i|^p)^{1/p}$, and for the Euclidean unit ball B_2^n we will often use the notation B^n.

Let $\mathcal{L}^n$ be the family of all n-dimensional lattices $\Lambda \subset \mathbb{R}^n$, i.e., Λ is a discrete subgroup generated by n linearly independent vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n \in \mathbb{R}^n$, called a basis of Λ. Hence, $\Lambda = B\mathbb{Z}^n$, where B is the matrix with columns $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$, and $\det \Lambda = |\det B|$ is called the determinant of the lattice.

A lattice $\Lambda \in \mathcal{L}^n$ is called a packing lattice of $K \in \mathcal{K}_o^n$ if $|\boldsymbol{b}|_K \geq 2$ for all $\boldsymbol{b} \in \Lambda \setminus \{\mathbf{0}\}$. In words, two different lattice translate $\boldsymbol{g}_1 + K$ and $\boldsymbol{g}_2 + K$, $\boldsymbol{g}_1 \neq \boldsymbol{g}_2 \in \Lambda$, have no interior points in common. The set of all lattice packings of $K \in \mathcal{K}_o^n$ will be denoted by $\mathcal{P}(K)$. For $\Lambda \in \mathcal{P}(K)$,

$$\delta(\Lambda, K) = \frac{\mathrm{vol}\,(K)}{\det \Lambda}$$

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is the *density of the lattice packing* $\Lambda + K$, i.e., the amount (percentage) of space which is covered by the packing $\Lambda + K$. Here $\mathrm{vol}\,(K)$ denotes the volume, i.e., the n-dimensional Lebesgue measure of K. The density of a densest lattice packing is denoted by $\delta(K)$, that is

$$\delta(K) = \sup\{\delta(\Lambda, K) : \Lambda \in \mathcal{P}(K)\}\,.$$

It is well known that this sup is attained and for further basic facts and notations from Geometry of Numbers we refer to [7, 23, 24].

Somehow contrary to the packing concept, a lattice $\Lambda \in \mathcal{L}^n$ is called a *covering lattice* of K if $K + \Lambda = \mathbb{R}^n$, i.e., the whole space is covered by the lattice translates. The smallest dilation factor μ such that a lattice Λ is a covering lattice of $\mu\,K$ is called *covering radius* or *inhomogeneous minimum* of Λ with respect to K and it is denoted by $\mu(\Lambda, K)$. Hence,

$$\begin{aligned}\mu(\Lambda, K) &= \min\{\mu \geq 0 : \Lambda + \mu\,K = \mathbb{R}^n\}\\ &= \max\{\mathrm{d}_K(\boldsymbol{x}, \Lambda) : \boldsymbol{x} \in \mathbb{R}^n\},\end{aligned}$$

where $\mathrm{d}_K(\boldsymbol{x}, \Lambda) = \min\{|\boldsymbol{x} - \boldsymbol{b}|_K : \boldsymbol{b} \in \Lambda\}$ is the distance of a point $\boldsymbol{x}$ from the lattice Λ. With respect to the lattice, the covering radius is homogeneous of degree 1, i.e., for $\lambda \in \mathbb{R}_{>0}$ we have

$$\mu(\lambda\,\Lambda, K) = \lambda\,\mu(\Lambda, K). \tag{1.1}$$

Again, for $K = B_p^n$ we write $\mathrm{d}_p(\boldsymbol{x}, \Lambda)$ and for $p = 2$ just $\mathrm{d}(\boldsymbol{x}, \Lambda)$. Points $\boldsymbol{h} \in \mathbb{R}^n$ having maximal distance to the lattice Λ, i.e., $\mathrm{d}_K(\boldsymbol{h}, \Lambda) = \mu(\Lambda, K)$ are called *deep holes*. The covering property $\Lambda + \mu(\Lambda, K)\,K = \mathbb{R}^n$ implies $\mathrm{vol}\,(\mu(\Lambda, K)\,K) \geq \det\Lambda$ and so for $\Lambda \in \mathcal{P}(K)$ we certainly have the bound

$$\delta(\Lambda, K) \geq \left(\frac{1}{\mu(\Lambda, K)}\right)^n. \tag{1.2}$$

Hence, upper bounds on the so called *packing-covering ratio* $\mu(K)$ of $K \in \mathcal{K}_o^n$ defined by

$$\mu(K) = \min\{\mu(\Lambda, K) : \Lambda \in \mathcal{P}(K)\}\,.$$

give lower bounds on $\delta(K)$. Rogers [33] seems to be the first who studied this ratio and among others he proved

$$\mu(K) \leq 3^{-1/n} \cdot 3 \tag{1.3}$$

which leads via (1.2) to the "bad" lower bound $\delta(K) \geq 3^{-(n-1)}$. The meaning of $\mu(K)$, however, is to measure covering properties of packing lattices, rather than proving good lower bounds on $\delta(K)$. Nevertheless, it was shown by Butler [5] by a probabilistic argument that

$$\mu(K) \leq 2 + o(1),$$

as n approaches infinity. Hence, this result gives "almost" the Minkowski-Hlawka bound

$$\delta(K) \geq 2^{-(n-1)}$$

which is still the best possible bound for general $K \in \mathcal{K}_o^n$ (cf., e.g., [23]). For l_p-balls, $p > 2$, and related highly symmetric convex bodies there are exponentially better bounds available (cf., e.g., [21, 36]). The best known lower bound for the sphere is $\delta(B_n) \geq (n-1)2^{-(n-1)}$ and it is due to Ball [1]. Another probabilistic-based approach to bound $\mu(K)$ is due to BOURGAIN [4] where also the convexity modulus of K plays a role.

