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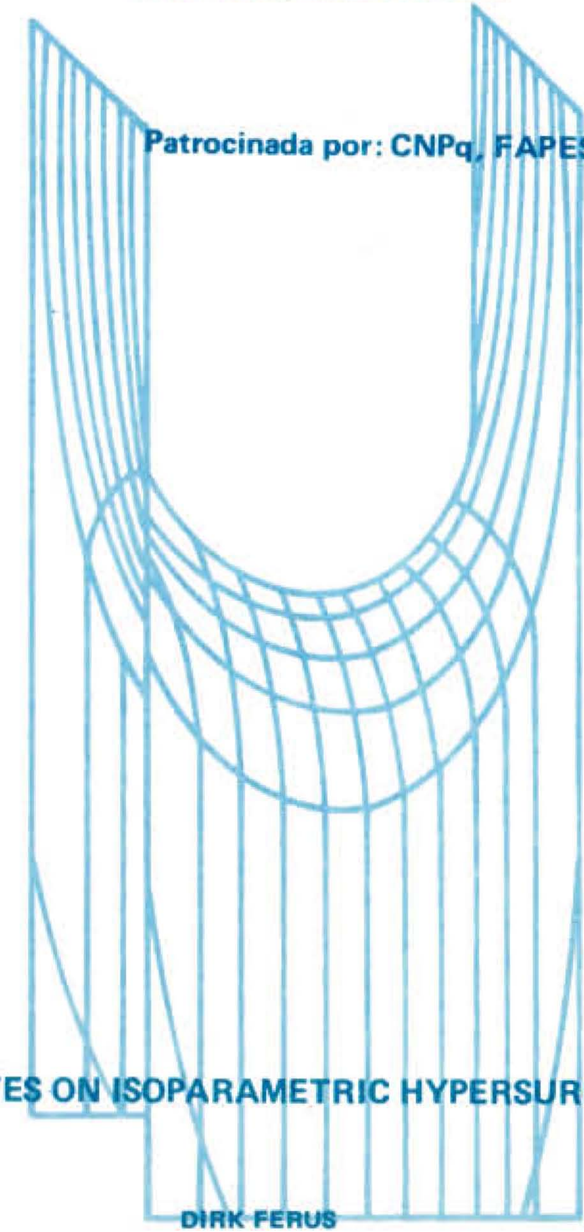

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NOTES ON ISOPARAMETRIC HYPERSURFACES

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Introduction

We are concerned with the study of hypersurfaces in a standard space of constant curvature: euclidean space, sphere or hyperbolic space. According to a fundamental theorem such hypersurfaces are uniquely determined up to congruence by their first and second fundamental form. Looking for scalar rather than tensorial invariants we are led to the principal curvatures. And if we want to understand their geometric relevance it seems desirable to first get a good understanding of the case of constant principle curvatures:

> How do hypersurfaces with constant principal curvatures look like?

This question was solved for hypersurfaces in euclidean space by Levi-Civita [10] and Segre [17] in the late thirties. At the same time E. Cartan solved the hyperbolic case. In both cases the number g of distinct principal curvatures is at most two, and the hypersurfaces looks like a tube around a totally geodesic subspace (i.e. around an affine subspace in the euclidean case), [1].

But in the spherical case Cartan found the situation quite different [2],[3],[4]. The orbits of a group of isometries acting on the sphere obviously have constant principal curvatures, *if they happen to be hypersurfaces*. Cartan was able to construct such homogeneous examples with up to $g = 4$ distinct principal curvatures. He could also show, that for $g \leq 3$ all examples must be homogeneous.

Remark. Hypersurfaces with constant principal curvatures are characterized as level hypersurfaces of functions for which the first and second "differential parameter" depends only on the value of the function itself. They are therefore called *isoparametric hypersurfaces*.

.fter Cartan the subject fell into oblivion until about 1970. .t that time Nomizu brought it up again with a survey on the nown results and open questions [14], Takagi and Takahashi [18] .oticed that the classification of homogeneous hypersurfaces n the sphere given by Hsiang and Lawson [8] solved the lassification of *homogeneous* isoparametric hypersurfaces, .nd Münzner [11,12] showed that every isoparametric hypersurface s algebraic with g = 1, 2, 3, 4, or 6 distinct principal cur-atures. These g-values occur among the homogeneous examples, nd no inhomogeneous examples were known until the surprising aper [15] of Ozeki and Takeuchi, where two infinite series of on-homogeneous hypersurfaces with g = 4 were constructed. tarting from this paper, Karcher, Münzner and me recently ound again a much larger number of (g = 4)-examples. They re constructed in a unified way using Clifford representations nd written in a form which easily yields detailed and, we think, ery interesting geometric information. In particular we find inite, but arbitrarily large families of compact riemannian anifolds which are not isometric, but have "the same" curvature ensor.

n my lectures I want to present a survey of the results entioned above. One appealing aspect of the subject is the ariety of methods applicable to it. After a short preliminary ection on the equations of Gauss and Codazzi, we begin with tensor analytic study of the shape operator on an isopara-etric hypersurface. Later we come to different approaches: nother geometric one, describing our objects as tubes around heir focal manifolds, and an analytic description that will initely take us into algebra.

shall not go into Münzner's proof [11] that only g = 1,2,3,4,6 re possible: it uses very intricate cohomology arguments which re beyond the scope of these lectures. And I shall not go into ll the details of the Clifford examples either. They will be ontained in a paper in preparation [7].

y notes are partially based on lecture notes by H.Karcher, and n Münzner [11].

The equations of GAUSS and CODAZZI.

The main intention of this preliminary section is to fix our notation.

Let M be a submanifold of an (n+1)-dimensional space $\tilde{M}$ of constant curvature $\tilde{c}$. By $\langle \dots , \dots \rangle$ we shall denote the riemannian metric of either manifold. The Levi-Civita covariant derivative on $\tilde{M}$ will be denoted by $\tilde{\nabla}$, that on M by ∇. If X and Y are tangent vector fields on M, then, with the usual identifications

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{1}$$

where $h(X,Y)$ is a normal vector field. This defines a symmetric bilinear map h from the tangent to the normal space called the *second fundamental form* of M in $\tilde{M}$. For each normal field ξ we obtain a symmetric endomorphism field on tangent vectors by

$$\langle S_\xi X, Y \rangle := \langle h(X,Y), \xi \rangle . \tag{2}$$

The tensor $S : \perp M \to \mathrm{End}(TM)$ is called the *shape operator* or *second fundamental tensor* of M in $\tilde{M}$.

Note: If M is a hypersurface with a distinguished unit normal field ξ, then we write

$h(X,Y)$ instead of $\langle h(X,Y), \xi \rangle$

and

S instead of S_ξ.

In this case h is real-valued and S simply a symmetric endomorphism field.

We shall repeatedly need the covariant derivative of tensor fields of various type. Without going into formal details we recall, that it is defined using the product rule as a guiding principle. For example, the covariant derivative of S is defined by

$$(\nabla_Z S)_\xi X := \nabla_Z (S_\xi X) - S_{\nabla_Z \xi} X - S_\xi \nabla_Z X, \tag{3}$$

where $\nabla_Z \xi$ is the normal covariant derivative (= normal component of $\tilde{\nabla}_Z \xi$).

Let R and $\tilde{R}$ be the curvature tensors of M and $\tilde{M}$. Then the equation of GAUSS reads

$$
\begin{aligned}
R(X,Y)Z &= \tilde{R}(X,Y)Z + S_{h(Y,Z)}X - S_{h(X,Z)}Y \\
&= \tilde{c}\,\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\} \\
&\quad + S_{h(Y,Z)}X - S_{h(X,Z)}Y . \qquad (4)
\end{aligned}
$$

For hypersurfaces with a distinguished unit normal field, this reduces to

$$
\begin{aligned}
R(X,Y)Z &= \tilde{c}\,\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\} \\
&\quad + \langle SY,Z\rangle SX - \langle SX,Z\rangle SY . \qquad (5)
\end{aligned}
$$

The equation of CODAZZI is

$$(\nabla_X S)_\xi Y = (\nabla_Y S)_\xi X , \qquad (6)$$

and ξ can be ommitted in the hypersurface case.

For a proof of (4) - (6) see for instance the book of Kobayashi and Nomizu, vol II.

If M is a hypersurface with a distinguished unit normal field, then the eigenvalues of S are called the *principal curvatures of* M in $\tilde{M}$. If they are constant, then M is called an *isoparametric hypersurface*.

The two local unit normal fields lead to opposite signs of the principal curvatures. Therefore the "constancy" of the principal curvatures and hence the notion "isoparametric" makes sense also for hypersurfaces *without* a distinguished normal field.

