APPROXIMATING THE VOLUME OF CONVEX BODIES

U. Betke and M. Henk

Abstract. It is a well known fact that for every polynomial time algorithm which gives an upper bound $\overline{V}(K)$ and a lower bound $\underline{V}(K)$ for the volume of a convex set $K \subset E^d$, the ratio $\overline{V}(K)/\underline{V}(K)$ is at least $(cd/\log d)^d$. Here we describe an algorithm which gives for $\epsilon > 0$ in polynomial time an upper and lower bound with the property $\overline{V}(K)/\underline{V}(K) \leq d!(1+\epsilon)^d$.

1. Introduction

Since it is hard to compute the volume of convex bodies in high dimensions one might ask for polynomial deterministic algorithms which give an upper bound $\overline{V}(K)$ and a lower bound $\underline{V}(K)$ for the volume $V(K)$ of a d-dimensional convex body K. Indeed such algorithms were given by Lovsz (cf. e.g. [GLS], pp. 122) with a ratio $\overline{V}(K)/\underline{V}(K) \leq d^{3d/2}$ and Applegate&Kannan [AK] — quoted by [DF] — with $\overline{V}(K)/\underline{V}(K) \leq 2^d \cdot d!(1+1/d^2)^d$. Here we give an algorithm which computes for any $\epsilon > 0$ in polynomial time $\overline{V}(K), \underline{V}(K)$ such that $\overline{V}(K)/\underline{V}(K) \leq d!(1+\epsilon)^d$.

The bounds given above appear to be very weak. But Brny&Fredi [BF] showed — see also Elekes [E] — that for any polynomial deterministic algorithm the ratio $\overline{V}(K)/\underline{V}(K)$ is at least $(cd/\log d)^d = d^{d(1-o(1))}$ for some constant c independent of d and we have by Stirling's formula $d! = \sqrt{2\pi d}(d/e)^d(1-o(1))$.

Algorithms of this kind are not only of interest in their own sake as the newly devised randomized algorithms for computing the volume need an approximative algorithm as a starting point (cf. e.g. [DF], [DFK]).

Thus it seems to be worth while not only to show polynomiality but to compare the running times somewhat closer. Here it turns out that the running time — in a sense made more precise below — of our algorithm is $1/d^2$ of the algorithm of Applegate&Kannan.

To make our ideas more precise we need some notation and we must state how the convex bodies are given: Let E^d denote the d-dimensional euclidean space and the set of all convex bodies — compact convex sets — in E^d is denoted by $\mathcal{K}^d$. e^i denotes the i-th canonical unit vector and for a vector $x \in E^d$ the i-th coordinate is denoted by x_i. Further $\|\cdot\|$ denotes the euclidean norm, $\|\cdot\|_\infty$ denotes the maximum norm and for $y^1, \dots, y^d \in E^d$ the determinant of the matrix with column vectors $y^1, \dots, y^d$ is denoted by $\det(y^1, \dots, y^d)$. Finally $L(y^1, \dots, y^i)$ denotes for $y^1, \dots, y^i \in E^d$ the linear space spanned by $y^1, \dots, y^i$.

For the description of a convex body K we adopt the oracle model as studied in detail in Grtschel, Lovsz and Schrijver[GLS]. This means that a convex body

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$K \in \mathcal{K}^d$ is given by a so called weak membership oracle `WMEMO`. This is a black box with the following properties:

> Given a point $y \in \mathbb{Q}^d$ and a positive rational number $\epsilon \in \mathbb{Q}$, the oracle answers that $y \in S(K, \epsilon)$ or that $y \notin S(K, -\epsilon)$,

where $S(K, \epsilon) = \{x \in E^d \mid \|x - y\| \leq \epsilon \text{ for some } y \in \mathcal{K}^d\}$ and $S(K, -\epsilon) = \{x \in E^d \mid S(x, \epsilon) \in \mathcal{K}^d\}$.

Moreover we must make the assumption that we have the following information about the convex body K, given by a `WMEMO`: Two rational numbers $R, r > 0$ and a point $a \in \mathbb{Q}^d$ with $S(a, r) \subset K \subset S(0, R)$. For simplification we may assume

$$S(0, r) \subset K \subset S(0, R). \tag{1.1}$$

By using a special version of the ellipsoid method Yudin&Nemirovskiĭ ([YN], [GLS], pp. 107) showed that there exist oracle-polynomial time algorithms that solve the following problems for a convex body given by `WMEMO` and (1.1) :

(1) The weak violation problem (`WVIOL`):

> Given a vector $c \in \mathbb{Q}^d$ and rational numbers γ, ϵ, $\epsilon > 0$, either
> assert that $c^T x \leq \gamma + \epsilon$ for all $x \in S(K, -\epsilon)$, or
> find a vector $y \in S(K, \epsilon)$ with $c^T y \geq \gamma - \epsilon$.