Rogers obtained (1.3) in a constructive manner; he showed that under the assumption $\mu(\Lambda, K) > 3$, i.e., we have "big" deep holes, one can find/construct a packing lattice $\Lambda_1 \in \mathcal{P}(K)$ containing Λ with $\det \Lambda_1 = \frac{1}{3} \det \Lambda$. Hence, as long as $\mu(\Lambda, K) > 3$ we can always refine our packing lattice and get a "smaller" one. Since the determinant of a packing lattice of K can not be smaller than the volume of K, this shows $\mu(K) < 3$. The additional factor of $3^{-1/n}$ in (1.3) comes from a final scaling argument. This successive refinement approach was rediscovered by BANASZCZYK [2], and he also pointed out that this approach leads to a finite algorithm for calculating "dense" packings of $K \in \mathcal{K}_o^n$. A detailed description of this algorithm is given by Micciancio [31, Theorem 4.4].

In a recent paper, Dadush [16, Theorem 15] uses this Greedy-algorithm of Rogers as one of many ingredients in a deterministic $2^{O(n)}$ time and poly(n) space algorithm which, in particular, for a "well-described" o-symmetric convex body computes a packing lattice of density at least 3^{-n}. This algorithm is also based on recent and quite involved results/concepts form Convex Geometry as well as from Computational Geometry of Numbers and there seems to be no efficient way to implement this approach, even in the special case of a ball.

Here we firstly observe that a slight modification of Rogers' refinement argumentation leads to better bounds for l_p-balls, $1 < p < \infty$. To this end, let $p \geq 2$, $q = p/(p-1)$ and for $x \in [0,1]$ let

$$f_1(p,x) = \frac{1}{2}(1 - x^q)^{1/q},$$
$$f_2(p,x) = \frac{1}{3}\left(2\,(1+x^p)^{q-1} - (1-x^p)^{q-1}\right)^{1/q}.$$

$f_1(p,x)$ is montonously decreasing in x whereas $f_2(p,x)$ is montonously increasing. Since $f_1(p,0) > f_2(p,0)$ and $f_1(p,1) < f_2(p,1)$ there exists an unique point $\beta_p \in (0,1)$ with $f_1(p,\beta_p) = f_2(p,\beta_p)$. For instance, for $p = 2$ we get

$$\beta_2 = \sqrt{\frac{5}{21}} \text{ and } f_1(2,\beta_2) = \sqrt{\frac{4}{21}} \sim \frac{1}{2.2913} \sim 2^{-1.1962}.$$

Moreover, by definition of $f_2(p,x)$ we also have $f_1(p,\beta_p) = f_2(p,\beta_p) > f_2(p,0) = \frac{1}{3}$ as well as $\lim_{p\to\infty} f_1(p,\beta_p) = \lim_{p\to\infty} f_2(p,\beta_p) = \frac{1}{3}$. Finally, we extend both functions to the range $1 < p \leq 2$ by setting $f_1(p,x) = f_1(q,x)$, $f_2(p,x) = f_2(q,x)$.

Proposition 1.1. *Let $\Lambda \in \mathcal{P}(B_p^n)$, $n \geq 2$. Then Λ is contained in a packing lattice $\overline{\Lambda} \in \mathcal{P}(B_p^n)$ with*

$$\mu(\overline{\Lambda}, B_p^n) < \frac{1}{f_1(p, \beta_p)},$$

and, hence, $\delta(\overline{\Lambda}, B_p^n) > f_1(p, \beta_p)^n$.

When writing this note we discovered that the case $p = 2$ was already proved by Davenport in 1952 [17]. Since the impact of this bound on $\mu(\overline{\Lambda}, B^n)$ to the optimal covering lattice density of spheres was inferior to the main result of Davenport's paper, this result of Davenport seems to have been lost. In fact, it was also reproved in the authors Habilitationsschrift [26] in relation to dense finite lattice sphere packings.

