INTEGRAL DECOMPOSITION OF POLYHEDRA AND SOME APPLICATIONS IN MIXED INTEGER PROGRAMMING

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Abstract. This paper addresses the question of decomposing an infinite family of rational polyhedra in an integer fashion. It is shown that there is a finite subset of this family that generates the entire family. Moreover, an integer analogue of Carathéodory's theorem carries over to this general setting. The integer decomposition of a family of polyhedra has different applications in integer and mixed integer programming. Three of them will be illuminated.

1. Introduction

This paper deals with the question of decomposing an infinite family of rational polyhedra that one associates with a fixed matrix and varying right hand side vectors into *irreducible* ones.

This problem has been studied in various fields of geometry when the right hand side vectors are arbitrary vectors. A major technique that is used in this context is the Minkowski sum of sets, i.e., the pointwise addition of points from different sets.

Our object of investigation is in fact the *integral decomposition* of polyhedra. More precisely, given a fixed matrix and an infinite family of integral right hand side vectors, we are interested in a subset of this family of integral right hand sides that generate the entire family of polyhedra with respect to an operation that is to be specified according to the concrete application.

It is proved that there always exists such a generating subset of finite cardinality. This extends naturally the result about the existence of a Hilbert basis for rational polyhedral cones. Here the family of polyhedra that one considers is just a family of points, i.e., a family of 0-dimensional polyhedra. Recall that the Hilbert basis of a rational polyhedral cone $\mathcal{C}$ is a minimal finite generating set of the integral points in $\mathcal{C}$ with respect to non-negative integral combinations. In other words, the Hilbert basis $\mathcal{H}(\mathcal{C})$ consists of a finite number of vectors $h^1, \dots, h^l \in \mathcal{C} \cap \mathbb{Z}^d \setminus \{0\}$ such that

$$\mathcal{C} \cap \mathbb{Z}^d = \left\{ \lambda_1 h^1 + \lambda_2 h^2 + \cdots + \lambda_l h^l : \lambda_i \in \mathbb{Z}_+,\ 1 \le i \le l \right\}.$$

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The existence of a Hilbert basis follows from the classical lemma of Gordan [Go1873], and it was shown by van der Corput [Cor31] that the Hilbert basis of a pointed cone C is uniquely determined by the following characterization.

Lemma 1.1 (van der Corput)**.** *The Hilbert basis of a rational polyhedral pointed cone $C \subset \mathbb{R}^n$ consists of all elements $h \in C \cap \mathbb{Z}^n$ which can not be written as $h = z^1 + z^2$ for some $z^i \in C \cap \mathbb{Z}^n \setminus \{0\}$.*

For further geometric and algorithmic properties of Hilbert bases we refer to [AWW00].

In the same vein it is shown in this paper that an integer analogue of Carathéodory's theorem that is known to hold for the integral points in a rational polyhedral cone carries over to the more general setting when one deals with families of polyhedra of arbitrary dimension with integral right hand sides.

The question of decomposing such a family of polyhedra with integral right hand sides in an integer fashion has different applications in integer and mixed integer programming. In the sequel we will illuminate three such settings that we consider relevant.

Application 1 addresses the question of whether there exists a finite certificate for testing that an infinite family of polyhedra with integral right hand sides is integral. Note that a rational polyhedron is called *integral* if each of its faces contains an integral point. To be more precise, let $A \in \mathbb{Z}^{m\times n}$ denote a given matrix and $b \in \mathbb{Z}^m$ a vector. By

$$P_b = \{\, x \in \mathbb{R}^n : Ax \leq b \,\}$$

we denote the polyhedron associated with A and b. Let $\mathcal{L}$ denote a lattice in $\mathbb{Z}^m$.

Theorem 1.1. *There exists a finite subset $\mathcal{F} \subseteq \mathcal{L}$ such that P_b is integral for all $b \in \mathcal{L}$ if and only if P_b is integral for all $b \in \mathcal{F}$.*

Theorem 1.1 follows immediately from the theory of integral decomposition of polyhedra that we will introduce in Chapter 3.

Application 2 extends the first application by considering totally dual integral systems of inequalities instead of integer polyhedra.

Definition 1.1. *Let $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$. The system of inequalities $Ax \leq b$ is called* totally dual integral (TDI) *if for every $c \in \mathbb{Z}^m$ such that $|\min\{\, b^\intercal y : A^\intercal y = c,\ y \geq 0 \,\}| < \infty$, there exists an integral vector $y^* \in \mathbb{Z}^m$, $A^\intercal y^* = c$, $y^* \geq 0$ such that $b^\intercal y^* = \min\{\, b^\intercal y : A^\intercal y = c,\, y \geq 0 \,\}$.*

Theorem 1.2. *There exists a finite subset $\mathcal{F} \subseteq \mathcal{L}$ such that $Ax \leq b$ is TDI for all $b \in \mathcal{L}$ if and only if $Ax \leq b$ is TDI for all $b \in \mathcal{F}$.*

In this sense, Theorem 1.2 shows the existence of a a finite certificate for testing that an infinite family of systems of inequalities with integral right hand sides is TDI. This is indeed a generalization of Theorem 1.1 because if $Ax \leq b$ is TDI and b is integral, then $\{\, x \in \mathbb{R}^n : Ax \leq b \,\}$ is integral, however, the converse is not true, see [EG77].