<u>The curvature foliations of isoparametric hypersurfaces.</u>

Let M be an isoparametric hypersurface of the $(n+1)$-space $\tilde{M}$. The following considerations beeing local we can assume that M has a distinguished unit normal field ξ. Let λ be one of the principal curvatures of M, and put

$$A := S - \lambda\,\mathrm{Id}. \qquad (7)$$

For vector fields X, Y, Z with $AX = AY = 0$ we find

$$
\begin{aligned}
\langle \nabla_X Y, AZ\rangle &= \langle A(\nabla_X Y), Z\rangle \\
&= \langle \nabla_X (AY), Z\rangle - \langle (\nabla_X A)Y, Z\rangle \\
&= -\langle (\nabla_X A)Z, Y\rangle \qquad \text{(symmetry of } \nabla_X A) \\
&= -\langle (\nabla_Z A)X, Y\rangle \qquad \text{(Codazzi)} \\
&= -\langle \nabla_Z (AX), Y\rangle + \langle A(\nabla_Z X), Y\rangle \\
&= \langle \nabla_Z X, AY\rangle \\
&= 0.
\end{aligned}
$$

Hence

$$SX = \lambda X \text{ and } SY = \lambda Y \text{ implies } S(\nabla_X Y) = \lambda(\nabla_X Y). \qquad (8)$$

Therefore the eigenspace distributions of S are autoparallel in M. Each principal curvature λ determines a foliation of M by totally geodesic submanifolds, the λ*-curvature leaves*. Since for X and Y tangent to such a leaf we have

$$
\begin{aligned}
\tilde{\nabla}_X Y &= \nabla_X Y + h(X,Y) \\
&= \nabla_X Y + \lambda \langle X,Y\rangle \xi ,
\end{aligned}
$$

these leaves are λ-umbilical in $\tilde{M}$ with mean curvature vector parallel to ξ. So M consists of mutually orthogonal families of umbilical submanifolds of $\tilde{M}$.

If the number of distinct principal curvatures is $g = 2$, then the hypersurface is obtained locally by taking a λ_1-umbilical submanifold of dimension equal to the multiplicity m_1 of λ_1, and ($\tilde{M}$-)parallel translating an orthogonal λ_2-umbilical sub-manifold of dimension m_2 along the first one.

If $g > 2$, then the situation becomes more complicated, because for principal curvatures $\lambda_i \neq \lambda_j$ the λ_j-leaves will not in general be parallel along the λ_i-leaves. But if we can get control over the rotation of the leaves around each other, we can again reconstruct M starting from a point x and from the eigenspaces of S at x.

We select a principal curvature λ, and a unit-speed geodesic γ tangent to a λ-curvature leaf L. We want to describe the behaviour of the μ-leaves, $\mu \neq \lambda$, along γ. For that purpose it is sufficient, to describe the behaviour of $S - \lambda \mathrm{Id}$ along γ. We shall see that this tensor field satisfies a first order linear differential equation along γ, which involves a certain tensor field C. This field in turn satisfies a Riccati equation. Putting together both equations, we arrive at a second order linear differential equation for $S - \lambda \mathrm{Id}$ which can be solved explicitely:

Let $\mathcal{H}$ and $\mathcal{V}$ denote the orthogonal projections onto $\ker(S - \lambda \mathrm{Id})$ and $\mathrm{im}(S - \lambda \mathrm{Id})$ respectively. Then

$$C_X Y := -\mathcal{V}\nabla_Y \mathcal{H} X \qquad (9)$$

defines a (2,1)-tensor field on M. Note that

$$\left.\begin{array}{l} \mathcal{H} \text{ and } \mathcal{V} \text{ are parallel along } L, \text{ and} \\ \qquad C_X \circ \mathcal{H} = 0 \end{array}\right\} \qquad (10)$$

by (8).

<u>Lemma 1.</u> Let X be a vector in $\ker(S - \lambda \mathrm{Id})$. Then

$$\nabla_X (S - \lambda \mathrm{Id}) = (S - \lambda \mathrm{Id}) \circ C_X . \qquad (11)$$

Proof: Extend X to a vector field in $\ker(S - \lambda \mathrm{Id})$, and let Y be any vector field. Then

$$\begin{aligned} (\nabla_X (S - \lambda \mathrm{Id}))Y &= (\nabla_Y (S - \lambda \mathrm{Id}))X \\ &= \nabla_Y (SX - \lambda X) - (S - \lambda \mathrm{Id}) \nabla_Y X \\ &= (S - \lambda \mathrm{Id})(-\nabla_Y \mathcal{H} X) \\ &= (S - \lambda \mathrm{Id})(-\mathcal{V} \nabla_Y \mathcal{H} X) \\ &= (S - \lambda \mathrm{Id}) \circ C_X Y. \end{aligned}$$

<u>Lemma 2.</u> If $X \in T_pM$ is a vector in $\ker(S - \lambda \mathrm{Id})$, then

$$(\nabla_X C)_X = C_X^{\ 2} + R_X , \qquad (12)$$

where $R_X Y := \mathcal{V} R(\mathcal{V} Y, X) X$.

Proof: All three tensors in (12) vanish on $\ker(S - \lambda \mathrm{Id}) = \mathrm{image}(\mathcal{H}_p)$. Now let Y be in $\ker(\mathcal{H}_p)$. We extend X and Y to vector fields on a neighborhood of p, such that

$$(\nabla_X X)_p = (\nabla_X Y)_p = 0 \quad \text{and} \quad X = \mathcal{H} X,\ Y = \mathcal{V} Y.$$

Using (10) several times, we find at p

$$\begin{aligned} (\nabla_X C)_X Y &= \nabla_X (C_X Y) \\ &= -\nabla_X \mathcal{V} \nabla_Y X \\ &= -\mathcal{V}(\nabla_X \nabla_Y X) \\ &= -\mathcal{V}(R(X,Y)X + \nabla_Y \nabla_X X + \nabla_{\nabla_X Y} X - \nabla_{\nabla_Y X} X) \\ &= -\mathcal{V} R(Y,X)X + \mathcal{V}(\nabla_{\nabla_Y X} X) - \mathcal{V}(\nabla_Y \nabla_X X + \nabla_{\underbrace{\nabla_X Y}_{=0}} X) \\ &= C_X^{\ 2} Y + R_X Y - \sum_i \langle \nabla_Y \nabla_X X, Y_i \rangle Y_i , \end{aligned}$$

where the Y_i are orthonormal vector fields with $\mathcal{V} Y_i = Y_i$.

But at p

$$\langle \nabla_Y \nabla_X X, Y_i \rangle = Y \cdot \overbrace{\langle \nabla_X X, Y_i \rangle}^{=0 \text{ by } (10)} - \langle \underbrace{\nabla_X X}_{=0}, \nabla_X Y_i \rangle = 0 .$$

This finishes the proof of Lemma 2.

Remark. Lemma 2 is true for any totally geodesic foliation. In the present geometric situation the Gauss equation yields more information on R_X:

$$R_X Y = R(\mathcal{V}Y,X)X = \langle X,X\rangle (\tilde{c} + \lambda S)\mathcal{V}Y. \qquad (13)$$

Now let γ be a unit-speed geodesic tangent to a λ-leaf L. We write $C := C_{\dot\gamma}$, $R := R_{\dot\gamma}$ and denote by $(..)'$ the covariant derivative along γ. Then (12) reads

$$C' = C^2 + R. \qquad (14)$$

Let Y be an L-normal vector field along γ such that

$$Y' + CY = 0. \qquad (15)$$

Then

$$\begin{aligned} 0 &= Y'' + C'Y + CY' \\ &= Y'' + C^2Y + RY - C^2Y \\ &= Y'' + RY. \qquad (16) \end{aligned}$$

Hence C defines (and is determined by) a certain class of Jacobi fields along γ, which are in a sense (specified for example in [6]) "adapted to the foliation".

We define

$$A(t) := \begin{cases} (S - \lambda \mathrm{Id})^{-1}_{\gamma(t)} & \text{on image } (\mathcal{H}_{\gamma(t)}) \\ 0 & \text{on ker } (\mathcal{H}_{\gamma(t)}). \end{cases} \qquad (17)$$

Then along γ

$$\mathcal{V} = A \circ (S - \lambda \mathrm{Id}),$$

and from (10) and (11)

$$\begin{aligned} 0 &= A'(S - \lambda \mathrm{Id}) + A(S - \lambda \mathrm{Id})' \\ &= A'(S - \lambda \mathrm{Id}) + A(S - \lambda \mathrm{Id})\, C \\ &= A'(S - \lambda \mathrm{Id}) + C \end{aligned}$$

or

$$A' + CA = 0. \qquad (18)$$

This is the tensor analog of (15) and implies the following analog of (16):

$$A'' + RA = 0. \qquad (19)$$

By (13) and $\|\dot\gamma\| = 1$ we find

$$A'' + (\tilde{c} + \lambda S)A = 0. \qquad (20)$$

Using $(S - \lambda \mathrm{Id})\cdot A = \mathcal{V}$ we conclude

$$A'' + (\tilde{c} + \lambda^2)A + \lambda\mathcal{V} = 0, \qquad (21)$$

and arrive at the following combination of Lemmas 1 and 2

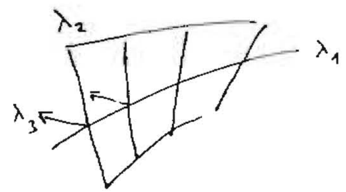
Proposition 3. Let γ be a unit-speed geodesic tangent to a λ-leaf L. Then, with $\mathcal{V}$ and A defined as above we have for $\tilde{c} + \lambda^2 \neq 0$

$$\big((\tilde{c} + \lambda^2)A + \lambda\mathcal{V}\big)'' + (\tilde{c} + \lambda^2)\big((\tilde{c} + \lambda^2)A + \lambda\mathcal{V}\big) = 0 \qquad (22)$$

and for $\tilde{c} + \lambda^2 = 0$

$$A'' + \lambda\mathcal{V} = 0. \qquad (22')$$

These differential equations are easily integrated explicitely. (Note that $\mathcal{V}$ is parallel along L.) But A determines S completely, and therefore our problem seems to be solved: we know the rotation of the λ-leaves around L. There is only one little point which unfortunately turns out to spoil much of our glorious victory. We don't know the initial conditons good enough. Besides A, which we can assume to be given at our starting point, we need its derivatives with respect to all L-tangent directions. At least for $\tilde{c} > 0$ it seems to be quite unclear, which data lead to a smooth field A on the sphere L. Even worse: after translating the λ_i-leaves, $i > 1$, along a λ_1-leaf, we have to translate the λ_j-leaves, $j > 2$, along all the λ_2-leaves. But for this we need the initial data for $\lambda = \lambda_2$ along the whole λ_1-leaf.