As mentioned before, Rogers' approach is constructive and, in particular, the proof of Proposition 1.1 will imply

Theorem 1.2. *Let $\Lambda \in \mathcal{P}(B_p^n)$, $n \geq 2$, and let $\gamma(p, \Lambda) = \delta(B_p^n)/\delta(\Lambda, B_p^n)$. A lattice $\overline{\Lambda} \in \mathcal{P}(B_p^n)$ as in Proposition 1.1 can be constructed via $O(3^n \gamma(p, \Lambda))$ calls of the function $\mathrm{d}_p(\boldsymbol{x}, \Lambda)$ and additional $O(n^2\, 3^n \log \gamma(p, \Lambda))$ arithmetic operations.*

For simple structured lattices like $\mathbb{Z}^n$, $D^n = \{\boldsymbol{z} \in \mathbb{Z}^n : \sum_{i=1}^n z_i \equiv 0 \bmod 2\}$ or $A^n = \{\boldsymbol{z} \in \mathbb{Z}^{n+1} : \sum_{i=1}^{n+1} z_i = 0\}$ the distance function $\mathrm{d}_p(\boldsymbol{x}, \Lambda)$ can easily be calculated, see, e.g., [14]. In particular, for D^n it can be done in $O(n)$ arithmetic operations. Moreover, $\sqrt{2}D^n$ is a packing lattice of the unit ball B^n with density

$$\begin{aligned}\delta(\sqrt{2}\, D^n, B^n) &= 2^{-n/2-1} \operatorname{vol}(B^n) = 2^{-n/2-1} \frac{\pi^{n/2}}{\Gamma(n/2+1)} \\ &\sim 2^{-n/2-1} \frac{\pi^{n/2}}{\sqrt{\pi\, n}(n/(2e))^{n/2}} = \frac{1}{2} \frac{\sqrt{\pi\, e}^n}{\sqrt{\pi\, n}\, n^{n/2}}\end{aligned}$$

by Stirling's formula. By the result of Kabatyanskiĭ and Levenshteĭn [28] (see also [13]) we know $\delta(B^n) < 2^{-0.599n+o(1)}$ and so Theorem 1.2 implies

Corollary 1.3. *A packing lattice $\overline{\Lambda} \in \mathcal{P}(B^n)$ with $\delta(\overline{\Lambda}, B_n) > 2^{-1.1962\, n}$ can be constructed in running time $O(n^{n/2})$.*

This bound on the density seems also to be a bit better then other known constructive results which are based on algebraic codes and exponential algorithms, cf., e.g., the bounds in [8, 29, 35].

We have also implemented an algorithm based on the construction in Theorem 1.2. The running time of this algorithm as well as the density of the constructed packing lattices depend on the lattice with which we start the algorithm, i.e., with which we begin the refinement process. The following three tables contain the numerical results for the start lattices $2\,\mathbb{Z}^n$, $\sqrt{2}\, D^n$ and $\sqrt{2}\, A^n$ in dimensions up to 25. In the tables we denote by $\overline{\Delta}$ the determinant of the packing lattice found by the algorithm. A bold and stared dimension indicates that the corresponding density coincides with the density of the densest (known) lattices packing of B^n, for which we refer to the book [15] or to the online catalogue [32].

n	$\overline{\Delta}$	n	$\overline{\Delta}$	n	$\overline{\Delta}$	n	$\overline{\Delta}$
2	4	**8***	16	14	32	**20***	8
3	8	9	32	15	32	21	8
4*	8	10	32	**16***	16	22	4
5	16	11	32	**17***	16	**23***	2
6	16	12	32	18	16	**24***	1
7*	16	13	32	19	16	25	2

TABLE 1.1. Density results for $2\,Z^n$

n	$\overline{\Delta}$	n	$\overline{\Delta}$	n	$\overline{\Delta}$	n	$\overline{\Delta}$
2	4	**8***	16	14	32	**20***	8
3*	$4\sqrt{2}$	**9***	$16\sqrt{2}$	**15***	$16\sqrt{2}$	**21***	$4\sqrt{2}$
4*	8	10	32	**16***	16	22	4
5*	$8\sqrt{2}$	11	$32\sqrt{2}$	17	$16\sqrt{2}$	23	$2\sqrt{2}$
6	16	12	32	18	16	**24***	1
7	$16\sqrt{2}$	13	$32\sqrt{2}$	**19***	$8\sqrt{2}$	**25***	$\sqrt{2}$

TABLE 1.2. Density results for $\sqrt{2}\,D^n$

n	$\overline{\Delta}$	n	$\overline{\Delta}$	n	$\overline{\Delta}$	n	$\overline{\Delta}$
2*	$2\sqrt{3}$	8	24	14	$8\sqrt{15}$	20	$2\sqrt{21}$
3*	$4\sqrt{2}$	9	$16\sqrt{5}$	**15***	$16\sqrt{2}$	21	$2\sqrt{11}$
4	$4\sqrt{5}$	10	$16\sqrt{11}$	16	$8\sqrt{17}$	22	$\sqrt{23}$
5	$8\sqrt{3}$	11	$16\sqrt{6}$	17	24	23	$2\sqrt{3}$
6	$8\sqrt{7}$	12	$16\sqrt{13}$	18	$4\sqrt{19}$	24	5
7*	16	13	$16\sqrt{7}$	19	$4\sqrt{10}$	25	$\sqrt{13}$

TABLE 1.3. Density results for $\sqrt{2}\,A^n$

The remaining "not best possible" dimensions in these tables are $6, 10, 11, 12, 13, 14, 18$ and 22. Except for dimension 22 we also found for these dimensions certain "simple structured" start lattices such that the algorithm ends with the best known lattice densities; details will be given in the last section.

