Errata - Friz, Victoir: Multidimensional stochastic processes as rough paths

Last modified: 16. February 2011

Errata Chapter 5.2

p 81. line 18. There is a small gap in the estimate. The left hand side [of line 18] must include (since $t \notin D$ in general!) an additional term $(*)$;

$$\sum_{t_i \in D_1 = (s = t_0 < t_1 < \cdots < t_{k-1})} d\left(x_{t_i}, x_{t_{i+1}}\right)^p + (*) + \omega_{x,p} t, t+h) > \omega_{x,p}(s, t+h) - \varepsilon$$

where $(*) = d\left(x_{t_k}, x_{t_{k+1}}\right)^p = d\left(x_{t_k}, x_t\right)^p + d\left(x_{t_k}, x_{t_{k+1}}\right)^p - d\left(x_{t_k}, x_t\right)^p$. Hence,

$$\omega_{x,p}(s,t) + \underbrace{d\left(x_{t_k}, x_{t_{k+1}}\right)^p - d\left(x_{t_k}, x_t\right)^p}_{\to 0 \text{ as } h \to 0} + \omega_{x,p}(t, t+h) > \omega_{x,p}(s, t+h) - \varepsilon$$

(since $t_{k+1} \in [t, t+h] \to t$ as $h \to 0$, and x is (uniformly) continuous on $[0,T]$). The rest of the argument is unchanged. (We also note that the proof that $\omega(t, t+) = 0$ can be a bit simplified; e.g. along the "outer continuity" argument in [2, page 12].)

p 91. line 1, include the word "with "after conclude

Errata Chapter 5.5

p 105. **Definition 5.50**. We need to modify the definition of $|f|_{p\text{-var}}$ to[1]

$$|f|_{p\text{-var};[s,t]\times[u,v]} := \sup_{\Pi \in \mathcal{P}([s,t]\times[u,v])} \left(\sum_{A \in \Pi} |f(A)|^p \right)^{1/p};$$

a **partition** Π of a rectangle $R \subset [0,T]^2$ is a finite set of (closed) rectangles, essentially disjoint**,** whose union is R; the family of all such partitions is denoted by $\mathcal{P}(R)$. We then maintain the definition that $C^{p\text{-var}}\left([0,T]^2, \mathbb{R}^d\right)$ denotes the space of continuous f with $|f|_{p\text{-var};[0,T]^2} < \infty$ and say that any such f has finite **controlled p-variation**. Indeed, lemma 5.52 (which is correct with

[1] Recall that $f(A)$ is the **rectangular increment** of $A = ((a,b),(c,d)) \in \Delta_T \times \Delta_T$,

$$f\begin{pmatrix} a,b \\ c,d \end{pmatrix} := f\begin{pmatrix} b \\ d \end{pmatrix} + f\begin{pmatrix} a \\ c \end{pmatrix} - f\begin{pmatrix} a \\ d \end{pmatrix} - f\begin{pmatrix} b \\ c \end{pmatrix},$$

and we regard A as (closed) **rectangle** $A \subset [0,T]^2$,

$$A := \begin{pmatrix} a,b \\ c,d \end{pmatrix} := [a,b] \times [c,d].$$

If $a = b$ or $c = d$ we call A degenerate; recall also that two rectangles are called **essentially disjoint** if their intersection is empty or degenerate.

this modified definition, see below) asserts that $\omega(R) := |f|^p_{p\text{-var};R}$ is a 2D control function such that (obviously)

$$\forall R \subset [0,T]^2 : |f(R)|^p \leq \omega(R).$$

Any continuous f which satisfies the above estimate for some 2D control ω is said to have finite p-variation **controlled** (equivalently: **dominated**) by ω; this is a quantitative way of saying that $f \in C^{p\text{-var}}\left([0,T]^2, \mathbb{R}^d\right)$ since super-addivity immediately gives

$$\forall R \subset [0,T]^2 : |f(R)|^p \leq \omega(R) \implies |f|_{p\text{-var};R} \leq \omega(R);$$

cf. part (ii) of the corrected lemma 5.52 below. We remark that the difference between this definition of controlled p-variation and our original one is that, in our original definition, the supremum is restricted to **grid-like partitions**,