Nevertheless, Proposition 3 gives valuable information that we are now going to exploit.

The equations (22), (22') have the solutions

$$(\tilde{c} + \lambda^2)A + \lambda\dot{\mathcal{V}} = \cos\sqrt{\tilde{c}+\lambda^2}\,t\,E + \sin\sqrt{\tilde{c}+\lambda^2}\,t\,\tilde{E} \qquad (23)$$

$$A = -t^2/2\,\lambda\,\mathcal{V} + t\tilde{E} + E \qquad (23')$$

with parallel tensorfields E, $\tilde{E}$ along γ, $E \circ \mathcal{H} = \tilde{E} \circ \mathcal{H} = 0$. (If $\tilde{c} + \lambda^2 < 0$ we substitute $\cosh\sqrt{|\tilde{c}+\lambda^2|}\,t$ etc. to get real solutions.) If the eigenvalues of S are

$$\lambda = \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_g$$

with multiplicities

$$m_1, m_2, m_3, \ldots, m_g$$

then the eigenvalues of A are 0 and $1/\lambda_i - \lambda$, and those of $(\tilde{c} + \lambda^2)A + \lambda\mathcal{V}$ are

$$\eta_1 = 0,\ \eta_2 = \frac{\tilde{c} + \lambda_2\lambda_1}{\lambda_2 - \lambda_1},\ \ldots,\ \eta_g = \frac{\tilde{c} + \lambda_g\lambda_1}{\lambda_g - \lambda_1} \qquad (24)$$

with the same multiplicities, and they are constant along γ. Note that the η_is are not necessarily distinct, because $\eta_i = 0$ may happen for an $i > 1$, but since we are finally interested only in $A\,|\ker(\mathcal{H})$ this will cause no trouble.

Therefore the eigenvalues on the right hand side of (23),(23') must be independent of t. This implies in particular that in case $\tilde{c} + \lambda^2 \neq 0$ the trace of E (and $\tilde{E}$) must be zero. In case $\tilde{c} + \lambda^2 = 0$ it follows $\lambda = 0$. In either case we have

Proposition 4. For an isoparametric hypersurface with distinct principal curvatures $\lambda_1, \ldots, \lambda_g$ of multiplicity $m_1, \ldots, m_g$ in a space of constant curvature $\tilde{c}$ we have

$$\sum_{\substack{j=1 \\ j \neq i}}^{g} m_j \frac{\tilde{c} + \lambda_j\lambda_i}{\lambda_j - \lambda_i} = 0 \qquad (25)$$

for all i.

(25) is Cartan's fundamental equation for isoparametric hypersurfaces. From it he obtained the classification in the cases of non-positive $\tilde{c}$.

Consider first the case $\tilde{c} < 0$, say $\tilde{c} = -1$. By changing the normal field if necessary, we may assume that there are positive λ_is. Let

$$\lambda := \sup\{\lambda_i;\ \ 0 < \lambda_i \leq 1\} \text{ and } \mu := \inf\{\lambda_i;\ \ 1 < \lambda_i\}.$$

Assume first that λ exists, and that there is no principal curvature in $]\lambda, \tfrac{1}{\lambda}[$. Then

$$\frac{-1 + \lambda\lambda_j}{\lambda_j - \lambda} = \lambda\,\frac{\lambda_j - 1/\lambda}{\lambda_j - \lambda}$$

is positive for all $\lambda_j \neq \lambda$, except possibly $\lambda_j = 1/\lambda$. Hence, if $g > 1$, by (25) we have $g = 2$, and $\lambda_1\lambda_2 = 1$.

Assume next that λ exists, and that there is a principal curvature in $]\lambda, \tfrac{1}{\lambda}[$. Then also $\mu \in]\lambda, \tfrac{1}{\lambda}[$, and there is no principal curvature in $]\tfrac{1}{\mu}, \mu[$. We can therefore apply the above argument to μ instead of λ.

Finally, if there is no λ_i in $]0,1]$, then μ exists, and again $]\tfrac{1}{\mu}, \mu[$ does not contain any λ_i.

For $\tilde{c} = 0$ a much simpler argument works, which I leave to the reader. We get

Theorem 5. If M is an isoparametric hypersurface in a space $\tilde{M}$ of constant curvature $\tilde{c} \leq 0$, then the number of distinct principal curvatures is $g \leq 2$. For $g=1$ M is totally umbilic and very well known, if $\tilde{M}$ is a standard space. For $g=2$ the two distinct curvatures satisfy

$$-\lambda_1\lambda_2 = \tilde{c}, \qquad (26)$$

and locally M is obtained by taking the two curvature leaves (which are totally umbilical submanifolds) through a given point, and then ($\tilde{M}$-)parallel translating one along the other.

This solves the local classification problem for $\tilde{c} \leq 0$. One can show that each connected isoparametric hypersurface is an open part of a complete one, and for those the classification is the same. In euclidean space the isoparametric hypersurfaces are (open parts of) spheres and hyperplanes ($g=1$), and spherical cylinders (g=2).

For $\tilde{c} > 0$ the above arguments fail, but proposition 4 still gives interesting information about the principal curvatures.

Proposition 6. Let M be an isoparametric hypersurface of the standard sphere S^{n+1} of curvature 1. Let M have the distinct principal curvatures $\lambda_1 < \lambda_2 < \ldots < \lambda_g$ with corresponding multiplicities $m_1, \ldots, m_g$. Then, for indices mod g, we have

$$m_i = m_{i+2} \tag{27}$$

and there exists $\alpha \in]0, \frac{\pi}{g}[$, such that

$$\lambda_i = \cot\left(\alpha + (g-i)\frac{\pi}{g}\right). \tag{28}$$

If g is odd, then all multiplicities are equal.

Proof: Put $\lambda_j := \cot \varphi_j$ with $\pi > \varphi_1 > \ldots > \varphi_g > 0$

Choose some $\lambda = \lambda_i$, and define A correspondingly. Then by (24) the eigenvalues of $(1+\lambda^2)A + \lambda\mathcal{V} \mid \ker(\mathcal{H})$ are

$$\frac{1+\lambda_i\lambda_j}{\lambda_j - \lambda_i} = \cot(\varphi_i - \varphi_j) \quad \text{with multipl. } m_j \tag{29}$$

Evaluating (23) for $t = 0$ and $\sqrt{1+\lambda^2}\, t = \pi$ shows that with $\cot(\varphi_i - \varphi_j)$ also $-\cot(\varphi_i - \varphi_j)$ is an eigenvalue, which by continuity must have the same multiplicity m_j.

From

$$\varphi_i - \varphi_{i+1} < \ldots < \varphi_i - \varphi_g < \pi + \varphi_i - \varphi_1 < \ldots < \pi + \varphi_i - \varphi_{i-1}$$

we obtain

$$\cot(\varphi_i - \varphi_{i+1}) > \ldots > \cot(\varphi_i - \varphi_g) > \cot(\varphi_i - \varphi_1) > \ldots > \cot(\varphi_i - \varphi_{i-1})$$

with corresponding multiplicities

$$m_{i+1} \quad \ldots \quad m_g \quad \ldots \quad m_{i-1}$$

Since the negative of the biggest eigenvalue is an eigenvalue, it must be the smallest one etc. Hence $m_{i+1} = m_{i-1}$, indices mod g . Moreover

$$\cot(\varphi_i - \varphi_{i+1}) = -\cot(\varphi_i - \varphi_{i-1})$$

implies

$$(\varphi_i - \varphi_{i+1}) + (\varphi_i - \varphi_{i-1}) \equiv 0 \mod \pi,$$

and

$$\varphi_i - \varphi_{i+1} \equiv \varphi_{i-1} - \varphi_i \mod \pi,$$

indices mod g. Hence the φ_i are equidistant mod π. This proves the proposition.

Parallel hypersurfaces, focal manifolds and isoparametric families

We concentrate on the spherical case as the most interesting one, but most of the following considerations carry over to the general situation.