Remark 1.4.

i) *As already mentioned, the running time of the algorithm depends on the start lattice and it grows exponentially with the dimension. For instance, the dimensions 18 and 24 with the start lattice* $2\mathbb{Z}^n$ *takes* 4.57 *and* $209,937$ *CPU seconds, respectively, and with the start lattice* $\sqrt{2}D^n$ *only* 0.28 *and* 23 *seconds*[1].

ii) *Except for dimensions* $11, 12, 13$ *the best known possible packing lattices in dimensions up to 25 are* laminated lattices *(see* [15]*). They are defined inductively and so it might be not surprising that a refinement procedure also ends up with these lattices. Also in dimensions* $11, 12, 13$ *the densities in Table 1.1 coincide with the maximal densities of laminated lattices in these dimensions (see* [15]*).*

The densest lattice packings are only known in dimensions 2 up to 8 by classical results of Lagrange, Gauss, Korkine&Zolotaref and Blichfeldt (see, e.g., [23]) and in dimension 24 where it was recently shown by Cohn&Kumar [12] that the Leech lattice is indeed the best one. For more information on the geometry of lattice sphere packings we refer to [20, 37] and the references within.

For the state of the art on upper bounds on (general) sphere packings in low dimensions and asymptotically, see [10, 11, 13]. It seems likely that in (very) small dimensions ≤ 8 optimal lattice sphere packings yield the optimal density among all sphere packings. So far this has only be verified by Thue in dimension 2 and by the proof of the Kepler conjecture by Hales in dimension 3 (see, e.g., [25]).

In recent years there has also been a lot of progress in the computational treatment of (general) packing problems for which refer to [16, 18, 19, 27, 30, 39, 40] and the references given there. In particular, the algorithm by Marcotte&Torquato [30] is a bit in the same spirit as the one described here, but there a given start lattice is properly "disturbed" instead of refined. Regarding concrete densities of (lattice) packings of convex bodies different from spheres not much seems to be known. For l_p-balls in dimension 3 see [39], for an algorithm for computing a densest lattice packing of 3-dimensional polytopes see [3] and for packing results of the l_1-balls we refer to [22].

Also the packing-covering ratio has been extensively studied in recent years in low dimensions, see, e.g., [42, 43, 44]. In particular, in [44] Chuanming Zong gives a best possible upper bound on $\mu(K)$ for two dimensional o-symmetric convex bodies.

Rogers [34] also studied the (lattice) packing-covering ratio of infinite dimensional Banach spaces and proved the analogous result there. Recently, Swanepoel [38] determined the packing-covering ratio of l_p-sequence spaces in the non lattice case.

[1] On a 3.1 GHz Intel Core i7 processor.

2. Proof of Proposition 1.1

Rogers proof of (1.3) as well as Banaszczyk's proof in [2] are based on the following observation: Suppose we have a packing lattice $\Lambda \in \mathcal{P}(K)$ of $K \in \mathcal{K}_o^n$ and assume that there exists an $\boldsymbol{a} \in \Lambda$ such that

$$\left|\frac{1}{3}\boldsymbol{a} - \boldsymbol{b}\right|_K \geq 2 \text{ for all } \boldsymbol{b} \in \Lambda. \tag{2.1}$$

Then

$$\Lambda_1 = \mathbb{Z}\,\frac{1}{3}\boldsymbol{a} + \Lambda = \Lambda \cup \left(\frac{1}{3}\boldsymbol{a} + \Lambda\right) \cup \left(\frac{2}{3}\boldsymbol{a} + \Lambda\right) \in \mathcal{P}(K), \tag{2.2}$$

i.e., Λ_1 is again a packing lattice of K. This follows immediately from $\Lambda \in \mathcal{P}(K)$, $|\frac{2}{3}\boldsymbol{a} - \boldsymbol{b}|_K = |\frac{1}{3}\boldsymbol{a} - (\boldsymbol{a} - \boldsymbol{b})|_K$ and (2.1).

Of course, the same reasoning shows that

$$\Lambda_1 = \mathbb{Z}\,\frac{1}{2}\boldsymbol{a} + \Lambda = \Lambda \cup \left(\frac{1}{2}\boldsymbol{a} + \Lambda\right) \in \mathcal{P}(K), \tag{2.3}$$

provided

$$\left|\frac{1}{2}\boldsymbol{a} - \boldsymbol{b}\right|_K \geq 2 \text{ for all } \boldsymbol{b} \in \Lambda. \tag{2.4}$$

Hence, as long as we can find an $\boldsymbol{a} \in \Lambda$ satisfying (2.1) or (2.4) we can refine our given packing lattice. In order to find those points we use the deep holes and for the results on the l_p-balls B_p^n we also need Clarkson's inequality (see, e.g., [6, pp. 117], [9])

$$|\boldsymbol{x} + \boldsymbol{y}|_p^q + |\boldsymbol{x} - \boldsymbol{y}|_p^q \geq 2\left(|\boldsymbol{x}|_p^p + |\boldsymbol{y}|_p^p\right)^{q-1} \tag{2.5}$$

for $2 \leq p < \infty$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Here $q = p/(p-1)$ and for $1 < p \leq 2$ the sign in (2.5) is reversed.