$$\left\{ \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} : 1 \leq i \leq m, 1 \leq j \leq n \right\},$$

where $D = (t_i : 1 \leq i \leq m) \in \mathcal{D}([s,t])$ and $D' = \left(t'_j : 1 \leq j \leq n\right) \in \mathcal{D}([u,v])$; i.e we consider continuous f such that[2]

$$V_p(f;[s,t]\times[u,v]) := \left(\sup_{(t_i)\in\mathcal{D}([s,t]),\left(t'_j\right)\in\mathcal{D}([u,v])} \sum_{i,j} \left| f\begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right|^p \right)^{\frac{1}{p}} < \infty.$$

Clearly, not every partition is grid-like (consider e.g. $[0,2]^2 = [0,1]^2 \cup [1,2]\times[0,1] \cup [0,2]\times[1,2]$) hence

$$\forall R \subset [0,T]^2 : V_p(f;R) \leq |f|_{p\text{-var};R}.$$

The trouble is that $V_p(f;\cdot)^p$ is not super-additive in 2D sense[3]*, hence not a 2D control, whereas $|f|^p_{p\text{-var};\cdot}$ based on all partitions does yield a 2D control; hence our modified definition 5.50.* Even so, the class of such functions remains important and we say that any f with $V_p\left(f;[0,T]^2\right) < \infty$ has **finite p-variation**. It is worth noting that this distinction is not seen when $p = 1$ (the short proof of this [2] is based on the idea that further refining of a partition to a grid-like partition can only increase the 1-variation; this is false for p-variation, $p > 1$), nor in the 1D case of course, and we are dealing with a phenomena specific to higher dimensional p-variation with $p > 1$. That said, it is possible to show ([2] for full details) that

$$\forall R \subset [0,T]^2 : |f|_{(p+\varepsilon)\text{-var};R} \leq c(p,\varepsilon) V_p(f;R)$$

so that the two notions of p-variations (controlled vs. genuine) are "ε-close".

p 105. **Definition 5.51** replace "which is super-additive in the sense that .... $\omega(R_1)+\omega(R_2) \leq \omega(R)$" by the more convenient "which is super-additive in the sense that

$$\sum_{i=1}^n \omega(R_i) \leq \omega(R), \text{ whenever } \{R_i : 1 \leq i \leq n\} \text{ is a partition of } R$$

[2] The notation $V_p(f)$ is consistent with the notation and terminology of [Towghi, 2002].

[3] We thank Bruce Driver for pointing this out by constructing an explicit counter-example.

p 105. **Lemma 5.52, part (i)** is correct as stated with the new definition of $|f|_{p\text{-var}}$; in **part (ii)** we need to insert the word "*controlled*" (the statement should read "f is of finite *controlled* p-variation if and only if there exists a 2D control ω such that for all $R : |f(R)|^p \le \omega(R)$.") We include the proof.

Proof. Let us start with the remark that the super-addivity of property of 2D controls, cf. our slightly modified definition 5.51 above, immediately gives

$$\forall R \subset [0,T]^2 : |f(R)|^p \le \omega(R) \implies |f|_{p\text{-var};R} \le \omega(R).$$

This settles **part (ii)** and we focus on part **(i).** Define $\omega(R) := |f|^p_{p\text{-var};R}$ where R is a rectangle of the form $[s,t] \times [u,v] \subset [0,T]^2$. By assumption $\omega(R) \le \omega\left([0,T]^2\right) < \infty$ and it is immediate from the definition of $|f|_{p\text{-var};R}$ that ω is zero on degenerate rectangles. We need to check super-addivity and continuity.

Super-additivity: Assume $\{R_i : 1 \le i \le n\}$ constitutes a partition of R. Assume also that Π_i is a partition of R_i for every $1 \le i \le n$. Clearly, $\Pi := \cup_{i=1}^n \Pi_i$ is a partition of R and hence

$$\sum_{i=1}^n \sum_{A \in \Pi_i} |f(A)|^p = \sum_{A \in \Pi} |f(A)|^p \le \omega(R)$$

Now taking the supremum over each of the Π_i gives the desired result.