Application 3 of the method of decomposing integral polyhedra in an integer fashion is directed towards the design of a primal approach for mixed integer programming. For integral matrices $A \in \mathbb{Z}^{m\times d}$, $B \in \mathbb{Z}^{m\times n}$ and integral vectors $b \in \mathbb{Z}^m$, $\alpha \in \mathbb{Z}^d$ and $\beta \in \mathbb{Z}^n$ the mixed integer linear programming problem (MIP) is the task to determine

$$\max\{\, \alpha^\intercal x + \beta^\intercal y : A\,x + B\,y \le b,\, x \in \mathbb{Z}^d,\, y \in \mathbb{R}^n \,\}. \tag{1.1}$$

By a primal approach we mean an augmentation algorithm that starting from any feasible point of the mixed integer program moves through an improving direction to a new feasible point of the program as long as possible. Of course, the set of all improving directions for all the feasible points of even a single pure integer program may not be finite. However, in the pure integer case, one can resort to a finite generating set for the family of all improving directions for an integer program. Such a set is commonly called a *test set*. A particularly nice geometric way of defining a test set has been presented by Graver [Gra75].

Definition 1.2. *For the family of integer programs of the form* (1.1) *with* $n = 0$ *associated with a fixed matrix* A *but varying data* $b \in \mathbb{Z}^m$, $\alpha \in \mathbb{R}^d$, *the* Graver test set $\mathcal{G}(A)$ *is defined as*

$$\mathcal{G}(A) = \bigcup_{\varepsilon\in\{-1,1\}^m} \mathcal{H}\left(\left\{x \in \mathbb{R}^d : A_\varepsilon\, x \le 0\right\}\right), \tag{1.2}$$

where A_ε *arises from* A *by multiplying the* i*-th row* A *with* ε_i.

One purpose of this note is to present an analogous test set approach for mixed integer linear programming problems. Here, the situation is however significantly more complicated, because a minimal test set with respect to inclusion may not be finite as the following trivial example demonstrates.

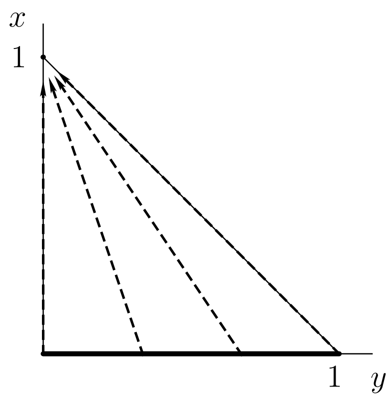

Figure 1. A mixed integer linear program with an infinite minimal test set

Example 1. We consider the mixed integer program

$$\begin{array}{llcl} \max & x & & \\ \text{s. t.} & x + y & \le & 1 \\ & -x & \le & 0 \\ & -y & \le & 0 \\ & x \in \mathbb{Z},\ y \in \mathbb{R} & & \end{array} \tag{1.3}$$

whose feasible set is the points of the unit triangle with integral x coordinate, i.e., a single point $(1,0)$ with objective value 1 and the points $(0,y)$ for $y \in [0,1]$ with objective value 0; see Figure 1. Every test set for (1.3) must contain all the points $(1,-y)$ for $y \in [0,1]$.

The way to overcome this difficulty is to resort to a finite representation of the infinite test set. This can indeed be done by applying the methods that we introduce in the context of integral decompositions of rational polyhedra. The proof of the following result will then follow.

Theorem 1.3. *There exists a finite set* $\mathcal{G}(A,B) \subset \{\, (x,\varepsilon,u)^\intercal : x \in \mathbb{Z}^d,\ \varepsilon \in \{-1,1\}^m,\ u \in \mathbb{Z}^m \,\}$*, depending only on A and B, such that for each non optimal point* $(\overline{x},\overline{y})^\intercal \in \{(x,y)^\intercal \in \mathbb{Z}^d \times \mathbb{R}^n : A\,x + B\,y \le b\}$ *of* (1.1) *there exists a* $(\widetilde{x},\varepsilon,u)^\intercal \in \mathcal{G}(A,B)$ *and a vector* $\widetilde{y} \in P_u^\varepsilon$ *satisfying*

$$\text{i) } A\,(\overline{x}+\widetilde{x}) + B\,(\overline{y}+\widetilde{y}) \le b, \quad \text{ii) } \alpha^\intercal(\overline{x}+\widetilde{x}) + \beta^\intercal(\overline{y}+\widetilde{y}) > \alpha^\intercal\overline{x} + \beta^\intercal\overline{y},$$

where $P_u^\varepsilon = \{\, y \in \mathbb{R}^n : B_\varepsilon\, y \le u \,\}$.