Proof of Proposition 1.1. We start with $p \geq 2$ and let $\Lambda \in \mathcal{P}(B_p^n)$ with

$$\overline{\mu} = \mu(\Lambda, B_p^n) \geq f_1(p, \beta_p)^{-1}. \tag{2.6}$$

In the following we show that the lower bound (2.6) implies that we can find a lattice of type (2.2) or (2.3) which is still a packing lattice of B_p^n. For simplification we just write $|\cdot|$ instead of $|\cdot|_p$ and β for β_p.

Let $\boldsymbol{h} \in \mathbb{R}^n$ be a deep hole of Λ with respect to B_p^n, i.e., $\overline{\mu} = \mathrm{d}_p(\boldsymbol{h}, \Lambda)$. Then

$$|\boldsymbol{h} - \boldsymbol{b}| \geq \overline{\mu} \text{ for all } \boldsymbol{b} \in \Lambda. \tag{2.7}$$

a) First we assume

$$\exists \boldsymbol{a} \in \Lambda \text{ with } \left|\boldsymbol{h} - \frac{1}{2}\boldsymbol{a}\right| \leq \beta\,\overline{\mu}. \tag{2.8}$$

With respect to this $\boldsymbol{a}$ we consider the refined lattice $\Lambda_1 = \Lambda \cup (\frac{1}{2}\boldsymbol{a} + \Lambda)$ and next we verify that it is still a packing lattice of B_p^n. To this end let $\boldsymbol{b} \in \Lambda$, $\boldsymbol{x} = \boldsymbol{h} - \boldsymbol{b}$ and $\boldsymbol{y} = \boldsymbol{h} - \frac{1}{2}\boldsymbol{a}$. By Clarkson's inequality (2.5) we obtain

$$\begin{aligned} |\boldsymbol{y}|^q + |\boldsymbol{x} - \boldsymbol{y}|^q &= \left|\boldsymbol{y} - \frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}\right|^q + \left|\boldsymbol{y} - \frac{1}{2}\boldsymbol{x} - \frac{1}{2}\boldsymbol{x}\right|^q \\ &\geq 2\left(\left|\boldsymbol{y} - \frac{1}{2}\boldsymbol{x}\right|^p + \left|\frac{1}{2}\boldsymbol{x}\right|^p\right)^{q-1}, \end{aligned}$$

and thus

$$\left|\frac{1}{2}\boldsymbol{a} - \boldsymbol{b}\right|^q = |\boldsymbol{x} - \boldsymbol{y}|^q \geq 2\left(\left|\boldsymbol{y} - \frac{1}{2}\boldsymbol{x}\right|^p + \left|\frac{1}{2}\boldsymbol{x}\right|^p\right)^{q-1} - |\boldsymbol{y}|^q.$$

In view of (2.7) we have $\left|\boldsymbol{y} - \frac{1}{2}\boldsymbol{x}\right|, \left|\frac{1}{2}\boldsymbol{x}\right| \geq \frac{1}{2}\overline{\mu}$ and so we obtain

$$\left|\frac{1}{2}\boldsymbol{a} - \boldsymbol{b}\right| \geq \overline{\mu}\,(1 - \beta^q)^{\frac{1}{q}} = 2\overline{\mu}\,f_1(p, \beta). \tag{2.9}$$

Hence, due to (2.6) and (2.4) this shows $\Lambda_1 \in \mathcal{P}(B_p^n)$.

b) Now we assume that there exists no $\boldsymbol{a} \in \Lambda$ such that (2.8) holds. Then for all $\boldsymbol{b} \in \Lambda$

$$\left|\frac{2}{3}\boldsymbol{h} - \frac{1}{3}\boldsymbol{b}\right| = \frac{2}{3}\left|\boldsymbol{h} - \frac{1}{2}\boldsymbol{b}\right| > \frac{2}{3}\beta\,\overline{\mu}. \tag{2.10}$$

Observe, in view of the homogeneity of the covering radius (cf. (1.1)) there always exists a $\boldsymbol{a} \in \Lambda$ with $|\boldsymbol{h} - \frac{1}{2}\boldsymbol{a}| \leq \frac{1}{2}\overline{\mu}$. Hence, in view of our assumption in this case b) we know

$$\beta \leq \frac{1}{2}. \tag{2.11}$$

Also by the homogeneity of the covering radius we find

$$\exists \boldsymbol{a} \in \Lambda \text{ with } \left|\boldsymbol{h} - \frac{1}{3}\boldsymbol{a}\right| \leq \frac{1}{3}\overline{\mu}. \tag{2.12}$$

With respect to this $\boldsymbol{a}$ we now consider the refined lattice $\Lambda_1 = \Lambda \cup (\frac{1}{3}\boldsymbol{a} + \Lambda) \cup (\frac{2}{3}\boldsymbol{a} + \Lambda)$ and as before we verify that it is still a packing lattice of B_p^n.