At last, we note that for similar ideas as in the 1D case (cf. p.81) give continuity of ω is a map from $\Delta_T \times \Delta_T \to [0,\infty)$; details can be found in [2]. ■

p 106. **Lemma 5.54** concerning the reduction of partitions based on $D \times D'$ to partitions based on $D \times D$ is also incorrect (because we use control argument for objects which are not controls), and should be removed. The lemma is used in Proposition 15.5 (variational regularity of fBM covariance), the (solution) to Exercise 15.6 and lemma 15.8. In each case the conclusion can obtained with an alternative argument; details are discussed at those places.

Errata Chapter 15.1

p 403. Lines 9,10, 18. Replace $|\cdot|_{\rho\text{-var}}$ by $V_\rho(\cdot)$. The comments and recalls on ρ-variation in 2D sense should also be extended such as to briefly mention *controlled* ρ-varation.

p 406. **Proposition 15.5:** replace "controlled by $\omega_H(.,.) = ...$)" by "i.e. $V_{1/(2H)}\left(R^H;[0,1]^2\right) < \infty$". Replace

$$\text{" } \left|R^H\right|_{\frac{1}{2H}\text{-var};[s,t]^2} \le C_H |t-s|^{2H} \text{ so that ....\"}$$

by

$$V_{\frac{1}{2H}}\left(R^H;[s,t]^2\right) \le C_H |t-s|^{2H}. \qquad (*)$$

Add: "This implies in particular that for $\rho > \frac{1}{2H}$, R^H is of finite (Hölder) controlled ρ-variation."

The original proof only considered $D = D'$. The arguments for general D, D' are similar; nonetheless we include full details:

New proof: (By fractional scaling it would suffice to consider $[s,t]=[0,1]$ in $(*)$ but this does not simplify the argument which follows.) Consider $D=(t_i), D'=(t'_j)\in\mathcal{D}[s,t]$. Clearly,

$$\begin{aligned}
3^{1-\frac{1}{2H}}\sum_j \left|E\left[\beta^H_{t_i,t_{i+1}}\beta^H_{t'_j,t'_{j+1}}\right]\right|^{\frac{1}{2H}} &\leq 3^{1-\frac{1}{2H}}\left|E\left[\beta^H_{t_i,t_{i+1}}\beta^H_{\cdot}\right]\right|^{\frac{1}{2H}}_{\frac{1}{2H}\text{-var};[s,t]} \\
&\leq \left|E\left[\beta^H_{t_i,t_{i+1}}\beta^H_{\cdot}\right]\right|^{\frac{1}{2H}}_{\frac{1}{2H}\text{-var};[s,t_i]} \qquad (1)\\
&\quad+\left|E\left[\beta^H_{t_i,t_{i+1}}\beta^H_{\cdot}\right]\right|^{\frac{1}{2H}}_{\frac{1}{2H}\text{-var};[t_i,t_{i+1}]} \qquad (2)\\
&\quad+\left|E\left[\beta^H_{t_i,t_{i+1}}\beta^H_{\cdot}\right]\right|^{\frac{1}{2H}}_{\frac{1}{2H}\text{-var};[t_{i+1},t]}, \qquad (3)
\end{aligned}$$

by super-additivity of (1D!) controls. The middle term (2) is estimated by

$$\begin{aligned}
\left|E\left[\beta^H_{t_i,t_{i+1}}\beta^H_{\cdot}\right]\right|^{\frac{1}{2H}}_{\frac{1}{2H}\text{-var};[t_i,t_{i+1}]} &= \sup_{(s_k)\in\mathcal{D}[t_i,t_{i+1}]}\sum_k\left|E\left[\beta^H_{t_i,t_{i+1}}\beta^H_{s_k,s_{k+1}}\right]\right|^{\frac{1}{2H}}\\
&\leq c_H\left|t_{i+1}-t_i\right|,
\end{aligned}$$