Let M be a hypersurface of the standard sphere S^{n+1}, and let ξ be a unit normal field along M. Then for most real numbers t the map

$$p \longmapsto \exp_p(\, t\,\xi(p)\,)$$

defines a *parallel hypersurface* M_t, and the shape operators S and S^t of M and M_t are related in a very simple way:
The eigen spaces of S^t are parallel (even in euclidean space) along the curve $t \longmapsto \exp\,(t\,\xi(p))$, and the eigen-value of S^t corresponding to the principal curvature λ on M is

$$\cot\,(\varphi - t), \quad \text{where} \quad \lambda = \cot\varphi . \tag{30}$$

Hence, if M is isoparametric, so are the parallel hypersurfaces M_t. Isoparametric hypersurfaces come in so-called *isoparametric families*.
If M has principal curvatures $\lambda_1 = \cot\varphi_1, \ldots, \lambda_g = \cot\varphi_g$ with multiplicities $m_1,\ldots,m_g$, then the mean curvature of M_t at the corresponding point is

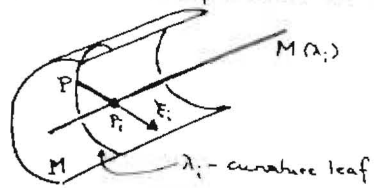
$$H^t = \frac{1}{n} \sum m_i \cot\,(\varphi_i - t) \tag{31}$$

If each M_t (for values t in some open interval) has constant mean curvature, then (31) is an analytic function of t, the poles of which determine uniquely mod π the φ_i, and hence the principal curvatures. This yields

Proposition 7. If all parallel hypersurfaces M_t of M for t sufficiently close to zero have constant mean curvature, then they have constant principal curvatures and are isoparametric.

The focal set of M is the set of singular values of the map $(t,p) \longmapsto \exp\,(t\,\xi(p))$. From the curvature foliations of M we see that to each m_i-fold principal curvature λ_i of an isoparametric hypersurface there corresponds an $(n - m_i)$-dimensional

smooth focal manifold $M(\lambda_i)$, and the hypersurface is (locally) a tube around $M(\lambda_i)$. The correspondence of shape operators described above remains true on the focal manifolds in the following sense: $p \in M$ determines a point $p_i \in M(\lambda_i)$ and a unit normal vector ξ_i of $M(\lambda_i)$ at p_i, namely the tangent of the normal great circle $t \longmapsto \exp_p(t\,\xi(p))$ at $t = \varphi_i$.



Then the eigen-values of the $M(\lambda_i)$-shape operator S^i with respect to ξ_i are

$$\cot\,(\varphi_j - \varphi_i)\;, \quad j \neq i, \quad \text{where } \lambda_j = \cot\varphi_j \tag{32}$$

and the eigen-spaces correspond to those of S at p under parallel translation along the normal great circle.
We see that on the focal manifolds of an isoparametric hypersurface the eigenvalues of the shape operator are constant: independent of the point and of the unit normal vector. Conversely, given a submanifold with this property, the tubes around it form an isoparametric family. The shape operators of the focal manifold in a fixed point correspond in the way described above to the shape operators of the hypersurface along a whole curvature leaf. Therefore the arguments based on the differential equation in Proposition 3 can also be carried out in the linear algebra context of focal manifolds, compare (29) and (31).

The very explicite knowledge of the principal curvatures also gives us a very good global picture: Let M be a *compact, connected* isoparametric hypersurface in S^{n+1} with unit normal field ξ and eigenvalues

$$\lambda_i = \cot\,(\alpha + \tfrac{g-i}{g}\pi) \quad \text{of multiplicity } m_i .$$

Then

$$\pi_i : p \longmapsto \exp\,((\alpha + \tfrac{g-i}{g}\pi)\,\xi(p))$$

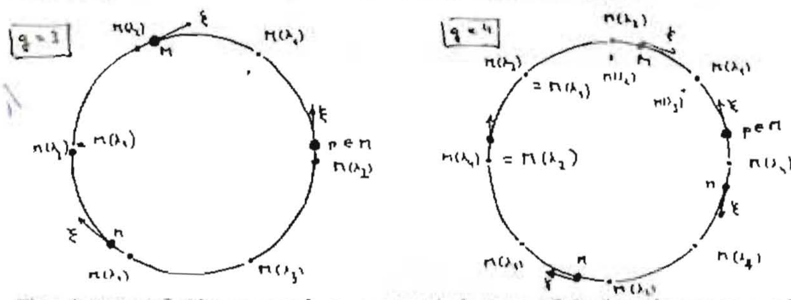
is a submersion of M onto the focal manifold $M(\lambda_i)$, which is therefore connected. M is a tube around each $M(\lambda_i)$ with the λ_i-leaves as π_i-fibres. If we fix $p \in M$, then the normal great circle $t \mapsto \exp(t\xi(p)) = (\cos t)p + (\sin t)\xi(p)$ will therefore meet the λ_i-leaf through p again at

$$\exp\left(2\left(\alpha + \frac{g-i}{g}\pi\right)\xi(p)\right).$$

This implies

$$M(\lambda_i) = M(\lambda_{i+2}) \qquad (33)$$

where by contrast with (27) the indices are not mod g.



The image of the normal exponential map of M is the union of these focal manifolds and of the parallel hypersurfaces of M, and therefore compact and open in S^{n+1}. Hence the isoparametric family determined by M fills the whole sphere. Each hypersurface divides S^{n+1} into two components. Therefore M has exactly two focal manifolds, and S^{n+1} is the union of two solid tubes around the focal manifolds which intersect in M.
This topological situation was studied by MÜNZNER [12] using cohomology theory. He obtained

Theorem 8. [12] If M is an isoparametric hypersurface of S^{n+1} with g distinct principal curvatures, then $g \in \{1,2,3,4,6\}$.

(The compactness-assumption for M is not needed in the theorem for reasons to become apparent in the following section.)

We close this section with a brief remark about minimality questions in connection with isoparametric families.

The mean curvature vector η of a submanifold is characterized by

$$n\langle \eta, \xi \rangle = \operatorname{trace} S_\xi$$

for each normal vector ξ. Since the right hand side is constant for isoparametric submanifolds, we obtain the first assertion [13] of the following theorem. The second one follows easily from (31).

Theorem 9. The focal manifolds of an isoparametric hypersurface are minimal submanifolds. Each isoparametric family contains minimal hypersurfaces.

Note that by (32) the minimality of the focal manifolds implies again Cartan's fundamental equation (25).

oparametric functions

ı this section, which very closely follows MÜNZNER [11], let be an isoparametric hypersurface of S^{n+1} with distinct principal curvatures

$$\lambda_i = \cot\left(\alpha + \frac{g-i}{g}\pi\right), \quad \text{multiplicity } m_i,$$

ıere $1 \leq i \leq g$, and the shape operator is taken with respect to unit normal field ξ. Then there is a smooth function

$$(a,b) : S^{n+1} \supset U \to \mathbb{R} \times M,$$

efined on an open neighborhood U of M, such that

$$q = \exp\left((\alpha - a(q))\,\xi(b(q))\right)$$

ınd $a|M = \alpha$. Up to an additive constant a is the oriented listance from M, and b is the nearest point map. Then

$$\|\operatorname{grad} a\|^2 = 1 \tag{34}$$

and

$$(\operatorname{Hess} a)\operatorname{grad} a = \nabla_{\operatorname{grad} a}\operatorname{grad} a = 0.$$

The mean curvature of the level hypersurfaces of a is given by [5]

$$n\,\|\operatorname{grad} a\|\,H = \Delta a - \frac{\langle (\operatorname{Hess} a)\operatorname{grad} a, \operatorname{grad} a\rangle}{\|\operatorname{grad} a\|^2}$$

or

$$nH = \Delta a.$$

But from (28) and (30) we know nH, and if we restrict ourselves to the slightly more complicated case of even $g = 2k$, we get from those two equations

$$\begin{aligned} nH &= \sum m_i \cot\left(\alpha + \frac{g-i}{g}\pi - (\alpha - a)\right) \\ &= \sum m_i \cot\left(a + \frac{g-i}{g}\pi\right) \\ &= m_{g-1} \sum_{g-i \equiv 1(2)} \cot\left(a + \frac{g-i}{g}\pi\right) + m_g \sum_{g-i \equiv 0(2)} \cot\left(a + \frac{g-i}{g}\pi\right) \end{aligned}$$

$$\begin{aligned} nH &= m_1 \sum_{i=0}^{k-1} \cot\left(a + \frac{i}{k}\pi\right) + m_2 \sum_{i=0}^{k-1} \cot\left(a + \frac{\pi}{g} + \frac{i}{k}\pi\right) \\ &= m_1 k \cot ka - m_2 k \tan ka. \end{aligned}$$

If we choose $\omega \in\,]0,\pi[$ such that

$$\cos^2\omega = \frac{m_1}{m_1+m_2}, \qquad \sin^2\omega = \frac{m_2}{m_1+m_2},$$

then we can continue

$$\begin{aligned} nH &= k(m_1+m_2)\left\{\cos^2\omega \cot ka - \sin^2\omega \tan ka\right\} \\ &= 2n\,\frac{\cos^2\omega\cos^2 ka - \sin^2\omega \sin^2 ka}{\sin ga} \\ &= n\left(\frac{\cos 2\omega}{\sin ga} + \cot ga\right). \end{aligned}$$

Therefore

$$\Delta a = n\left(\cot ga + \frac{\cos 2\omega}{\sin ga}\right).$$

This becomes still simpler if we substitute $f = \cos ga$ for a.

$$\begin{aligned} \Delta f &= \Delta \cos ga \\ &= -g \sin ga\,\Delta a - g^2 \cos ga\,\|\operatorname{grad} a\|^2 \\ &= -gn \cos ga - gn \cos 2\omega - g^2 \cos ga \\ &= -gn f - g^2 f - gn\,\frac{m_1-m_2}{m_1+m_2}, \end{aligned}$$

or

$$\Delta f + g(n+g) f = g^2\,\frac{m_2-m_1}{2}. \tag{35}$$

The same equation is true for odd g, and the proof is quite simil... . The equation (34) implies

$$\|\operatorname{grad} f\|^2 = g^2(1-f^2). \tag{36}$$

Finally we extend f as a positive-homogeneous function F of degree g onto the open cone $\{tq;\ t>0,\ q \in U\}$. Then, using the *euclidean* differential operators instead of the spherical ones, we obtain the first claim of eorem 10. [11] Let M be an isoparametric hypersurface of S^{n+1} th g distinct principal curvatures, and F defined as above. en, with $r(x) = \|x\|$,

i) F satisfies

$$\|\mathrm{grad}\ F\|^2 = g^2\, r^{2g-2} \qquad (37)$$

$$\Delta F = c\, r^{g-2}, \qquad (38)$$

here $c := g^2(m_2-m_1)/2$ (= 0 for g odd).