Here we set $\boldsymbol{y} = \boldsymbol{h} - \frac{1}{3}\boldsymbol{a}$ and $\boldsymbol{x} = \boldsymbol{h} - \boldsymbol{b}$ for a fixed but arbitrary $\boldsymbol{b} \in \Lambda$. Then by (2.5) we get

$$\begin{aligned} |\boldsymbol{y} - \boldsymbol{x}|^q + \left|\boldsymbol{y} + \frac{1}{3}\boldsymbol{x}\right|^q &\geq 2\left(\left|\boldsymbol{y} - \frac{1}{3}\boldsymbol{x}\right|^p + \left|\frac{2}{3}\boldsymbol{x}\right|^p\right)^{q-1}, \\ \left|\boldsymbol{y} - \frac{1}{3}\boldsymbol{x}\right|^p + \left|\boldsymbol{y} + \frac{1}{3}\boldsymbol{x}\right|^p &\leq 2\left(|\boldsymbol{y}|^q + \left|\frac{1}{3}\boldsymbol{x}\right|^q\right)^{p-1}, \end{aligned}$$

where for the second inequality we plug $\boldsymbol{y} - \frac{1}{3}\boldsymbol{x}$ and $\boldsymbol{y} + \frac{1}{3}\boldsymbol{x}$ into the right hand side of (2.5). Substituting $\left|\boldsymbol{y} + \frac{1}{3}\boldsymbol{x}\right|^q$ from the second into the first inequality leads to

$$|\boldsymbol{y}-\boldsymbol{x}|^q \geq 2\left(\left|\boldsymbol{y}-\frac{1}{3}\boldsymbol{x}\right|^p + \left|\frac{2}{3}\boldsymbol{x}\right|^p\right)^{q-1} - \left(2\left(|\boldsymbol{y}|^q + \left|\frac{1}{3}\boldsymbol{x}\right|^q\right)^{p-1} - \left|\boldsymbol{y}-\frac{1}{3}\boldsymbol{x}\right|^p\right)^{q-1}.$$

By (2.10) and (2.12) we conclude

$$\left|\frac{1}{3}\boldsymbol{a}-\boldsymbol{b}\right|^q = |\boldsymbol{y}-\boldsymbol{x}|^q \geq 2\left(\left(\frac{2}{3}\beta\overline{\mu}\right)^p + \left|\frac{2}{3}\boldsymbol{x}\right|^p\right)^{q-1} - \left(2\left(\left(\frac{1}{3}\overline{\mu}\right)^q + \left|\frac{1}{3}\boldsymbol{x}\right|^q\right)^{p-1} - \left(\frac{2}{3}\beta\overline{\mu}\right)^p\right)^{q-1}.$$

According to (2.7) we may substitute $|\boldsymbol{x}|$ by $\lambda\overline{\mu}$ with $\lambda \geq 1$ which leads to

$$\begin{aligned}\left|\frac{1}{3}\boldsymbol{a}-\boldsymbol{b}\right|^q &\geq \left(\frac{2}{3}\overline{\mu}\right)^q\left(2\left(\beta^p+\lambda^p\right)^{q-1} - \left(2^{1-p}\left(1+\lambda^q\right)^{p-1} - \beta^p\right)^{q-1}\right)\\ &\geq \left(\frac{2}{3}\overline{\mu}\right)^q\left(2\left(\beta^p+\lambda^p\right)^{q-1} - \left(\lambda^p-\beta^p\right)^{q-1}\right),\end{aligned}$$

where the last inequality follows from $1+\lambda^q \leq 2\lambda^q$, since $\lambda \geq 1$. Calculating the derivative with respect to λ of the function on the right hand side and taking into account $\beta \leq 1/2$ (cf. (2.11)), $1 \leq q \leq 2$, shows that it is monotonously increasing for $\lambda \geq 1$. Hence, we finally get

$$\left|\frac{1}{3}\boldsymbol{a}-\boldsymbol{b}\right| \geq 2\overline{\mu}\frac{1}{3}\left(2(1+\beta^p)^{q-1} - (1-\beta^p)^{q-1}\right)^{1/q} = 2\overline{\mu}f_2(p,\beta). \tag{2.13}$$

Thus, again due to (2.6) and (2.1) this shows $\Lambda_1 \in \mathcal{P}(B_p^n)$.

So we have shown that as long as (2.6) holds the packing lattice Λ is contained in a packing lattice Λ_1 with $\det\Lambda_1 \in \{\frac{1}{2}\det\Lambda, \frac{1}{3}\det\Lambda\}$. Since the determinant of a packing lattice can not be too small we can repeat this refinement process only finitely many times.

Finally, we remark that in the case $1 < p \leq 2$ the sign in Clarkson's inequality (2.5) is reversed and so we get the same bounds as in (2.9) and (2.13) with p and q interchanged.

□

3. A simple algorithm

The proof of Proposition 1.1 immediately suggests the following simple algorithm for finding a packing lattice $\overline{\Lambda}$ with $\delta(\overline{\Lambda}, B_p^n) > f_1(p,\beta_p)^n$ (cf. also [31, Theorem 4.4]):

`S0` Input: $\Lambda^{(0)} \in \mathcal{P}(B_p^n)$ given by a basis $\boldsymbol{c}_1^{(0)},\ldots,\boldsymbol{c}_n^{(0)} \in \mathbb{Q}^n$ and for which we can evaluate $\mathrm{d}_p(\boldsymbol{x},\Lambda^{(0)})$.