where we used that $[s_k,s_{k+1}]\subset[t_i,t_{i+1}]$ implies $\left|E\left[\beta^H_{t_i,t_{i+1}}\beta^H_{s_k,s_{k+1}}\right]\right|\leq c_H\left|s_{k+1}-s_k\right|^{2H}$. The first term (1) and the last term (3) are estimated by exploiting the fact that disjoint increments of fractional Brownian motion have negative correlation when $H<1/2$ (resp. zero correlation in the Brownian case, $H=1/2$); that is, $E\left(\beta^H_{c,d}\beta^H_{a,b}\right)\leq 0$ whenever $a\leq b\leq c\leq d$. We can thus estimate (1) as follows;

$$\begin{aligned}
\left|E\left[\beta^H_{t_i,t_{i+1}}\beta^H_{\cdot}\right]\right|^{\frac{1}{2H}}_{\frac{1}{2H}\text{-var};[s,t_i]} &= \left|E\left[\beta^H_{t_i,t_{i+1}}\beta^H_{s,t_i}\right]\right|^{\frac{1}{2H}}\\
&\leq 2^{\frac{1}{2H}-1}\left(\left|E\left[\beta^H_{t_i,t_{i+1}}\beta^H_{s,t_i}\right]\right|^{\frac{1}{2H}}+E\left[\left|\beta^H_{t_i,t_{i+1}}\right|^2\right]^{\frac{1}{2H}}\right).
\end{aligned}$$

The covariance of fractional Brownian motion gives immediately $E\left[\left|\beta^H_{t_i,t_{i+1}}\right|^2\right]^{\frac{1}{2H}}=c_H\left(t_{i+1}-t_i\right)$. On the other hand, $[t_i,t_{i+1}]\subset[s,t_{i+1}]$ implies $\left|E\left[\beta^H_{t_i,t_{i+1}}\beta^H_{s,t_i}\right]\right|^{\frac{1}{2H}}\leq c_H\left|t_{i+1}-t_i\right|$; hence

$$\left|E\left[\beta^H_{t_i,t_{i+1}}\beta^H_{\cdot}\right]\right|^{\frac{1}{2H}}_{\frac{1}{2H}\text{-var};[s,t_i]}\leq c_H\left|t_{i+1}-t_i\right|.$$

As already remarked, the last term is estimated similarly. It only remains to sum up and to take the supremum over all dissections D and D'.

p 407. **Exercise 15.6.** It should be assumed that $H\in(0,1/2]$ and the exercise should be rephrased as: show that R_X,the covariance of X has finite $1/2H$-variation. More precisely, show that there exists a constant C_H such that for all $s<t$ in $[0,1]$,

$$V_{\frac{1}{2H}}\left(R_X;[s,t]^2\right)\leq C_H\left|t-s\right|^{2H}.$$

Proof. Let β^H be a fractional Brownian motion with Hurst parameter H. The argument that leads to

$$E\left(|X_{s,t}|^2\right) \leq c_H E\left(\left|\beta^H_{s,t}\right|^2\right),$$

and, for $s \leq t \leq u \leq v$,

$$|E(X_{s,t}X_{u,v})| \leq c_H \left|E\left(\beta^H_{s,t}\beta^H_{u,v}\right)\right| \tag{4}$$

is unchanged. To prove that the covariance of X has finite $\frac{1}{2H}$-variation, we follow the (above) proof of proposition 15.5 and see that we need (replace $[t_i, t_{i+1}]$ by some generic $[u,v] \subset [s,t]$)

$$\begin{aligned}
|E[X_{u,v}X_\cdot]|^{\frac{1}{2H}}_{\frac{1}{2H}\text{-var};[s,u]} &\leq c_H |v-u| \\
|E[X_{u,v}X_\cdot]|^{\frac{1}{2H}}_{\frac{1}{2H}\text{-var};[u,v]} &\leq c_H |v-u| \\
|E[X_{u,v}X_\cdot]|^{\frac{1}{2H}}_{\frac{1}{2H}\text{-var};[v,t]} &\leq c_H |v-u|.
\end{aligned}$$

The first and third inequality follow from (4) and the corresponding fBM estimates contained in the (above) proof of proposition 15.5. So it only remains to establish the "middle" estimate, after renaming $[u,v] \rightsquigarrow [s,t]$, we need, for any $s < t$ in $[0,1]$

$$|E[X_{s,t}X_\cdot]|^{\frac{1}{2H}}_{\frac{1}{2H}\text{-var};[s,t]} \leq c_H |t-s|.$$