2. Linear decomposition of polytopes

In the following let $W \in \mathbb{Z}^{m\times n}$ be a fixed but arbitrary integral matrix with row vectors $w^i \in \mathbb{Z}^n$, $1 \le i \le m$. We assume that

$$\operatorname{pos}\{w^1,\dots,w^m\} = \mathbb{R}^n,$$

where pos denotes the positive hull. Thus for every $u \in \mathbb{R}^m$ the set $P_u = \{\, y \in \mathbb{R}^n : Wy \le u \,\}$ is a polytope. Now we are interested in the family of all non empty polytopes arising in this way and therefore we set

$$\mathcal{U}(W) = \{\, u \in \mathbb{R}^m : P_u \ne \emptyset \,\}.$$

This set has been investigated by various authors in different contexts (cf. [Grü67, p. 316], [KLS90], [McM73], [Mey74], [Smi87] and the references within). It is not hard to see that the dual set

$$\mathcal{U}^*(W) = \{\, s \in \mathbb{R}^m : s^\intercal u \ge 0, \text{ for all } u \in \mathcal{U}(W) \,\}$$

is given by

$$\mathcal{U}^*(W) = \{\, s \in \mathbb{R}^m : W^\intercal s = 0 \text{ and } s \ge 0 \,\}.$$

This shows, in particular, that $\mathcal{U}(W)$ is a rational polyhedral m-dimensional cone. Since we do not want to distinguish between a polytope P_u and its translate $t + P_u = P_{u+Wt}$ for $t \in \mathbb{R}^n$, we choose as a representative of each equivalence class P_{u+Wt}, $t \in \mathbb{R}^n$, the right hand side $\tilde{u}$ satisfying $W^\intercal\tilde{u} = 0$. In other words, $\tilde{u}$ is the orthogonal projection of u onto the orthogonal complement of the space $W\mathbb{R}^n$. Hence

$$\widetilde{\mathcal{U}}(W) = \big\{\, u \in \mathbb{R}^m : P_u \ne \emptyset \text{ and } W^\intercal u = 0 \,\big\}$$

is a rational polyhedral $(m-n)$-dimensional cone and we have

$$\mathcal{U}(W) = \widetilde{\mathcal{U}}(W) \oplus W\mathbb{R}^n.$$

Since the maximal linear subspace of $\mathcal{U}(W)$ consists of the space $W\mathbb{R}^n$ the cone $\widetilde{\mathcal{U}}(W)$ is even pointed. Moreover, the dual cone $\widetilde{\mathcal{U}}^*(W)$ of $\widetilde{\mathcal{U}}(W)$ w.r.t. the subspace $\{\, s \in \mathbb{R}^m : W^\intercal s = 0 \,\}$ reads

$$\widetilde{\mathcal{U}}^*(W) = \big\{\, s \in \mathbb{R}^m : s^\intercal u \ge 0 \text{ for all } u \in \widetilde{\mathcal{U}}(W),\ W^\intercal s = 0 \,\big\}.$$

This shows that $\widetilde{\mathcal{U}}^*(W)$ coincides with $\mathcal{U}^*(W)$. By the primal-dual relationship we obtain the following characterization of the facets of $\widetilde{\mathcal{U}}(W)$.

Remark 2.1. *For $s \in \widetilde{\mathcal{U}}^*(W)$ let W_s be the matrix consisting of all rows w^i with $s_i > 0$ and let r_s be the number of rows. Then the outer normal vector of the facets of $\widetilde{\mathcal{U}}(W)$ are given by $\{\, s \in \widetilde{\mathcal{U}}^*(W) : \operatorname{rank}(W_s) = r_s - 1 \,\}$.*

Let us mention that there is also a nice geometrical meaning of the outer normal vectors of the facets of $\widetilde{\mathcal{U}}(W)$. Namely, via the so called Minkowski's existence theorem of polytopes [Mi1897] one can assign to each outer normal vector s a $(\#W_s - 1)$-dimensional unique (up to dilations) simplex such that each polytope P_u for $u \in \widetilde{\mathcal{U}}(W)$ can be written as the Blaschke sum of at most $m - n$ of those simplices (see [Grü67, p. 331]).

Another interesting aspect of $\widetilde{\mathcal{U}}(W)$ arises from the study of *(in)decomposable* polytopes. Two polytopes $P_1, P_2 \subset \mathbb{R}^n$ are called *homothetic* if $P_1 = \rho P_2 + t$ for some $t \in \mathbb{R}^n$ and $\rho > 0$. A polytope $P \subset \mathbb{R}^n$ is called *decomposable* if two polytopes P_1 and P_2 exist with $P = P_1 + P_2$, and P_i, $i = 1, 2$, are not homothetic to P. Otherwise P is *indecomposable.* Here $P_1 + P_2$ denotes the usual Minkowski sum of two sets. Observe that, in this general setting, every point is indecomposable. Moreover a polytope P_1 is called a *summand* of a polytope P ($P_1 \prec P$) if there exist a positive scalar ρ and a polytope P_2 such that $P = \rho P_1 + P_2$.