In the lth step, $l \geq 0$, we want to find a $\boldsymbol{a}^{(l)} \in \Lambda^{(l)}$ such that (2.4) or (2.1) is satisfied, i.e., we want to find a point of the lattice $\frac{1}{2}\Lambda^{(l)}$ or $\frac{1}{3}\Lambda^{(l)}$ having maximal distance to $\Lambda^{(l)}$. This can easily be done by examine the 2^n (3^n) points of $\frac{1}{2}\Lambda^{(l)}$ ($\frac{1}{3}\Lambda^{(l)}$) contained in a fundamentell cell of $\Lambda^{(l)}$. Thus we determine

S1 For $k = 2, 3$ determine

$$\sigma_k^{(l)} = \max\left\{ \mathrm{d}_p\left(\frac{1}{k}\sum_{i=1}^{n} w_i\, \boldsymbol{c}_i^{(l)}, \Lambda^{(l)}\right) : w_i \in \{0, 1, k-1\}, 1 \leq i \leq n \right\},$$

and let

$$\boldsymbol{a}_k^{(l)} = \sum_{i=1}^{n} \overline{w_i}\, \boldsymbol{c}_i^{(l)} \in \Lambda^{(l)} \tag{3.1}$$

be a point attaining this maximal distance (after scaling by $1/k$). If $\sigma_k(l) \geq 2$ then the lattice

$$\Lambda^{(l+1)} = \bigcup_{m=0}^{k-1} \left(\frac{m}{k}\boldsymbol{a}_k^{(l)} + \Lambda^{(l)}\right) \tag{3.2}$$

is a new packing lattice of B_p^n of determinant $\frac{1}{k}\det\Lambda^{(l)}$ (cf. (2.4), (2.1)).

A basis $\boldsymbol{c}_1^{(l+1)}, \ldots, \boldsymbol{c}_n^{(l+1)}$ of $\Lambda^{(l+1)}$ is obtained by replacing a $\boldsymbol{c}_i^{(l)}$ with $\overline{\mathrm{w}}_i \neq 0$ in the representation (3.1) by the vector $\frac{1}{k}\boldsymbol{a}_k^{(l)}$. Due to the coset representation (3.2) we can also easily compute the distance to $\Lambda^{(l+1)}$ by setting

$$\mathrm{d}_p(\boldsymbol{x}, \Lambda^{(l+1)}) = \min\left\{ \mathrm{d}_p\left(\boldsymbol{x} - \frac{m}{k}\boldsymbol{a}_k^{(l)}, \Lambda^{(l)}\right), m = 0, .., k-1 \right\}. \tag{3.3}$$

S2 Determine a basis of $\Lambda^{(l+1)}$ and update the distance function according to (3.3).

In order to estimate the running time we assume that we first consider only lattice refinements of type (2.3), i.e., $k = 2$, and then we continue with $k = 3$, i.e., lattices of type (2.2).

The determination of $\sigma_2^{(l)}$ costs 2^n calls of $\mathrm{d}_p\left(\frac{1}{2}\sum_{i=1}^{n} w_i\, \boldsymbol{c}_i^{(l)}, \Lambda^{(l)}\right)$ which by the recursion (3.3) leads to

$$2^n\, 2^l \text{ calls of } \mathrm{d}_p\left(\boldsymbol{x}, \Lambda^{(0)}\right).$$

The calculation of a vector of the type $\frac{1}{2}\sum_{i=1}^{n} w_i\, \boldsymbol{c}_i^{(l)}$ takes $O(n^2)$ arithmetic operations and assuming that we execute L_2 times the refinement with $k = 2$, then the running time of this part of the algorithm is bounded by

$$O(n^2\, 2^n\, L_2) \text{ arithmetic operations } + O(2^{n+L_2}) \text{ calls of } \mathrm{d}_p\left(\boldsymbol{x}, \Lambda^{(0)}\right).$$

Next we continue with refinements of lattices of type (2.2). The calculation of $\sigma_3^{(l)}$ costs 3^n calls of $\mathrm{d}_p\left(\frac{1}{3}\sum_{i=1}^n w_i\, \boldsymbol{c}_i^{(l)}, \Lambda^{(l)}\right)$, which by the recursion (3.3) leads in the r-th execution of this second refinements to

$$3^n\, 2^{L_2}\, 3^r \text{ calls of } \mathrm{d}_p\left(\boldsymbol{x}, \Lambda^{(0)}\right).$$

Assuming that we perform L_3 steps of these second refinements of the algorithm, the running time of this part is bounded by

$$O(n^2\, 3^n\, L_3) \text{ arithmetic operations } + O(3^n\, 3^{L_3} 2^{L_2}) \text{ calls of } \mathrm{d}_p\left(\boldsymbol{x}, \Lambda^{(0)}\right).$$

Regarding the overall numbers of steps of the algorithm, we observe that in each step the density is either doubled ($k = 2$) or tripled ($k = 3$) and so we have

$$2^{L_2}\, 3^{L_3} \leq \frac{\delta(B_p^n)}{\delta(\Lambda^{(0)}, B_p^n)}.$$

This implies the running time in Theorem 1.2.