Let again $[u,v] \subset [s,t]$. The triangle inequality gives

$$\begin{aligned}
|E(X_{s,t}X_{u,v})| &\leq |E(X_{s,u}X_{u,v})| + \left|E\left(|X_{u,v}|^2\right)\right| + |E(X_{v,t}X_{u,v})| \\
&\leq c_H \left(\left|E\left(\beta^H_{s,u}\beta^H_{u,v}\right)\right| + E\left(\left|\beta^H_{u,v}\right|^2\right) + \left|E\left(\beta^H_{v,t}\beta^H_{u,v}\right)\right|\right) =: \Delta
\end{aligned}$$

But using the structure of fractional Brownian motion (using $H \leq 1/2$) we see that

$$\begin{aligned}
\Delta &= c_H \left(-E\left(\beta^H_{s,u}\beta^H_{u,v}\right) + E\left(\left|\beta^H_{u,v}\right|^2\right) - E\left(\beta^H_{v,t}\beta^H_{u,v}\right)\right) \\
&= c_H \left(-E\left(\beta^H_{s,t}\beta^H_{u,v}\right) + 2E\left(\left|\beta^H_{u,v}\right|^2\right)\right) \\
&\leq c_H \left|E\left(\beta^H_{s,t}\beta^H_{u,v}\right)\right| + 2c_H E\left(\left|\beta^H_{u,v}\right|^2\right).
\end{aligned}$$

Hence,for a suitable constant $\tilde{c} = \tilde{c}(H)$ which may change from line to line,

$$\begin{aligned}
|E(X_{s,t}X_{u,v})|^{\frac{1}{2H}} &\leq \tilde{c}_H \left|E\left(\beta^H_{s,t}\beta^H_{u,v}\right)\right|^{\frac{1}{2H}} + \tilde{c}_H E\left(\left|\beta^H_{u,v}\right|^2\right)^{\frac{1}{2H}} \\
&= \tilde{c}_H \left|E\left(\beta^H_{s,t}\beta^H_{u,v}\right)\right|^{\frac{1}{2H}} + \tilde{c}_H |v-u|
\end{aligned}$$

and then

$$|E[X_{s,t}X_\cdot]|^{\frac{1}{2H}}_{\frac{1}{2H}\text{-var};[s,t]} \leq \tilde{c}_H \left|E\left[\beta^H_{s,t}\beta^H_\cdot\right]\right|^{\frac{1}{2H}}_{\frac{1}{2H}\text{-var};[s,t]} + \tilde{c}_H |t-s|.$$

Since $\left|E\left[\beta^H_{s,t}\beta^H_{\cdot}\right]\right|^{\frac{1}{2H}}_{\frac{1}{2H}\text{-var};[s,t]} = \tilde{c}_H\,|t-s|$, this was seen in the (above) proof of proposition 15.5, the exercise is now completed. ■

p 415. **Lemma 15.8** The second claimed estimate ("$|R^A|_{2\text{-var};[s,t]^2} \leq |R|_{2\text{-var};[s,t]^2}$") should be rephrased as[4]

$$V_2\left(R^A;[s,t]^2\right) \leq V_2\left(R;[s,t]^2\right).$$

The given proof (we may take $[s,t]^2 = [0,1]^2$ without loss of generality) of lemma 15.8 actually only shows this estimate when $\sup_{D,D'\in\mathcal{D}[0,1]}$ in the definition of V_2 is replaced by the sup over all $D = D' \in \mathcal{D}[0,1]$. This gap is closed by the following (new) lemma which may be interesting in its own right.

Lemma 1 *Define $R(s,t) := E(X_sX_t)$ for some stochastic process $(X_t : t\in[0,1])$. Then*

$$\sup_{D,D'\in\mathcal{D}[0,1]}\sum_{i,j}\left|R\begin{pmatrix} t_i,t_{i+1}\\ t'_{j'},t'_{j+1}\end{pmatrix}\right|^2 = \sup_{D\in\mathcal{D}[0,1]}\sum_{i,j}\left|R\begin{pmatrix} t_i,t_{i+1}\\ t_j,t_{j+1}\end{pmatrix}\right|^2,$$

where we write $D=(t_i)$ and $D'=(t'_j)$.