For a polytope P_u, $u \in \mathcal{U}(W)$, each summand admits a representation as P_v for a certain $v \in \mathcal{U}(W)$. Again, since we may neglect translations, the polytope P_u, $u \in \widetilde{\mathcal{U}}(W)$, is decomposable if and only if there exist $u^1, u^2 \in \widetilde{\mathcal{U}}(W)$ such that $P_u = P_{u^1} + P_{u^2}$ and P_{u^i}, $i = 1, 2$, are not homothetic to P_u. In fact, one can even establish a stronger relation to the cone $\widetilde{\mathcal{U}}(W)$. To this end we denote for $u \in \mathcal{U}(W)$ by $\eta(u) \in \mathcal{U}(W)$ the *support vector* of the polytope P_u, i.e.,

$$\eta(u)_i = \max\left\{(w^i)^\intercal y : y \in P_u\right\}, \quad 1 \le i \le m.$$

$\eta(u)$ is the componentwise least right hand side which yields the same polytope, $P_u = P_{\eta(u)}$. One can show that (cf. e.g. [Grü67])

$$P_u = P_{u^1} + P_{u^2} \Leftrightarrow \eta(u) = \eta(u^1) + \eta(u^2) \text{ and } P_{u^1}, P_{u^2} \prec P_u. \tag{2.1}$$

Furthermore, it follows from works of McMullen [McM73] and Meyer [Mey74] that $\widetilde{\mathcal{U}}(W)_u = \{\, \eta(v) : v \in \widetilde{\mathcal{U}}(W) \text{ with } P_v \prec P_u \,\}$ is a rational polyhedral subcone of $\widetilde{\mathcal{U}}(W)$, where the extreme rays of $\widetilde{\mathcal{U}}(W)_u$ correspond to indecomposable polytopes. As a nice consequence of this construction we get by Carathéodory's theorem and (2.1)

Theorem 2.1 (McMullen, Meyer)**.** *Every polytope P_u for $u \in \widetilde{\mathcal{U}}(W)$ can be written as the Minkowski sum of at most $m - n$ indecomposable polytopes.*

Indeed, depending on certain affine dependencies of the rows of the matrix W one can give stronger bounds (cf. [Mey74], [McM73], [Smi87]). The relative interior of this cone $\widetilde{\mathcal{U}}(W)_u$ consists of all polytopes P_v which are strongly combinatorially equivalent to P_u. Recall that two polytopes

$P, Q \subset \mathbb{R}^n$ are called *strongly combinatorially equivalent* if for all $v \in \mathbb{R}^n$ the following equality holds:

$$\dim\left(\{\, y \in P : v^\intercal y = \max\{\, v^\intercal y : y \in P \,\}\,\}\right) \\ = \dim\left(\{\, y \in Q : v^\intercal y = \max\{\, v^\intercal y : y \in Q \,\}\,\}\right).$$

Since the cone $\widetilde{\mathcal{U}}(W)$ possesses finitely many strongly combinatorially non equivalent polytopes we can extend Theorem 2.1 to the cone $\mathcal{U}(W)$.

Corollary 2.1. *There exist finitely many vectors $u^1, \dots, u^k \in \mathcal{U}(W)$, corresponding to indecomposable polytopes, such that each polytope P_u for $u \in \mathcal{U}(W)$ can be written as a non-negative linear combination of at most m of the polytopes P_{u^i}.*

Proof. By the foregoing remarks and Theorem 2.1 we know that there exist finitely many $u^1, \dots, u^l \in \widetilde{\mathcal{U}}(W)$, corresponding to indecomposable polytopes, such that for each $\tilde{u} \in \widetilde{\mathcal{U}}(W)$ the polytope $P_{\tilde{u}}$ can be written as a non-negative linear combination of at most $(m-n)$ polytopes P_{u^i}, $i \in \{1, \dots, l\}$. Let us select $u^{l+1}, \dots, u^{l+n+1} \in W\mathbb{R}^n$ such that $\operatorname{pos}\{u^{l+1}, \dots, u^{l+n+1}\} = W\mathbb{R}^n$. Obviously, $P_{u^{l+i}}$ is indecomposable. Let $u \in \mathcal{U}(W)$ and let $\tilde{u} \in \widetilde{\mathcal{U}}(W)$ be its orthogonal projection onto the orthogonal complement of $W\mathbb{R}^n$. Then there exist non-negative scalars λ_i such that $u - \tilde{u} = \lambda_1 u^{l+1} + \dots + \lambda_{n+1} u^{l+n+1}$, where at least one λ_i vanishes. We have $P_u = P_{\tilde{u}} + \lambda_1 P_{u^{l+1}} + \cdots \lambda_{n+1} P_{u^{l+n+1}}$ and hence the vectors $u^1, \dots, u^{l+n+1}$ have the desired property. □

So one may say that the vectors u^i from the corollary above form a minimal generating system of $\mathcal{U}(W)$ (or of all polytopes P_u for $u \in \mathcal{U}(W)$).