4. Numerical results

The differences between our implementation of the algorithm and its description in the last section are essentially the following:

i) Instead of computing the maximum $\sigma_k^{(l)}$, $k = 2, 3$, we take as $\boldsymbol{a}^{(l)}$ the "first" vector $\boldsymbol{a}^{(l)} = \sum_{i=1}^n w_i\, \boldsymbol{c}_i^{(l)}$ for which $\mathrm{d}_p\left(\frac{1}{k}\boldsymbol{a}^{(l)}, \Lambda^{(l)}\right) \geq 2$. Here "first" refers to the reverse lexicographic order on the coefficient vectors $\overline{\boldsymbol{w}}$.

ii) We first look for refinements with $k = 2$ and if we can not find a vector $\boldsymbol{a}_2^{(l)}$ with $\mathrm{dist}_p(\frac{1}{2}\boldsymbol{a}_l^{(l)}, \Lambda^{(l)}) \geq 2$ we continue with $k = 3$. If already the first execution with $k = 3$ does not lead to a refinement, then we go one step back and try to get an improvement in the step before with a factor $k = 3$ instead of $k = 2$ and we then continue.

In order to also obtain – in addition to the results in Tables 1.1, 1.2 – the densities of the densest known packing lattices in the dimensions $6, 10, 11, 12, 13, 14, 18$ via this implementation of the algorithm we need a bit more notation. Let Λ_{k_i}, $1 \leq i \leq r$, be k_i-dimensional lattices, then $(\Lambda^{k_1}, \ldots, \Lambda^{k_r})$ denotes the $k_1 + \cdots + k_r$-dimensional lattice where each Λ^{k_i} is embedded into the linear space generated by unit vectors $\boldsymbol{e}_j$, $k_1 + \cdots + k_{i-1} + 1 \leq j \leq k_1 + \cdots + k_i$. For instance, $(2\mathbb{Z}^1, 2\sqrt{3}\mathbb{Z}^1)$ denotes the 2-dimensional lattice with basis $(2, 0)^\intercal$, $(0, 2\sqrt{3})^\intercal$.

$n = \mathbf{6}^\star$ Start lattice: (A^2, A^2, A^2).
Here only one step with factor $1/3$ is carried out, and the resulting lattice is the well-known lattice E^6 of determinant $8\sqrt{3}$. We can use the coset representation in order to compute the distance function $\mathrm{d}_p(\boldsymbol{x}, E^6)$.

$n = \mathbf{10}^\star$ Start lattice: $(\sqrt{2}\,D^4, E^6)$.
Output: Lattice with $\det\overline{\Lambda} = 16\sqrt{3}$
$n = \mathbf{11}^\star$ Start lattice: $(A^2, A^2, A^2, A^2, A^2, 2\,\mathbb{Z}^1)$.
Output: Lattice with $\det\overline{\Lambda} = 18\sqrt{3}$
$n = \mathbf{12}^\star$ Start lattice: $(A^2, A^2, A^2, A^2, A^2, A^2)$.
Output: Lattice with $\det\overline{\Lambda} = 27$
$n = \mathbf{13}^\star$ Start lattice: $(A^2, A^2, A^2, A^2, A^2, A^2, \sqrt{12}\,\mathbb{Z}^1)$.
Output: Lattice with $\det\overline{\Lambda} = 18\sqrt{3}$
$n = \mathbf{14}^\star$ Start lattice: $(\sqrt{2}\,D^{12}, A^2)$.
Output: Lattice with $\det\overline{\Lambda} = 16\sqrt{3}$
$n = \mathbf{18}^\star$ Start lattice: $(\sqrt{2}\,D^{16}, A^2)$.
Output: Lattice with det $\det\overline{\Lambda} = 8\sqrt{3}$

Unfortunately, we could not find a suitable start lattice for getting the best known packing lattice in dimension 22. We have also done few experiments in dimension $26-31$ and in Table 4.1 are the best results/determinants we obtained so far.

n	start lattice	$\overline{\Delta}$
$\mathbf{26}^\star$	$(\sqrt{2}\,D^{25}, \sqrt{6}\,\mathbb{Z}^1)$	$\sqrt{3}$
27	$(\sqrt{2}\,D^{24}, 2\,\mathbb{Z}^3)$	2
28	$\sqrt{2}\,D^{28}$ or $(\sqrt{2}\,D^{24}, 2\,\mathbb{Z}^4)$	2
29	$(\sqrt{2}\,D^{24}, 2\,\mathbb{Z}^5)$	2
30	$\sqrt{2}\,D^{30}$ or $(\sqrt{2}\,D^{24}, 2\,\mathbb{Z}^6)$	2
31	$\sqrt{2}\,D^{31}$	$\sqrt{2}$

Table 4.1. Some results in dimensions $26-31$

In dimensions $26, 27, 28, 29, 31$ these determinants coincide with the best *laminated* packing lattices. We have also verified/double checked the packing property of the constructed lattices via the program `shvec` by Frank Vallentin [41].

We conclude with a problem (question): Can the lattice $\sqrt{2}D^n$ always be refined to a packing lattice of B^n of density $\geq 2^{-n}$?

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