Proof. We only need to show "$\leq$". Set $X_i = X_{t_i,t_{i+1}}$ and $X_{j'} = X_{t'_{j'},t'_{j+1}}$ so that

$$R\begin{pmatrix} t_i,t_{i+1}\\ t'_{j'},t'_{j+1}\end{pmatrix} = E(X_iX_{j'}).$$

Consider an IID copy of X, say $\tilde{X}$, so that

$$\left|R\begin{pmatrix} t_i,t_{i+1}\\ t'_{j},t'_{j+1}\end{pmatrix}\right|^2 = E(X_iX_{j'})\,E\left(\tilde{X}_i\tilde{X}_{j'}\right) = E\left(X_iX_{j'}\tilde{X}_i\tilde{X}_{j'}\right).$$

It follows that

$$\begin{aligned}\sum_{i,j}\left|R\begin{pmatrix} t_i,t_{i+1}\\ t'_{j},t'_{j+1}\end{pmatrix}\right|^2 &= \sum_{i,j}E\left[X_iX_{j'}\tilde{X}_i\tilde{X}_{j'}\right]\\ &= E\left[\left(\sum_i X_i\tilde{X}_i\right)\left(\sum_j X_{j'}\tilde{X}_{j'}\right)\right]\\ &\leq \sqrt{E\left[\left(\sum_i X_i\tilde{X}_i\right)^2\right]}\sqrt{E\left[\left(\sum_j X_{j'}\tilde{X}_{j'}\right)^2\right]}\end{aligned}$$

[4] Recall $V_2\left(R;[s,t]^2\right) = \sup_{D,D'\in\mathcal{D}[s,t]}\sum_{i,j}\left|R\begin{pmatrix} t_i,t_{i+1}\\ t'_{j'},t'_{j+1}\end{pmatrix}\right|^2$.

where we used Cauchy-Schwarz. Since

$$E\left[\left(\sum_i X_i\tilde{X}_i\right)^2\right] = E\left[\sum_{i,k} X_i\tilde{X}_iX_k\tilde{X}_k\right] = \sum_{i,k} E\left[X_iX_k\right]E\left[\tilde{X}_i\tilde{X}_k\right] = \sum_{i,k}\left|R\begin{pmatrix} t_i,t_{i+1} \\ t_k,t_{k+1}\end{pmatrix}\right|^2,$$

$$E\left[\left(\sum_j X_{j'}\tilde{X}_{j'}\right)^2\right] = \cdots \text{ (as above) } \cdots = \sum_{j,l}\left|R\begin{pmatrix} t'_{j'},t'_{j+1} \\ t'_{l},t'_{l+1}\end{pmatrix}\right|^2$$

we see that

$$\left(\sum_{i,j}\left|R\begin{pmatrix} t_i,t_{i+1} \\ t'_j,t'_{j+1}\end{pmatrix}\right|^2\right)^2 \le \sum_{i,k}\left|R\begin{pmatrix} t_i,t_{i+1} \\ t_k,t_{k+1}\end{pmatrix}\right|^2 \times \sum_{j,l}\left|R\begin{pmatrix} t'_{j'},t'_{j+1} \\ t'_{l},t'_{l+1}\end{pmatrix}\right|^2.$$

Since we managed to factorize the dependence on $D=(t_i)\in\mathcal{D}[0,1]$ and $D'=\left(t'_j\right)\in\mathcal{D}[0,1]$ on the right-hand-side the conclusion follows immediately upon taking $\sup_{D,D'}$ first on the right-hand-side, then on the left-hand-side. ■