3. Integral decomposition of polytopes

Here we want to study an analogous relation for polytopes P_u with an integral right hand side, i.e., $u \in \mathcal{U}(W) \cap \mathbb{Z}^m$, and with respect to non-negative integral combinations. To this end we need one more result on Hilbert bases of rational polyhedral pointed cones that we will refer to as the *weak integer Carathéodory property.*

Theorem 3.1 (Sebö, [Seb90])**.** *Each integral vector of a rational polyhedral pointed cone $\mathcal{C} \subset \mathbb{R}^n$ can be written as a non-negative integral combination of at most $2n-2$ elements of its Hilbert basis $\mathcal{H}(\mathcal{C})$.*

Moreover, it was also shown by Sebö that in the 3-dimensional case 3 elements of the Hilbert basis are sufficient. However, there does not exist a strong integral counterpart to Carathéodory's theorem since, in general, one needs more elements of the Hilbert basis than the dimension of the cone [B-W99]. In order to establish an analogous statement to Corollary 2.1 we need to express precisely what we mean by the integral decomposition of a polytope P_z, $z \in \mathcal{U}(W) \cap \mathbb{Z}^m$. In view of our application to test sets we define

Definition 3.1. *A polytope P_z, $z \in \mathcal{U}(W) \cap \mathbb{Z}^m$, is called* integral decomposable *if there exist P_{z^1}, P_{z^2} not homothetic to P_z such that $P_z = P_{z^1} + P_{z^2}$*

and $z = z^1 + z^2$, $z^i \in \mathcal{U}(W) \cap \mathbb{Z}^m$. *Otherwise* P_z *is called* integral indecomposable.

Observe that in contrast to the usual definition of indecomposability this definition does not depend on the polytope only but also on the right hand side, i.e., on the representation of the polytope. It may indeed happen that for some $z^1, z^2 \in \mathcal{U}(W) \cap \mathbb{Z}^m$ the polytopes P_{z^1} and P_{z^2} coincide, although P_{z^1} is decomposable, whereas P_{z^2} is indecomposable. Of course, the reason lies in the representation of redundant facets. These do not play any role in the non integral case, since we may always fix the representation of the polytope via the support vector. If we restrict our attention to polytopes P_z, $z \in \mathcal{U}(W) \cap \mathbb{Z}^m$, with integral support vectors, then we have the following

Corollary 3.1. *There exist finitely many vectors* $h^i \in \mathcal{U}(W) \cap \mathbb{Z}^m$, *corresponding to integral indecomposable polytopes, such that for each polytope* P_z, $z \in \mathcal{U}(W) \cap \mathbb{Z}^m$, *with* $\eta(z) = z$ *there exist* $h^{j_1}, \ldots, h^{j_{2m-2-n}}$ *and non-negative integers* $\lambda_{j_1}, \ldots, \lambda_{j_{2m-2-n}}$ *with*

$$P_z = \sum_{i=1}^{2m-2-n} \lambda_{j_i} P_{h^{j_i}} \quad \text{and} \quad z = \sum_{i=1}^{2m-2-n} \lambda_{j_i} h^{j_i}.$$

Before giving the proof we fix some notation to which we shall also refer later. Let $w_{\mathrm{c}}^1, \ldots, w_{\mathrm{c}}^n$ be the column vectors of the matrix W. Choose points $w_{\mathrm{c}}^{n+1}, \ldots, w_{\mathrm{c}}^m \in \mathbb{Z}^m$ such that the vectors $w_{\mathrm{c}}^1, \ldots, w_{\mathrm{c}}^m$ form a basis of the lattice $\mathbb{Z}^m$. Then for each $u \in \mathcal{U}$, $u = \sum_{i=1}^m \lambda_i w_{\mathrm{c}}^i$, let $\overline{u}$ be the vector given by $u = \sum_{i=n+1}^m \lambda_i w_{\mathrm{c}}^i$. We define

$$\overline{\mathcal{U}}(W) = \{\, \overline{u} : u \in \mathcal{U}(W) \,\}. \tag{3.1}$$

$\overline{\mathcal{U}}(W)$ is a rational polyhedral pointed $(m-n)$-dimensional cone. In particular, we have

$$\big(\overline{\mathcal{U}}(W) \cap \mathbb{Z}^m\big) + W\mathbb{Z}^n = \mathcal{U}(W) \cap \mathbb{Z}^m.$$

We are now ready for the proof of Corollary 3.1.

Proof of Corollary 3.1. Let $z \in \mathcal{U}(W) \cap \mathbb{Z}^m$ with $\eta(z) = z$ and thus $\overline{z} = \eta(\overline{z}) \in \mathbb{Z}^m$. As in the general case we consider the cone

$$\overline{\mathcal{U}}(W)_z = \{\, \overline{\eta(v)} : v \in \mathcal{U}(W) \text{ and } P_v \prec P_z \,\}.$$

$\overline{\mathcal{U}}(W)_z$ is a rational polyhedral pointed cone. Therefore, it possesses a unique Hilbert basis that we denote by $h^1, \ldots, h^k$. First we claim, that the polytopes P_{h^i} are integral indecomposable. For if not, then there exist $z^1, z^2 \in \mathcal{U}(W) \cap \mathbb{Z}^m \setminus \{0\}$ such that $P_{h^i} = P_{z^1} + P_{z^2}$ and $h^i = z^1 + z^2$. This yields (see (2.1))

$$z^1 + z^2 = h^i = \eta(h^i) = \eta(z^1) + \eta(z^2) \leq z^1 + z^2.$$

In other words, $\eta(z^1) = z^1$ and $\eta(z^2) = z^2$. Thus z^1, z^2 belong to the cone $\overline{\mathcal{U}}(W)_z$. This contradicts the definition of a Hilbert basis, cf. Theorem 1.1.