(2) The weak optimization problem (`WOPT`):

> Given a vector $c \in \mathbb{Q}^d$ and a rational number $\epsilon > 0$, either
> find $y \in \mathbb{Q}^d$ such that $y \in S(K, \epsilon)$ and $c^T x \leq c^T y + \epsilon$ for all $x \in S(K, -\epsilon)$, or
> assert that $S(K, -\epsilon)$ is empty.

The result of Applegate&Kannan [AK] can now be stated as follows: Given a convex body $K \in \mathcal{K}^d$ by a `WMEMO` and (1.1). Then we can find a parallelepiped P and a simplex S, such that $S \subset K \subset P$ and $V(P)/V(S) \leq d! 2^d (1 + 1/d^2)^d$. Beside some elementary matrix operations the running time of this algorithm is dominated by at most $2d^3 \ln(2dR/r)$ calls of the `WVIOL`.

Briefly our algorithm can be described in the following way: We construct a parallelepiped which contains the given convex body K and a polytope which is contained in K such that the volume of the parallelepiped is not greater than $d!$ times the volume of the inscribed polytope. To do this we need $2d$ calls of the `WOPT`. Since we have no exact arithmetic our main result is

Theorem 1. *There exists an oracle-polynomial time algorithm that, for a convex body $K \in \mathcal{K}^d$ given by a* `WMEMO` *and* (1.1) *and for every $\epsilon > 0$, computes by $2d$ calls of the* `WOPT` *an upper bound $\overline{V}(K)$ and a lower bound $\underline{V}(K)$ of the volume of K such that*

$$\frac{\overline{V}(K)}{\underline{V}(K)} \leq d! \cdot (1 + \epsilon)^d.$$

Let us remark, that if we want to compute for a convex body $K \in \mathcal{K}^d$ given by a `WMEMO` and (1.1) nontrivial upper and lower bounds of the volume by using the `WOPT` we need at least $d + 1$ calls of the `WOPT`: After d calls of the `WOPT` we have the information, that K is contained in some unbounded polyhedron and contains

d points, which lie in a suitable affine hyperplane. Together with (1.1) we only get upper and lower bounds which depend on the input data R, r. From this point of view our algorithm is best possible up to a factor 2.

Further the running time of our algorithm — measured in the number of calls of the `WVIOL` — is approximately $1/d^2$ of that of APPLEGATE&KANNAN as pointed out in the third part of this paper where the proof of Theorem 1. is given. In the second part we describe our algorithm in a more geometrical form. From this presentation we deduce our basic theoretical result (Theorem 2.). As a theoretical application of our algorithm we get an inequality connecting the volume of a convex body and certain successive diameters and widths. This result is indicated at the end of the second part and is a special case of a series of inequalities concerning successive diameters and widths, which are described in more detail in [BH].

2. The Algorithm

Geometrical version. *Let $K \in \mathcal{K}^d$.*

(1) *let $c^1 \in E^d$ and $i = 1$;*
(2) *find $\overline{z}^i, \underline{z}^i \in K$ such that for all $x \in K$ holds*
$$(c^i)^T \underline{z}^i \le (c^i)^T x \le (c^i)^T \overline{z}^i;$$
(3) *let $y^i = \overline{z}^i - \underline{z}^i$;*
(4) *if $(i = d)$ then STOP;*
(5) *find $c^{i+1} \in E^d$ such that c^{i+1} is orthogonal to $L(y^1, \ldots, y^i)$;*
(6) *let $i = i + 1$;*
(7) *GOTO* (2).

Theorem 2. *With the notation above we have*

$$\frac{|\det(y^1, \ldots, y^d)|}{d!} \le V(K) \le |\det(y^1, \ldots, y^d)|.$$

Proof. Let P be the parallelepiped given by $P = \{x \in E^d \mid (c^i)^T \underline{z}^i \le (c^i)^T x \le (c^i)^T \overline{z}^i,\ 1 \le i \le d\}$ and C the polytope with vertices $\underline{z}^1, \overline{z}^1, \ldots, \underline{z}^d, \overline{z}^d$. We obviously have $C \subset K \subset P$ and in the following we shall prove

$$V(P) = |\det(y^1, \ldots, y^d)| \quad \text{and} \quad V(C) \ge |\det(y^1, \ldots, y^d)|/d!. \tag{2.1}$$

This will be done by induction with respect to the dimension. For $d = 1$ (2.1) is trivial. Hence we may assume $d \ge 2$. Let $H = \{x \in E^d \mid (y^1)^T x = 0\}$ and let $\overline{z}^i_p, \underline{z}^i_p$ be the images of the points $\overline{z}^i, \underline{z}^i$ under the orthogonal projection onto H, $1 \le i \le d$. Application of the STEINER symmetrization ([BoF], pp.69) to P and C with respect to the hyperplane H gives convex bodies P_S, C_S with $V(C_S) = V(C)$ and $V(P_S) = V(P)$.