(ii) F is the restriction of a homogeneous polynomial of degree g.

Conversely, for each homogeneous polynomial F of degree g, that satisfies (37) and (38) the level hypersurfaces of $F|S^{n+1}$ form an isoparametric family. We call such polynomials *isoparametric functions*.

Proof: (i) is a trivial consequence of (36), (35), and the formulas relating the euclidean and the spherical grad and Δ.
(ii) Put $G := F - s\, r^g$ for some real number s. Then

$$\|\mathrm{grad}\ G\|^2 = u\, r^{2g-2} + v\, r^{g-2}G, \qquad (39)$$

since G is homogeneous. Choose s such that $\Delta G = 0$. This is possible by (38), and for odd g we have $s = v = 0$.
From (39)

$$\Delta^g \|\mathrm{grad}\ G\|^2 = 0.$$

On the other hand, since the partials of G are harmonic,

$$0 = \Delta^g \|\mathrm{grad}\ G\|^2 = 2^g \sum (\partial_{i_0} \ldots \partial_{i_g} G)^2,$$

whence all partial derivatives of G of order $g+1$ vanish. Hence G, and therefore F is a polynomial of degree g.
(iii) is a simple computation using Proposition 7.

Corollary 11. Each connected isoparametric hypersurface of S^{n+1} is an open part of a compact isoparametric hypersurface imbedded in a "global" isoparametric family.

According to Theorem 10 (ii) the determination of all isoparametric hypersurfaces in the sphere is an algebraic problem, though a difficult one, because (37) is non-linear.

Homogeneous examples

The Lie algebra $\mathfrak{g}$ of $SU(3)$ decomposes into a direct sum of the subalgebra $\mathfrak{k} = \mathfrak{so}(3)$, and the vector subspace $\mathfrak{p} = \{X \in \mathfrak{g}\ ;\ \bar{X} = -X\}$. Using $\langle X,Y\rangle := -\mathrm{trace}\,(X\bar{Y})$ as an inner product on $\mathfrak{g}$, this decomposition is orthogonal, and $SO(3)$ acts isometrically on the euclidean 5-space $\mathfrak{p}$ by inner automorphisms. Consider the orbit M of

$$x := \frac{1}{\sqrt{2}}\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{p}.$$

For $X \in \mathfrak{k}$ we have

$$[X, x] = 0 \iff X = 0,$$

and it follows that

$$k \longmapsto \mathrm{Ad}(k)x = k\, x\, k^{-1}$$

is a covering map from $SO(3)$ onto M. Hence $\dim M = 3$, and M is a homogeneous and therefore isoparametric hypersurface of S^4.
The tangent space of M at x is

$$T_xM = [\mathfrak{k}, x].$$

The geodesic in M with initial vector $[X,x]$ at x is given by $(\exp tX)\, x\, (\exp -tX)$. Therefore the shape operator at x with respect to a unit normal vector ξ is given by

$$S\,[X,x] = -[X,\xi]. \qquad (40)$$

Using

$$\xi := \frac{1}{\sqrt{6}}\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}$$

a simple computation shows that the principal curvatures are $\pm 1/\sqrt{3}$ and 0. Hence M has $g = 3$ distinct principal curvatures, each of multiplicity 1.
According to our considerations on page 16, the orbit through ξ is a focal manifold, corresponding to $\lambda = 0$. The isotropy group of ξ is $S(O(2)\times O(1))$. Hence the focal manifold is a real projective plane $SO(3)/S(O(2)\times O(1))$, imbedded as a so-called

Veronese surface.

The above example of a homogeneous hypersurface can be generalized very much. Instead of $(SU(3),SO(3))$ one can start with any Riemannian symmetric pair of compact type, and look for an orbit of maximal dimension of the isotropy representation. Its euclidean codimension turns out to be the rank of the symmetric pair. Hence for pairs of rank 2 we get isoparametric hypersurfaces of the sphere. The principal curvatures and their multiplicities can be computed from the roots of the pair. In this way one obtains homogeneous examples with the following values of g and (m_1,m_2):

g	(m_1,m_2)	Remarks
1	-	M is a hypersphere
2	$(k,n-k)$	M is a product of two round spheres
3	$(1,1),(2,2)$ $(4,4),(8,8)$	M is a tube around a projective plane over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{C}$ay.
4	$(1,k-1)$ $(2,2k-1)$ $(4,4k-1)$ $(2,2)$ $(4,5)$ $(9,6)$	k any natural number
6	$(1,1),(2,2)$	

A more detailed list, giving also the Riemannian symmetric pairs, is contained in [18], where you can also find details of the construction outlined above. Moreover [18] contains the proof of

Theorem 12. [18], [8] Each homogeneous (isoparametric) hypersurface of the sphere is an orbit of the isotropy representation of a Riemannian symmetric pair of rank 2, and hence contained in the list given in [18].

Clifford examples for $g = 4$

An $(n+1)$-tuple $(P_o,\dots,P_m)$ of symmetric endomorphisms of $\mathbb{R}^{2l}$ is called a *Clifford system*, if

$$P_iP_j + P_jP_i = 2\,\delta_{ij}\,\mathrm{Id}. \tag{41}$$

We have the following

Theorem 13. [7] . Given a Clifford system $(P_o,\dots,P_m)$ on $\mathbb{R}^{2l}$, such that $m_1 := m$ and $m_2 := l - m - 1$ are positive, then

$$F(x) := \|x\|^4 - 2\sum \langle P_ix,x\rangle^2 \tag{42}$$

is an isoparametric function defining an isoparametric family with $g = 4$, and multiplicities (m_1,m_2).

Proof: We have

$$\mathrm{grad}_x F = 4\langle x,x\rangle x - 8\sum\langle P_ix,x\rangle P_ix$$

$$\|\mathrm{grad}_x F\|^2 = 16\langle x,x\rangle^3 + 64\sum\langle P_ix,x\rangle\langle P_jx,x\rangle\langle P_ix,P_jx\rangle$$
$$- 64\sum\langle P_ix,x\rangle^2\langle x,x\rangle$$
$$= 16\langle x,x\rangle^3,$$

because $\langle P_ix,P_jx\rangle = \langle P_jP_ix,x\rangle = \langle x,x\rangle\delta_{ij}$. Note that P_jP_i is skew-symmetric for $i\neq j$.

Furthermore

$$\Delta_x F = 4(2l+2)\langle x,x\rangle - 2\sum(2\langle P_ix,x\rangle\Delta\langle P_ix,x\rangle + 2\|\mathrm{grad}\langle P_ix,x\rangle\|^2)$$
$$= 8(m_2-m_1)\langle x,x\rangle.$$

The assertion follows from Theorem 10.

We shall now study the following questions:
How many Clifford systems are there?
To what extent is a Clifford system determined by the induced isoparametric family?
Which Clifford examples of isoparametric hypersurfaces are homogeneous?

We begin with the study of Clifford systems. From $P_i^2 = Id$ it follows that the eigenvalues of P_i are ± 1. From $P_iP_j + P_jP_i = 0$, $i \neq j$, it follows that P_j interchanges the eigenspaces $E_+(P_i)$ of P_i, whence $\dim E_+(P_i) = \dim E_-(P_i) = l$. For $j \geq 2$ we have $P_o(P_1P_j) = (P_1P_j)P_o$. Therefore we can define

$$E_i := P_1P_{i+1}|E_+(P_o) : E_+(P_o) \longrightarrow E_+(P_o),$$

for $1 \leq i \leq m-1$. Then the E_i are skew-symmetric and satisfy

$$E_iE_j + E_jE_i = -2\,\delta_{ij}Id. \tag{42}$$

In other words, the E_i give an orthogonal representation of the Clifford algebra C_{m-1} on the l-space $E_+(P_o)$. Conversely, given skew-symmetric $E_1,\dots,E_{m-1}$ on $\mathbb{R}^l$, we define symmetric $P_o,\dots,P_m$ on $\mathbb{R}^{2l}$ by

$$P_o(x,y) := (x,-y),\quad P_1(x,y) := (y,x),\quad P_{i+1}(x,y) := (E_iy,-E_ix).$$

Then we obtain a Clifford system.

Two Clifford systems $(P_o,\dots,P_m)$ and $(Q_o,\dots,Q_m)$ on $\mathbb{R}^{2l}$ are called *algebraically equivalent*, if they are conjugate under an orthogonal transformation A of $\mathbb{R}^{2l}$: $Q_i = AP_iA^t$.