On account of (2.1) and Theorem (3.1), there exist h^{j_i} and non-negative integers λ_{j_i}, $1 \le i \le 2(m-n)-2$, such that

$$P_{\overline{z}} = \sum_{i=1}^{2m-2-n} \lambda_{j_i} P_{h^{j_i}} \quad \text{and} \quad \overline{z} = \sum_{i=1}^{2m-2-n} \lambda_{j_i} h^{j_i}.$$

Finally, we note that $z - \overline{z}$ can be written as a non-negative integral combination of at most n vectors from the set $w_{\mathrm{c}}^1, \dots, w_{\mathrm{c}}^n, -(w_{\mathrm{c}}^1 + \cdots + w_{\mathrm{c}}^n)$. As mentioned before, the relative interior points of the cone $\overline{\mathcal{U}}(W)_z$ correspond to polytopes which are strongly combinatorially equivalent to P_z and since there are only finitely many strongly combinatorially non equivalent polytopes P_u, $u \in \mathcal{U}(W)$, we are done. □

In the general case we are not aware of a nice geometric description of a family of indecomposable polytopes such that each polytope can be written as a non-negative integral combination of a *fixed number* of polytopes. However, a finite generating system for all P_z, $z \in \mathcal{U}(W) \cap \mathbb{Z}^m$, with the weak integer Carathéodory property can always be constructed.

Theorem 3.2. *There exist finitely many vectors $g^1, \dots, g^k \in \mathcal{U}(W) \cap \mathbb{Z}^m$ such that for each polytope P_z, $z \in \mathcal{U}(W) \cap \mathbb{Z}^m$, there exist $g^{j_1}, \dots, g^{j_{2m-2-n}}$ and non-negative integers $\lambda_{j_1}, \dots, \lambda_{j_{2m-2-n}}$ such that*

$$P_z = \sum_{i=1}^{2m-2-n} \lambda_{j_i} P_{g^{j_i}} \quad \textit{and} \quad z = \sum_{i=1}^{2m-2-n} \lambda_{j_i} g^{j_i}.$$

Proof. For an index set $I = \{i_1, \dots, i_n\}$ corresponding to linearly independent row vectors $w^{i_1}, \dots, w^{i_n}$ of the matrix W, let W_I be the regular submatrix consisting of these rows and let Π_I be the matrix representing the projection of an m-dimensional vector b onto its coordinates $b_{i_1}, \dots, b_{i_n}$. Let $I_1, \dots, I_k$ be all possible index sets of this type and let

$$M = \left(W\,(W_{I_1})^{-1}\Pi_{I_1} - E_m, \dots, W\,(W_{I_k})^{-1}\Pi_{I_k} - E_m\right)^{\mathsf{T}} \in \mathbb{R}^{k\,m \times m},$$

where E_m is the $m \times m$ identity matrix. Let $\overline{\mathcal{U}}(W)$ be the cone constructed in (3.1) and for a given orthant $\mathcal{O}$ in $\mathbb{R}^{k\,m}$ let

$$\overline{\mathcal{U}}(W)_{\mathcal{O}} = \{\, \overline{z} \in \overline{\mathcal{U}}(W) : M\overline{z} \in \mathcal{O} \,\}.$$

$\overline{\mathcal{U}}(W)_{\mathcal{O}}$ is a rational polyhedral pointed cone of dimension at most $m-n$. Next we claim,

$$P_{\overline{z}^1} + P_{\overline{z}^2} = P_{\overline{z}^1 + \overline{z}^2}, \quad \text{for } z^1, z^2 \in \overline{\mathcal{U}}(W)_{\mathcal{O}}. \tag{3.2}$$

To this end let $\mathcal{I}_i$ be the set of all index sets I satisfying

$$W(W_I)^{-1}\Pi_I \overline{z}^i - \overline{z}^i \le 0,$$

and for an arbitrary index set let $v_i^I = (W_I)^{-1}\Pi_I \overline{z}^i$, $i = 1, 2$. In particular we have

$$P_{\overline{z}^i} = \operatorname{conv}\left\{\, v_i^I : I \in \mathcal{I}_i \,\right\}.$$

Let I be an index set such that $v^I = (W_I)^{-1}\Pi_I(\overline{z}^1 + \overline{z}^2)$ is a vertex of the polytope $P_{\overline{z}^1+\overline{z}^2}$. Of course, we have $v^I = v_1^I + v_2^I$ and since the vectors $W(v_1^I), W(v_2^I)$ lie in the same orthant $\mathcal{O}$ we get $I \in \mathcal{I}_1 \cap \mathcal{I}_2$. Thus

$$P_{\overline{z}^1+\overline{z}^2} = \operatorname{conv}\{\, v_1^I + v_2^I : I \in \mathcal{I}_1 \cap \mathcal{I}_2 \,\} \subset P_{\overline{z}^1} + P_{\overline{z}^2} \subset P_{\overline{z}^1+\overline{z}^2},$$

which shows (3.2).