By definition of this symmetrization C_S contains the polytope with vertices $\overline{z}^1_p + \frac{1}{2}y^1, \overline{z}^1_p - \frac{1}{2}y^1, \overline{z}^2_p, \underline{z}^2_p, \ldots, \overline{z}^d_p, \underline{z}^d_p$ and hence we have

$$V(C) \ge \frac{\|y^1\|}{d} \cdot V(\overline{C}; H), \tag{2.2}$$

where $\overline{C}$ is the polytope with vertices $\overline{z}_p^2, \underline{z}_p^2, \dots, \overline{z}_p^d, \underline{z}_p^d$ and $V(\overline{C}; H)$ denotes the volume of $\overline{C}$ with respect to the euclidean space H. On account of the choice of the directions c^i we have $P_S = \{x \in E^d \mid -\frac{\|y^1\|^2}{2} \le (y^1)^T x \le \frac{\|y^1\|^2}{2},\ (c^i)^T \underline{z}_p^i \le (c^i)^T x \le (c^i)^T \overline{z}_p^i,\ 2 \le i \le d\}$ and thereby

$$V(P) = \|y^1\| \cdot V(\overline{P}; H), \tag{2.3}$$

with $\overline{P} = \{x \in H \mid (c^i)^T \underline{z}_p^i \le (c^i)^T x \le (c^i)^T \overline{z}_p^i,\ 2 \le i \le d\}$. Now, the situation for $\overline{P}, \overline{C}$ in the space H is the same as for P, C and hence the assertion follows from (2.2) and (2.3) by using the induction hypothesis. □

Remark. A first way to choose the directions c^i from a theoretical point of view is as follows: Choose the c^i in (1), (5) such that the breadth in direction c^i becomes minimal (maximal). If we do this it is easy to see that we can find $\overline{z}^i, \underline{z}^i$ such that $L(c^1, \dots, c^i) = L(y^1, \dots, y^i)$. Further $\|\overline{z}^1 - \underline{z}^1\|$ gives the width (diameter) of the convex body and for $i = 2, \dots, d$ the length of the projections of $\overline{z}^i - \underline{z}^i$ onto the orthogonal complement of $L(y^1, \dots, y^i)$ gives the width (diameter) of the projection of K onto this space. Thus we obtain upper and lower bounds for the volume with respect to the product of 'iterated' widths (diameters). This essentially proves the main theorem in [BH] for the case of projections.

3. Proof of Theorem 1.

First we state the algorithm in its computational form.

Input: *A rational number $\epsilon > 0$ and a convex body $K \in \mathcal{K}^d$ given by a* `WMEMO` *and* (1.1)

Output: *An upper bound $\overline{V}(K)$ and a lower bound $\underline{V}(K)$ of the volume of K with the property $\overline{V}(K)/\underline{V}(K) \le d!(1+\epsilon)^d$.*

[1] *let* $\delta := \min\{\frac{r}{3}, \frac{\epsilon r}{6+\epsilon}\}$, $\alpha := \frac{r}{r+\delta}$ *and* $\beta := \frac{r}{r-\delta}\left(1 + \frac{\delta}{r-2\delta}\right)$;
[2] *let* $c^1 := e^1$ *and* $i := 1$;
[3] *find* $\overline{z}^i, \underline{z}^i \in S(K, \delta)$ *such that for all* $x \in S(K, -\delta)$ *holds*
$$(c^i)^T \underline{z}^i - \delta \le (c^i)^T x \le (c^i)^T \overline{z}^i + \delta;$$
[4] *let* $y^i := \overline{z}^i - \underline{z}^i$;
[5] *if* $(i = d)$ *then*
$\underline{V}(K) := \alpha^d |\det(y^1, \dots, y^d)|/d!$ *and* $\overline{V}(K) := \beta^d |\det(y^1, \dots, y^d)|$;
STOP.
[6] *find* $c^{i+1} \in \mathbb{Q}^d$ *such that* c^{i+1} *is orthogonal to* $L(y^1, \dots, y^i)$ *and* $\|c^{i+1}\| \ge 1$;
[7] *let* $i := i + 1$;
[8] *GOTO* [3];