Given two Clifford systems $(P_o,\dots,P_m)$ on $\mathbb{R}^{2l}$ and $(Q_o,\dots,Q_m)$ on $\mathbb{R}^{2k}$, then $P_i \oplus Q_i$: $(x,y) \longmapsto (P_ix, Q_iy)$ defines a Clifford system on $\mathbb{R}^{2(l+k)}$, the *direct sum* of the P-system and Q-system. A Clifford system, which cannot be written as a non-trivial direct sum (up to algebraic equivalence) will be called *irreducible*. Obviously each Clifford system is the direct sum of irreducible ones, and the latter are obtained in the indicated way from irreducible representations of Clifford algebras.

Now from the Clifford representation theory, see for example [9], we obtain the following facts:

(i) There are irreducible Clifford systems $(P_o,\dots,P_m)$ on $\mathbb{R}^{2l}$ for, and only for the following values of m and $l = \delta(m)$:

m	1	2	3	4	5	6	7	8	$m+8$
$\delta(m)$	1	2	4	4	8	8	8	8	$16\,\delta(m)$

(43)

(ii) For $m \not\equiv 0 \bmod 4$ there is exactly one irreducible system up to algebraic equivalence, for $m \equiv 0 \bmod 4$ there are two, which can be distinguished as follows: for $m \equiv 0 \bmod 4$, irreducible systems $(P_o,\dots,P_m)$ and $(Q_o,\dots,Q_m)$ are algebraically equivalent if and only if

$$\operatorname{trace}(P_o\dots P_m) = \operatorname{trace}(Q_o\dots Q_m) = \pm\, 2\,\delta(m).$$

It follows that for $m \not\equiv 0 \bmod 4$ and $l = k\delta(m)$ there is only one algebraic equivalence class of Clifford systems, for $m \equiv 0 \bmod 4$ there are $k+1$.

We now come to our second question: To what extent is the Clifford system determined by the induced isoparametric family, or rather by the congruence class of that family?
It follows immediately from

$$\langle AP_iA^tx, x\rangle = \langle P_iA^tx, A^tx\rangle$$

that algebraically equivalent systems induce congruent families. But the converse is not true. To see this, let (α_{ij}) be an orthogonal $(m+1)$-matrix, and let $(P_o,\dots P_m)$ be a Clifford system. Put

$$Q_j = \sum_{i=0}^{m}\alpha_{ij}P_i .$$

A straight-foreward computation shows that $(Q_o,\dots,Q_m)$ is again a Clifford system, and induces the same isoparametric function as $(P_o,\dots,P_m)$. But taking $(\alpha_{ij}) = -Id$ gives algebraically inequivalent systems for $m \equiv 0 \bmod 4$, $l = \delta(m)$.

As a consequence of these considerations the isoparametric function depends only on $\operatorname{span}(P_o,\dots,P_m)$ in the space of symmetric endomorphisms, and each orthonormal basis of this span is a Clifford system. We call the unit sphere in $\operatorname{span}(P_o,..,P_m)$ the *Clifford sphere* determined by $P_o,\dots,P_m$ and denote it by

$(P_o,\dots,P_m)$. We are thus led to define: Two Clifford systems $(P_o,\dots,P_m)$ and $(Q_o,\dots,Q_m)$ in $\mathbb{R}^{2l}$ are *geometrically equivalent* if there exists an orthogonal transformation A of $\mathbb{R}^{2l}$ such that the two Clifford spheres are conjugate under A:

$$\Sigma(P_o,\dots,P_m) = A\,\Sigma(Q_o,\dots,Q_m)A^t.$$

By our earlier remarks the irreducible Clifford systems for $m \equiv 0 \bmod 4$ are geometrically equivalent: just replace one P_i by its negative. One can easily show that

$$|\operatorname{trace}(P_o\dots P_m)|$$

is invariant under geometric equivalence, and therefore there are $\left[\frac{k}{2}\right]+1$ geometric equivalence classes of Clifford systems with $l = k\delta(m)$, $m \equiv 0 \bmod 4$.

The congruence class of the isoparametric family depends only on the geometric equivalence class, and the converse of this is now true in "most" cases:

Theorem 14. [7] Let M be an isoparametric hypersurface of S^{n+1}, $n+2 = 2l$, with multiplicities $m_1 \le m_2$. If there exists a Clifford system $(P_o,\dots,P_m)$ inducing M with $m = m_1$, then $\Sigma(P_o,\dots,P_m)$ is uniquely determined by M *in a geometric way*.

Remark. It follows from (43), that for Clifford examples $m = m_1 \le m_2 = k\delta(m) - m - 1$ except for finitely many (namely 8) exceptions. Therefore the assumption $m = m_1 \le m_2$ is not very restrictive. We shall come back later to the exceptional cases. Note moreover, that the congruence class of $\Sigma(P_o,\dots,P_m)$ is already determined by m and l, unless $m \equiv 0 \bmod 4$. But even in the case $m \not\equiv 0 \bmod 4$ the following proof gives a nice geometric description of $\Sigma(P_o,\dots,P_m)$.

Proof of Theorem 14: Let F be the isoparametric function determined by M. Then the two focal manifolds of M are

$$M_\pm := F^{-1}(\{\pm 1\})$$

We are interested in

$$M_- = \left\{x \in S^{n+1};\ \sum_{i=0}^{m} \langle P_i x, x\rangle^2 = 1\right\}.$$

Now for $\|x\| = 1$ the vectors $P_o x,\dots,P_m x$ are orthonormal, whence

$$M_- = \left\{x \in S^{n+1};\ \text{exists } P \in \Sigma(P_o,\dots,P_m) \text{ s.t. } Px = x\right\}. \tag{44}$$

Since any orthogonal $P, Q \in \Sigma(P_o,\dots P_m)$ anticommute, the P in (44) is uniquely determined by x. Hence M_- is foliated by the intersection of S^{n+1} with the (+1)-eigenspaces $E_+(P)$, $P \in \Sigma(P_o,\dots,P_m)$. We shall show that this foliation can be characterized geometrically: the tangent space of the leaf through $x \in M_-$ is spanned by the kernels of the non-zero shape operators of M_- at x. But the foliation determines the $E_+(P)$, and therefore $\Sigma(P_o,\dots,P_m)$.
We have to compute the shape operators of M_-. But M_- consists of singular points of F, and the direct computation of its geometrical data is somewhat awfull. We therefore first compute them for the simpler M_+ instead of M_-, and then use the fact, that the eigenspaces of the shape operators are parallel along normal great circles of our family, and the change of eigenvalues is explicitely known. M_+ is simpler to handle than M_-, because

$$M_+ = \{ y \in S^{n+1};\ \langle P_o y, y\rangle = \dots = \langle P_m y, y\rangle = 0 \}$$

admits a set of m+1 independent defining equations.
At $y \in M_+$

$$\perp_y M_+ = \operatorname{span}(P_o y,\dots P_m y) \tag{45}$$

$$T_y M_+ = \{ X;\ \langle X, y\rangle = \langle X, P_o y\rangle = \dots = \langle X, P_m y\rangle = 0\} \supset \{P_i P_j y;\ i \ne j\}. \tag{46}$$

The shape operator S_i^+ with respect to the normal field $y \mapsto P_i y$ is easily computed:

$$S_i^+ X = 0 \quad \text{for } X = P_i P_j y,\ j \ne i \tag{47}$$

$$S_i^+ X = -P_i X \quad \text{for } X \in T_y M,\ \langle X, P_i P_j y\rangle = 0 \text{ for all } i,j \tag{48}$$

Now, given $x \in M_-$ and , say, $P_0 x = x$, put

$$N(x) := \{ \eta \in E_-(P_0);\ \langle \eta, P_1 x\rangle = \dots = \langle \eta, P_m x\rangle = 0 \}.$$

This space has dimension $l-m = m_2+1$, and for $\eta \in N(x)$, $\|\eta\| = 1$, we have

$$y := (x-\eta)/\sqrt{2} \ \in M_+$$

$$P_0 y = (x+\eta)/\sqrt{2} \ \in \perp_y M_+$$

and

$$\perp_y M_+ = \operatorname{span}(P_0 y, \dots, P_m) = \operatorname{span}(P_0(x-\eta), \dots, P_m(x-\eta)) \tag{49}$$

$$\ker S_0^+ = \operatorname{span}(P_0P_1 y, \dots, P_0P_m y) = \operatorname{span}(P_1(x+\eta), \dots) \tag{50}$$

$$E_-(S_0^+) = \{ X \in E_+(P_0);\ \langle X, y\rangle = \langle X, P_0 y\rangle = \dots = \langle X, P_m y\rangle = 0 \}$$

$$= \{ X \in E_+(P_0);\ \langle X, x\rangle = \langle X, P_1\eta\rangle = \dots = \langle X, P_m\eta\rangle = 0 \} \tag{51}$$

$$E_+(S_0^+) = \{ X \in E_-(P_0);\ \langle X, y\rangle = \langle X, P_0 y\rangle = \dots = \langle X, P_m y\rangle = 0 \}$$

$$= \{ X \in E_-(P_0);\ \langle X, \eta\rangle = \langle X, P_1 x\rangle = \dots = \langle X, P_m x\rangle = 0 \}. \tag{52}$$