Now let $g^1, \dots, g^l$ be the Hilbert basis of the cone $\overline{\mathcal{U}}(W)_{\mathcal{O}}$. Then by Theorem 3.1 and (3.2) we know that each $P_{\overline{z}}$ for $\overline{z} \in \overline{\mathcal{U}}(W)_{\mathcal{O}}$ can be written as a non negative integral combination of at most $2(m-n)-2$ polytopes of the form P_{g^i} and the corresponding right hand sides sum up. Finally, let $g^1, \dots, g^k$ be the union of all Hilbert bases with respect to all orthants. Then these vectors form a generating system satisfying the requirements of the theorem for all $z \in \overline{\mathcal{U}}(W) \cap \mathbb{Z}^m$. Finally, as in the proof of Corollary 3.1, we may conclude that $g^1, \dots, g^k, w^1, \dots, w^n, -(w^1 + \dots + w^n)$ have the desired property for all $z \in \mathcal{U}(W) \cap \mathbb{Z}^m$. □

Next we want to consider a slight variant of the above problem. To this end let $W \in \mathbb{Z}^{m\times n}$ as above and let $\mathcal{L} \subset \mathbb{Z}^m$ be a d-dimensional lattice. Then we are interested in the set

$$\mathcal{U}(\mathcal{L}, W) = \{\, l \in \mathcal{L} : P_l \neq \emptyset \,\}. \tag{3.3}$$

In other words we restrict the class of all possible right hand sides to a sublattice. It is not hard to see that also in this case we can easily find via the methods described in the proof of Theorem 3.2 a finite generating system. More precisely we have

Remark 3.1. *There exists a finite set of lattice points $\mathcal{H}(\mathcal{L}, W) \subset \mathcal{U}(\mathcal{L}, W)$ such that for each $l \in \mathcal{U}(\mathcal{L}, W)$ there exists at most $2m-2-n$ elements $h^1, \dots, h^k \in \mathcal{H}(\mathcal{L}, W)$ and positive integers $\lambda_1, \dots, \lambda_k$ such that*

$$P_l = \lambda_1 P_{h^1} + \dots + \lambda_k P_{h^k} \quad \text{and} \quad l = \lambda_1 h^1 + \dots + \lambda_k h^k.$$

4. Proofs of the Theorems

The proof of Theorem 1.1 is an immediate consequences of the previous section.

Proof of Theorem 1.1. Let $\mathcal{H}(\mathcal{L}, W)$ as in Remark 3.1. Then we obviously have that P_l is integral for all $l \in \mathcal{L}$ if and only if P_h is integral for all $h \in \mathcal{H}(\mathcal{L}, W)$. □

For the proof of Theorem 1.2 we need a well-known characterization of TDI-systems due to Giles and Pulleyblank [GP79].

Theorem 4.1. *Let $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$. The system $Ax \le b$ is TDI if and only if for every minimal face of P_b the set of row vectors that are tight at this face determine a Hilbert basis of the cone that they generate.*

Proof of Theorem 1.2. Thereom 4.1 implies that for two strongly combinatorially equivalent polytopes P_{l^1} and P_{l^2}, $l^i \in \mathbb{Z}^m \cap \mathcal{L}$ it holds, that P_{l^1} is TDI if and only if P_{l^2} is TDI. Since there are only finitely many strongly combinatorially non equivalent polytopes of the type P_l, $l \in \mathcal{L}$, we have a finite certificate to test whether all polytopes P_l, $l \in \mathcal{L}$, are TDI. $\square$

Now we come to the proof of the main Theorem. We consider the mixed integer linear program as described in (1.1). Let $a^1, \dots, a^m$ and $b^1, \dots, b^m$ be the row vectors of the matrix A and B. For $\varepsilon \in \{-1,1\}^m$ let A_ε and B_ε be the matrices with row vectors $\varepsilon_1 a^1, \dots, \varepsilon_m a^m$ and $\varepsilon_1 b^1, \dots, \varepsilon_m b^m$, respectively. We may assume that $\operatorname{rank}(A) = d$ by replacing the columns of A with a basis for the lattice $A\mathbb{Z}^d$.

Proof of Theorem 1.3. For each $\varepsilon \in \{-1,1\}^m$ let

$$\bar{A}_{-\varepsilon} = \begin{pmatrix} A_{-\varepsilon} & & \\ & -E_n & \\ & & E_n \end{pmatrix} \quad \text{and} \quad \bar{B}_\varepsilon = \begin{pmatrix} B_\varepsilon \\ -E_n \\ E_n \end{pmatrix},$$

where E_n is the $n \times n$ identity matrix, and let $\mathcal{H}(\bar{A}_{-\varepsilon}, \bar{B}_\varepsilon) \subset \mathbb{Z}^d$ be a finite generating set of all non empty polytopes of the form

$$\begin{aligned} P^{\varepsilon}_{\bar{A}_{-\varepsilon}(x,\mathfrak{l},\mathfrak{u})^\intercal} &= \{\, y \in \mathbb{R}^n : \bar{B}_\varepsilon y \le \bar{A}_{-\varepsilon}(x,\mathfrak{l},\mathfrak{u})^\intercal \,\} \\ &= \{\, y \in \mathbb{R}^n : B_\varepsilon y \le A_{-\varepsilon}x,\ \mathfrak{l} \le y \le \mathfrak{u} \,\} \end{aligned}$$