Proof of Theorem 1. First we study the correctness of the algorithm above. Since $S(0, r) \subset K$ we have by simple geometric arguments $\frac{r}{r+\delta} x \in K$ for all $x \in S(K, \delta)$ and $\frac{r-\delta}{r} x \in S(K, -\delta)$ for all $x \in K$. Hence the polytope with vertices $\alpha\overline{z}^1, \alpha\underline{z}^1, \dots, \alpha\overline{z}^d, \alpha\underline{z}^d$ is contained in K. From (2.1) follows

$$V(K) \ge \frac{\alpha^d}{d!} \cdot |\det(y^1, \dots, y^d)|. \tag{3.1}$$

On the other hand we have for all $x \in K$ the relations

$$\frac{r}{r-\delta}((c^i)^T \underline{z}^i - \delta) \leq (c^i)^T x \leq \frac{r}{r-\delta}((c^i)^T \overline{z}^i + \delta), \quad 1 \leq i \leq d.$$

Since $\|c^i\| \geq 1$ and $S(0, r-\delta) \subset S(K, -\delta)$ we have $(c^i)^T \overline{z} + \delta \geq r - \delta$ and $(c^i)^T \underline{z} - \delta \leq \delta - r$ and hence the convex body K is contained in the parallelepiped $\{x \in E^d \mid (c^i)^T \beta \underline{z}^i \leq (c^i)^T x \leq (c^i)^T \beta \overline{z}^i,\ 1 \leq i \leq d\}$. Again, from (2.1) we get

$$V(K) \leq \beta^d \cdot |\det(y^1, \ldots, y^d)|. \tag{3.2}$$

On account of the choice of δ we have by (3.1) and (3.2) the bound $\overline{V}(K)/\underline{V}(K) \leq d!(1+\epsilon)^d$.

Next we consider the running time of the algorithm. To this end let $\langle\rangle$ denote the numbers of bits needed to write down a rational object ([GLS], pp. 30). The size of the input of the algorithm is $\langle K, \epsilon \rangle = d + \langle r \rangle + \langle R \rangle + \langle \epsilon \rangle$. Step [3] of the algorithm can be done with the `WOPT` in oracle polynomial time with respect to the input size $d + \langle r \rangle + \langle R \rangle + \langle \delta \rangle + \langle c^i \rangle$. The size of the output of the `WOPT` oracle depends on the precision needed by the `WOPT` to carry out its arithmetic operations. As pointed out in [GLS] the number of binary digits which are needed by the `WOPT` is a polynomial in $d + \langle r \rangle + \langle R \rangle + \langle \delta \rangle$ and hence a polynomial in $\langle K, \epsilon \rangle$. This means that all the calculated points $\overline{z}^i, \underline{z}^i$ are of a fixed size and by using the well-known Gaussian elemination we can find in polynomial time with respect to $\langle K, \epsilon \rangle$ a vector c which is orthogonal to $L(y^1, \ldots, y^i)$. In particular the size of c is bounded by a polynomial in $\langle K, \epsilon \rangle$. If we use a suitable normalization to get $\|c\| \geq 1$, we see that we can find appropriate directions c^i in polynomial time. Since the sizes of these directions are bounded by a polynomial in $\langle K, \epsilon \rangle$ the running time of the `WOPT` is also bounded by a polynomial in $\langle K, \epsilon \rangle$. So we have an oracle polynomial time algorithm. □

Remarks.

(1) Using binary search one can easily see that the `WOPT` in Step [3] of our algorithm can be solved by at most $\log_2(3\|c^i\|R/\delta)$ calls of the `WVIOL`. Hence, if we take the special normalization $\|c^i\|_\infty = 1$ we obtain that the running time of our algorithm is dominated by at most

$$2d\log_2(3\sqrt{d}R/r) + 2d\log_2(6/\epsilon + 1)$$

calls of the `WVIOL`. This shows that the running time of our algorithm — measured in the number of calls of the `WVIOL` — is approximately $1/d^2$ of the running time of the algorithm of Applegate&Kannan.

(2) If we take, for example, as direction c^{i+1}, $1 \leq i \leq d-1$, the vector which is orthogonal to $L(y^1, \ldots, y^i, e^{i+2}, \ldots, e^d)$ and satiesfies $c^{i+1}_{i+1} = 1$, we do not need to compute $\det(y^1, \ldots, y^d)$, since

$$|\det(y^1, \ldots, y^d)| = \textstyle\prod_{i=1}^d (c^i)^T y^i.$$

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Mathematisches Institut, Universität Siegen, Hölderlinstrasse 3, D-W-5900 Siegen, Federal Republic of Germany.

E-mail address: `henk@hrz.uni-siegen.dbp.de`