p 431-432. **Exercise 15.36** is corrected as stated (in particular, under the assumption of Theorem 15.33 concerning finite controlled ρ-variation of the covariance). However, if one wants to apply this to fractional Brownian motion with $H<1/2$, say, one has to work with $\tilde{\rho}$-variation, $\tilde{\rho}:=1/(2H)+\varepsilon$, any $\varepsilon>0$, rather than $\rho:=1/(2H)$. The conclusion in part (iii) of this exercise, finite $\psi_{2\tilde{\rho},\tilde{\rho}}$-variation, then is not optimal: one wants (optimal) finite $\psi_{2\rho,\rho}=\psi_{\frac{1}{H},\frac{1}{2H}}$-variation. In fact, one *does* get this result upon realizing that by fractional scaling $\tilde{\omega}\left([s,t]^2\right)^{\frac{1}{\tilde{\rho}}}=(\text{const})\times|t-s|^{\frac{1}{\rho}}$. In particular, (15.20) applied with $\tilde{\omega}$ then yields

$$|d(\mathbf{X}_s,\mathbf{X}_t)|_{L^q}\le C\sqrt{q}\tilde{\omega}\left([s,t]^2\right)^{\frac{1}{2\tilde{\rho}}}=C\sqrt{q}\,|t-s|^{\frac{1}{2\rho}};$$

finite $\psi_{2\rho,\rho}$-variation of sample paths is then a standard consequence of the results in section A.4. (In a similar spirit one can show that the assumption of finite ρ-variation, rather than finite *controlled* ρ-variation, leads to $\psi_{2\rho,\rho}$-variation of the sample paths.)

p 438. Replace "$\left|R^A\right|_{2\text{-var};[s,t]^2}\le|R|_{2\text{-var};[s,t]^2}$" (15.28) by

$$V_2\left(R^A;[s,t]^2\right)\le V_2\left(R;[s,t]^2\right). \tag{(15.28)}$$

p 440, remove the text from line 7 until the end of the proof, and replace it with the following argument.

For $u<v$ in $[s,t]$, define $Q^1_{u,v}=\begin{pmatrix} u,v \\ u,v\end{pmatrix}$, $Q^2_{u,v}=\begin{pmatrix} s,u \\ u,v\end{pmatrix}$, $Q^3_{u,v}=\begin{pmatrix} u,v \\ s,u\end{pmatrix}$; rewrite the previous equation as

$$f_{u,v}=R_{X^{A^c};i}\left(Q^1_{u,v}\right)+R_{X^{A^c};i}\left(Q^2_{u,v}\right)+R_{X^{A^c};i}\left(Q^3_{u,v}\right).$$

It follows that, for $\varepsilon>0$ and $c_1=3^{2+\varepsilon-1}$,

$$\begin{aligned} 3^{1-(2+\varepsilon)}\left|f_{u,v}\right|^{2+\varepsilon} &\le \left|R_{X^{A^c};i}\left(Q^1_{u,v}\right)\right|^{2+\varepsilon}+\left|R_{X^{A^c};i}\left(Q^2_{u,v}\right)\right|^{2+\varepsilon}+\left|R_{X^{A^c};i}\left(Q^3_{u,v}\right)\right|^{2+\varepsilon} \\ &\le \left|R_{X^{A^c};i}\right|^{2+\varepsilon}_{(2+\varepsilon)\text{-var};[u,v]^2}+\left|R_{X^{A^c};i}\right|^{2+\varepsilon}_{(2+\varepsilon)\text{-var};[s,t]\times[u,v]}+\left|R_{X^{A^c};i}\right|^{2+\varepsilon}_{(2+\varepsilon)\text{-var};[u,v]\times[s,t]}. \end{aligned}$$

Since the last line is (1D) super-additive in $[u,v]$ it follows that

$$\begin{aligned}|f|^{2+\varepsilon}_{(2+\varepsilon)\text{-var};[s,t]} &\leq 3^{2+\varepsilon}\left|R_{X^{A^c;i}}\right|^{2+\varepsilon}_{(2+\varepsilon)\text{-var};[s,t]^2} \\ &\leq c_1 V_2\left(R_{X^{A^c;i}};[s,t]^2\right) \\ &\leq c_1 V_2\left(R_{X^i};[s,t]^2\right) \text{ use (5.28)} \\ &\leq c_1 \left|R_X\right|_{2\text{-var};[s,t]^2}.\end{aligned}$$

By taking $\varepsilon > 0$ small enough, the proof is finished with the same argument, namely the Young-Wiener estimate of Proposition 15.39.

References

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