for arbitrary $x \in \mathbb{Z}^d$ and lower and upper bounds $\mathfrak{l}, \mathfrak{u} \in \mathbb{Z}^n$ as defined in Remark 3.1. Note that $P^{\varepsilon}_{\bar{A}_{-\varepsilon}(x,\mathfrak{l},\mathfrak{u})^\intercal} \subset P^{\varepsilon}_{A_{-\varepsilon}x}$ for all $\mathfrak{l}, \mathfrak{u} \in \mathbb{Z}^n$, where $P^{\varepsilon}_{A_{-\varepsilon}x}$ is the polyhedron defined in Theorem 1.3. Finally we set

$$\mathcal{G}(A,B) = \bigcup_{\varepsilon \in \{-1,1\}^m} \{\, (x, \varepsilon, u) : x \in \mathcal{H}(\bar{A}_{-\varepsilon}, \bar{B}_\varepsilon),\ u = A_{-\varepsilon}x \,\}$$

and claim that this set has the properties required in the theorem. To this end, let $(\bar{x}, \bar{y})^\intercal$ be a feasible but non-optimal solution of (MIP) and let $(x^*, y^*)^\intercal$ be an optimal solution of the problem. Now let I_1, I_2 be a partition of the indices $\{1, \dots, m\}$ such that

$$\begin{aligned} (a^i)^\intercal \bar{x} + (b^i)^\intercal \bar{y} &\le (a^i)^\intercal x^* + (b^i)^\intercal y^*, \quad i \in I_1, \\ (a^i)^\intercal \bar{x} + (b^i)^\intercal \bar{y} &\ge (a^i)^\intercal x^* + (b^i)^\intercal y^*, \quad i \in I_2. \end{aligned}$$

Define $\varepsilon \in \{-1,1\}^m$ according to $\varepsilon_i = -1$ for $i \in I_1$ and $\varepsilon_i = 1$ for $i \in I_2$. Then we have

$$B_\varepsilon(y^* - \bar{y}) \le A_{-\varepsilon}(x^* - \bar{x})$$

and thus $(x^* - \bar{x}, \mathfrak{l}, \mathfrak{u})^\intercal \in \mathcal{U}(\bar{A}_{-\varepsilon}, \bar{B}_\varepsilon)$ for some $\mathfrak{l}, \mathfrak{u} \in \mathbb{Z}^n$. Hence there exist some $(g^1, \mathfrak{l}^1, \mathfrak{u}^1)^\intercal, \dots, (g^k, \mathfrak{l}^k, \mathfrak{u}^k) \in \mathcal{H}(\bar{A}_{-\varepsilon}, \bar{B}_\varepsilon)$ and positive integers $\lambda_1, \dots, \lambda_k$ such that

$$\begin{pmatrix} x^* - \bar{x} \\ \mathfrak{l} \\ \mathfrak{u} \end{pmatrix} = \sum_{j=1}^k \lambda_i \begin{pmatrix} g^j \\ \mathfrak{l}^j \\ \mathfrak{u}^j \end{pmatrix} \quad \text{and} \quad P^{\varepsilon}_{\bar{A}_{-\varepsilon}(x^*-\bar{x},\mathfrak{l},\mathfrak{u})^\intercal} = \sum_{j=1}^k \lambda_j P^{\varepsilon}_{\bar{A}_{-\varepsilon}(g^j,\mathfrak{l}^j,\mathfrak{u}^j)^\intercal}.$$

Thus there exist $y^j \in P^{\varepsilon}_{A_{-\varepsilon}g^j}$ such that

$$\begin{pmatrix} x^* - \bar{x} \\ y^* - \bar{y} \end{pmatrix} = \sum_{j=1}^k \lambda_j \begin{pmatrix} g^j \\ y^j \end{pmatrix}.$$

Now we show that for each $j \in \{1,\ldots,k\}$ the vector $(\bar{x}+g^j, \bar{y}+y^j)$ is feasible. For $i \in \{1,\ldots,m\}$ we have

$$b_i \geq (a^i, b^i)\begin{pmatrix} x^* \\ y^* \end{pmatrix} = (a^i, b^i)\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} + \sum_{j=1}^{k} \lambda_j (a^i, b^i)\begin{pmatrix} g^j \\ y^j \end{pmatrix}. \qquad (4.1)$$

By definition we have

$$(a^i, b^i)\begin{pmatrix} g^j \\ y^j \end{pmatrix} \geq 0 \quad \text{for } i \in I_1 \quad \text{and} \quad (a^i, b^i)\begin{pmatrix} g^j \\ y^j \end{pmatrix} \leq 0 \quad \text{for } i \in I_2.$$

Since all the scalars in (4.1) are positive, every vector $(\bar{x}+g^j, \bar{y}+y^j)$ for $j \in \{1,\ldots,k\}$ is feasible. Finally we observe that $\alpha^\intercal x^* + \beta^\intercal y^* > \alpha^\intercal \bar{x} + \beta^\intercal \bar{y}$. Thus there exists at least one index j such that

$$\alpha^\intercal g^j + \beta^\intercal y^j > 0.$$

This finally shows that the vector $(g^j, y^j)^\intercal$ is both applicable at $(\bar{x}, \bar{y})^\intercal$ and improving. □

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