Using the normal great circle $t \mapsto (\cos t)y + (\sin t)P_0 y$ to translate the data to $x = (\cos \pi/4)y + (\sin \pi/4)P_0 y$ we obtain

$$E_-(S_\eta^-) = \operatorname{span}(P_1(x-\eta), \dots, P_m(x-\eta)) \tag{49'}$$

$$E_+(S_\eta^-) = \operatorname{span}(P_1(x+\eta), \dots, P_m(x+\eta)) \tag{50'}$$

$$\ker S_\eta^- = \{ X \in E_+(P_0);\ \langle X, x\rangle = \langle X, P_1\eta\rangle = \dots = \langle X, P_m\eta\rangle = 0 \} \tag{51'}$$

$$\perp_x M_- = \{ X \in E_-(P_0);\ \langle X, x\rangle = \langle X, P_1 x\rangle = \dots = \langle X, P_m x\rangle = 0 \}. \tag{52'}$$

Hence

$$T_x M_- = \operatorname{span}(P_1 x, \dots, P_m x) \oplus \operatorname{span}(P_1\eta, \dots, P_m\eta) \oplus \ker S_\eta^- \tag{53}$$

and

$$\operatorname{span}\{\ker S_\eta^-;\ \eta \neq 0\} = \Big(\bigcap_{\eta \neq 0} \operatorname{span}(P_1 x, \dots, P_m x) \oplus \operatorname{span}(P_1\eta, \dots, P_m\eta) \Big)^\perp$$

$$= \Big(\operatorname{span}(P_1 x, \dots, P_m x) \oplus \bigcap_{\eta \neq 0} \operatorname{span}(P_1\eta, \dots, P_m\eta) \Big)^\perp \tag{54}$$

since $P_1 x, \dots, P_m x \in E_-(P_0)$, $P_1\eta, \dots, P_m\eta \in E_+(P_0)$.
Now

$$\bigcap_{\eta \neq 0} \operatorname{span}(P_1\eta, \dots, P_m\eta) = 0 . \tag{55}$$

This can be seen as follows: If we had $0 \neq u \in \operatorname{span}(P_1\eta, \dots)$, then for every $\eta \neq 0$ in $\perp_x M_-$ there would exist $P \in \operatorname{span}(P_1, \dots, P_m)$ such that

$$P\eta = u$$

or

$$\|P\|^2 \eta = P^2 \eta = Pu.$$

But then the linear map

$$\operatorname{span}(P_1, \dots, P_m) \longrightarrow \mathbb{R}^{2l}, \quad P \longmapsto Pu$$

would have rank greater or equal to $\dim \perp_x M_- = m_2+1$, contradicting $m = m_1 \leq m_2$.

From (54) and (55) we have

$$\operatorname{span}\{\ker S_\eta^-;\ \eta \neq 0\} = (\operatorname{span}(P_1 x, \dots, P_m x))^\perp,$$

the orthogonal complement being taken in $T_x M_-$. But this orthogonal complement contains only vectors in $E_+(P_0)$, see (53) and (51'), and has dimension $2m_1 + m_2 - m_1 = m_1 + m_2 = l - 1$. Therefore

$$E_+(P_0) = \operatorname{span}\{\ker S_\eta^-;\ 0 \neq \eta \in \perp_x M_-\} \oplus \mathbb{R}x,$$

and the theorem follows.

From the theorem and the results on Clifford systems mentioned earlier we see:

For $m \equiv 0 \bmod 4$ there are $\left[\frac{k}{2}\right] + 1$ incongruent isoparametric families on euclidean $2k\delta(m)$-space.

The only possible exceptions are $m = 4$, $k = 2$, and $m = 8$, $k = 2$, because for these $m_1 > m_2$. It can however be shown, that in these cases too the two geometric equivalence classes of Clifford systems lead to incongruent families, see the following section p. 34 ff. Besides the problem of the $m_1 > m_2$-cases, there is a related problem not touched upon so far: Can two Clifford examples with multiplicities (m_1, m_2) and $(\tilde{m}_1, \tilde{m}_2) = (m_2, m_1)$ be congruent? We just mention the results without going into proofs. Since this $m_1 > m_2$ - irregularity occurs only for small dimensions, we first present a list of the low-dimensional Clifford examples. Here $(\underline{m_1, m_2})$, $(\underline{\underline{m_1, m_2}})$, ... means that there are two, three, ... geometric equivalence classes with multiplicity (m_1, m_2).

–	–	–	–	(5,2)	(6,1)	–	–	–	(9,6)
–	(2,1)	(3,4)	$(\underline{4,3})$	(5,10)	(6,9)	(7,8)	$(\underline{8,7})$	(8,7)	...
(1,1)	(2,3)	(3,8)	$(\underline{4,7})$	...	...	...	...	...	...
(1,2)	(2,5)	(3,12)	$(\underline{\underline{4,11}})$	...	...	...	...	...	...
(1,3)	(2,7)	...	....	...	...	...	...	...	...
...	...	...	....	...	...	...	...	...	...

Hence $m_1 > m_2$ only for $(2,1), (5,2), (6,1), (\underline{8,7}), (9,6), (\underline{4,3})$. The families with the first three multiplicities can be shown to be congruent to $(1,2), (2,5), (1,6)$ respectively, while $(9,6)$ and the two $(8,7)$s are not congruent with $(6,9)$ or $(7,8)$. Finally, one of the two $(4,3)$s (the indefinite one with trace $P_0 \ldots P_4 = 0$) is congruent to the $(3,4)$, while the other is not, see the following section.

Before turning to our third question I want to stop briefly to discuss a very fascinating consequence of the above. Note that our Clifford hypersurfaces have at least three principal curvatures different from zero, and are therefore rigid in the sphere. But if we take two incongruent families with the same multiplicities, and from each we choose a hypersurface, such that these two have the same principal curvatures (possible by (28), (30)), then on both hypersurfaces the shape operator and (equation of Gauss) the curvature operator behave pointwise the same.

To make this precise, let us define: Two riemannian manifolds M and M' have the same curvature tensor at $x \in M$ and $x' \in M'$, if there exists an isometry $j: T_x M \longrightarrow T_{x'} M'$ such that for the respective curvature tensors we have

$$jR(X,Y)Z = R'(jX, jY)jZ.$$

Then we have the following consequence of Theorem 14:

Corollary 15. For $m \equiv 0 \bmod 4$, and any natural number k, there exist $[k/2] + 1$ non-isometric compact riemannian manifolds with the same curvature tensor (at any two points of any two of them). The dimension of these manifolds is $2k\delta(m) - 2$.

We now turn to the third question: Which Clifford examples are homogeneous? Obviously, most are not: look at the multiplicities. But we can extend our question, and also ask for local geometric invariants which prove inhomogeneous examples to be inhomogeneous. Let $P_0, \ldots, P_m$ be a Clifford system on $\mathbb{R}^{21}$ with $m = m_1 \geq 3$, $m_2 = 1 - m - 1 > 0$, and consider the focal manifold M_+. Let N_+ be the set of all points $y \in M_+$ such that there are orthonormal vectors $\eta_1, \eta_2 \in \perp_y M_+$ such that

$$\dim(\ker S_{\eta_1} \cap \ker S_{\eta_2}) > 1.$$

Then we have

Lemma 16. N_+ is the set of all $y \in M_+$ for which there are orthonormal $Q_0, \ldots, Q_3$ in $\operatorname{span}(P_0, \ldots, P_m)$ with

$$Q_0 \ldots Q_3 y = y.$$

Proof: First let $y \in N_+$. By replacing $P_0, \ldots, P_m$ by another orthonormal basis of their span we may assume that

$$\ker S_{P_0 y} \cap \ker S_{P_1 y} = \operatorname{span}\{P_0 P_i y;\ i \neq 0\} \cap \operatorname{span}\{P_0 P_i;\ i \neq 1\}$$

has dimension greater than 1, compare (45), (47). Hence in this intersection there is a unit vector u orthogonal to $P_0 P_1 y$. Then $u = P_0 Q_2 y = P_1 Q_3 y$ with $Q_2, Q_3 \in \operatorname{span}(P_0, \ldots, P_m)$,

$\langle P_o,Q_2\rangle = \langle P_1,Q_3\rangle = 0$. Now for $|x| = 1$ the map $Q \longrightarrow Qx$ is isometric, whence $Q_o := P_o$, $Q_1 := P_1$, Q_2, Q_3 are orthonormal, and

$$Q_o\ldots Q_3 y = -Q_oQ_2Q_1Q_3 y = -Q_oQ_2Q_oQ_2 y = y.$$

Conversely, let $Q_o,\ldots,Q_3 \in \mathrm{span}(P_o,\ldots,P_m)$ be orthonormal, $Q_o\ldots Q_3 y = y$. Then $Q_oQ_1 y$ and $Q_oQ_2 y = Q_1Q_3 y$ are independent vectors in the intersection of the kernels of the shape operators corresponding to the normal directions $Q_o y$, $Q_1 y$.

Theorem 17. [7] Suppose $9 \leq 3m_1 < m_2 + 9$, and in case $m=4$ moreover $P_o\ldots P_4 \neq \pm Id$. Then

$$\emptyset \neq N_+ \neq M_+.$$

Hence the isoparametric family is inhomogeneous.

Proof: Put $P := P_o\ldots P_3$. Then P is symmetric, and anticommutes with each P_i, $0 \leq i \leq 3$, and commutes with any other P_i.
Put $E_+'(P) := E_+(P) \cap S^{n+1}$.
For $x \in E_+(P)$ we have

$$F(x) = \langle x,x\rangle^2 - 2\sum_{i=4}^{m}\langle P_i x,x\rangle^2$$

For $m=3$ we have $E_+'(P) \subset M_+$.
For $m=4$ by our assumption P_4 is indefinite on $E_+(P)$, whence $E_+(P) \cap M_+ = \{x \in E_+'(P)\ ;\ \langle P_4 x,x\rangle = 0\}$ has dimension $l - 2$.
For $m > 4$ we have a Clifford system $P_4,\ldots,P_m$ on $E_+(P)$ whose (+1)-focal manifold has dimension $l-m+2$, and is just $E_+(P) \cap M_+$.
Thus in all three cases $N_+ \supset E_+(P) \cap M_+ \neq \emptyset$.

On the other hand, it is not too hard to show that $P = Q_o\ldots Q_3$ for any other orthonormal basis $Q_o,\ldots,Q_3$ of $\mathrm{span}(P_o,\ldots,P_3)$. Since the Grassmannian $G_4(\mathrm{span}(P_o,\ldots,P_m))$ has dimension $4(m-3)$, the dimension of N_+ is atmost $4(m-3) + \dim E_+(P) \cap M_+ = 4m_1 + m_2 - 9$. If we compare to $\dim M_+ = m_1 + 2m_m$, we see

$$4m_1 + m_2 - 9 < m_1 + 2m_2$$

if (and only if)

$$3m_1 < m_2 + 9.$$

This proves Theorem 17.

The exceptions not covered by the theorem are:

A) $m_1 \leq 2$, $(4,4k-1)$ and $P_o\ldots P_4 = \pm Id$, $(5,2)$, $(6,1)$, $(9,6)$

B) $(4,3)$ and $P_o\ldots P_4 \neq \pm Id$, $(6,9)$, $(7,8)$, $(8,7)$, $(8,15)$, $(10,21)$.

The cases A) are homogeneous. One possible way of proving this is to use explicite Clifford systems (obtained from real division algebras), and then construct sufficiently many isometries leaving the isoparametric function invariant, see the following section.
On the other hand the cases B) are inhomogeneous. This can be shown similar to Theorem 17. Only in the cases $(8,15)$, $(10,21)$ we don't have such a proof. But their multiplicities do not occur in the list of homogeneous examples.

Besides the homogeneous examples there were two inhomogeneous series of multiplicity $(3,4k)$ and $(7,8k)$ known before, [15,16] . These series coincide with our Clifford series of the same multiplicities. Again this can be shown using explicite Clifford systems, and checking conditions (A),(B) of [15].

The only isoparametric families known that are not Clifford are the homogeneous examples of multiplicities $(2,2)$ and $(4,5)$.

Specific Clifford examples

In this section we discuss some of the questions concerning Clifford examples using explicite representations.

Let i_1, i_2, i_3 be the imaginary units of the quaternions $\mathbb{H}$. Define

$$E_j : \mathbb{H}^n \longrightarrow \mathbb{H}^n, \quad u \longmapsto i_j u \tag{56}$$

to be left-multiplication by i_j. Then the E_j are skew-symmetric, satisfy (42), and hence define a Clifford system $P_0^{(n)}, \dots, P_4^{(n)}$ on $\mathbb{H}^n \oplus \mathbb{H}^n = \mathbb{R}^{2 \cdot 4n}$.

Put $c_1 := 1$, $c_j := i_{j-1}$, $2 \le j \le 4$. Then the isoparametric function associated with $P_0^{(n)}, \dots, P_4^{(n)}$ is

$$F(u,v) = (\|u\|^2 + \|v\|^2)^2 - 2\{(\|u\|^2 - \|v\|^2)^2 + 4\sum \langle u, c_i v\rangle^2\} \tag{57}$$

where $u, v \in \mathbb{H}^n$. If $n=1$ then $F = -1$, and $m_2 = 4-4-1 < 0$. This does not give an isoparametric family. For $n > 1$ however we get one, and we concentrate on the case $n=2$.

Put $P_i := P_i^{(2)}$. Then

$$|\operatorname{trace} P_0 \dots P_4| = |\operatorname{trace}(-\mathrm{Id})| = 16,$$

and the family corresponding to $(P_0, \dots, P_4)$ has multiplicities $(4, 8-4-1) = (4,3)$.

Now F is invariant under the following isometries

$$\{(\cos s)P_0 + (\sin s)P_1 \; ; \; s \in \mathbb{R}\} \tag{58}$$

$$\{\mathrm{Id} \oplus \alpha\,\mathrm{Id} \; ; \; \alpha \in \mathbb{H}, \; |\alpha| = 1\} \tag{59}$$

$$\{A \oplus A \; ; \; A \in Sp(2)\}. \tag{60}$$

The invariance is trivial for (60) and easily verified for (58). For (59) note that the last sum in (57) is the square of $\|v\|$-times the length of the orthogonal projection of u onto $\mathbb{H}v = \alpha\mathbb{H}v$.

A point $(u,v) \in S^{15} \subset \mathbb{H}^2 \oplus \mathbb{H}^2$ can be moved by an isometry of type (58) into a point (u',v') with $\|u'\|^2 = \|v'\|^2 = 1/2$. Using (60) we may therefore assume that $(u,v) = (1/\sqrt{2}, 0, v_1, v_2)$. By (59) we can moreover have $v_1 \in \mathbb{R}$, $v_1 \ge 0$, and, using (60) again, we see that (u,v) is equivalent with a point

$$(\bar{u}, \bar{v}) = (1, 0, \cos t, \sin t)/\sqrt{2} \quad , \quad 0 \le t \le \pi/2.$$

But then $F(u,v) = F(\bar{u},\bar{v}) = \cos 2t$. Therefore the isometries (58) - (60) generate a group acting transitively on the level hypersurfaces of the function: *the family is homogeneous.* (Note that this can be proved similarly for arbitrary n as well as for $\mathbb{R}$ or $\mathbb{C}$ instead of $\mathbb{H}$.)

On the other hand, if we omit P_0 we get a Clifford system $P_1, \dots, P_4$ on $\mathbb{H}^2 \oplus \mathbb{H}^2 = \mathbb{R}^{16}$ which induces a function

$$F(u,v) = (\|u\|^2 + \|v\|^2)^2 - 8\sum \langle u, c_i v\rangle^2,$$

and a family of multiplicities (3,4). Consider the focal manifold M_+ of this family. Obviously $y := (1,0,0,1)/\sqrt{2}$ is a point of M_+, and at this point we have by (47)

Can simply use Thm 17

$$\begin{aligned} \ker S^+_{P_i y} &= \operatorname{span}\{P_j P_i y \; ; \; j \ne i\} \\ &= \operatorname{span}\{(c_j \bar{c}_i, 0, 0, \bar{c}_j c_i) \; ; \; j \ne i\}, \end{aligned}$$

whence

$$\bigcap_i \ker S^+_{P_i y} = \{0\}.$$

On the other hand for $z := (1,0,0,0) \in M_+$

$$\begin{aligned} \ker S^+_{P_i z} &= \operatorname{span}\{(c_j \bar{c}_i, 0, 0, 0) \; ; \; j \ne i\} \\ &= \{(x,0,0,0) \; ; \; \bar{x} = -x\} \end{aligned}$$

is independent of i, and hence the kernels intersect in a 3-dimensional space. Therefore M_+, and hence the family is *not homogeneous.*

Finally we can do the above construction for the Cayley numbers $\mathbb{C}\mathrm{ay}$ instead of the quaternions. For $n=1$ we obtain a Clifford system $P'_0,\dots,P'_8$ on $\mathbb{C}\mathrm{ay} \oplus \mathbb{C}\mathrm{ay} = \mathbb{R}^{16}$. Then $P'_0,\dots,P'_4$ and $P'_5,\dots,P'_8$ are two Clifford systems leading to families of multiplicities (4,3) and (3,4) respectively. Since

$$\begin{aligned}\sum_i \langle P'_i(u,v),(u,v)\rangle^2 &= (\|u\|^2 - \|v\|^2)^2 + 4\sum\langle u, c_i v\rangle^2 \\ &= (\|u\|^2 - \|v\|^2)^2 + 4\|u\|^2\|v\|^2 \\ &= (\|u\|^2 + \|v\|^2)^2\end{aligned}$$

we see that for $x = (u,v) \in \mathbb{R}^{16}$

$$\|x\|^4 - 2\sum_{i=0}^{4}\langle P'_i x, x\rangle^2 = -(\|x\|^4 - 2\sum_{i=5}^{8}\langle P'_i x, x\rangle^2).$$

Hence the (4,3)-family and the (3,4)-family coincide.

By the algebraic results cited earlier $(P_1,\dots,P_4)$ must be equivalent with $(P'_5,\dots,P'_8)$ (Exercise: Construct an explicite equivalence!) By contrast, since $P'_0\dots P'_4$ anticommutes with P'_5, its trace must be zero, and $(P'_0,\dots,P'_4)$ is *not* geometrically equivalent with $(P_0,\dots,P_4)$ above.

To sum up our results: There are two Clifford examples with multiplicities (4,3) or (3,4). The definite (4,3) is homogeneous (as are the definite (4,4k-1)s), while the indefinite (4,3) coincides with the (3,4), and is inhomogeneous.

Final remark.

We know more about the geometry and topology of the Clifford examples than discussed in these notes, see [7]. The central problem however remains open: the classification of isoparametric families with $g=4$ and $g=6$.

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