On the Riemann Mapping Theorem via Dirichlet Principle

Let G be a simply connected region in $\mathbb{C}$ with a non-trivial sufficiently smooth boundary ∂G parametrized by $c : [0,1] \to \mathbb{C}$, and let $z_0 \in G$. We are looking for a continuous function $f : \bar{G} \to \mathbb{C}$, holomorphic on G, with $f(z_0) = 0$ and

$$|f(z)| = 1 \text{ for all } z \in \partial G. \tag{1}$$

By the maximum principle then $f(G) \subset \mathbb{D}$. Moreover, the winding number

$$n(f \circ c, a) = \frac{1}{2\pi i} \int_0^1 \frac{f'(c(t))c'(t)}{f(c(t)) - a} dt = \frac{1}{2\pi i} \int_c \frac{f'(z)}{f(z) - a} dz$$

is constant on $\mathbb{D}$, all values $a \in \mathbb{D}$ are attained the same number of times. With the ansatz

$$f(z) = (z - z_0)e^{g(z)} \tag{2}$$

$f : G \to \mathbb{D}$ has exactly one zero, and hence is bijective.

How to find such an f?

We split g into real and imaginary part:

$$g = u + iv.$$

Then (1) is equivalent with

$$\log|z - z_0| + u(z) = 0 \text{ on } \partial G.$$

We therefore want a continuous function u on $\bar{G}$, harmonic on G, with prescribed boundary values $u_0(z) := -\log|z - z_0|$ on ∂G. This is now commonly called a *Dirichlet problem.*

Put $\mathcal{F} := \left\{\tilde{u} : \bar{G} \to \mathbb{R} \;\middle|\; \tilde{u}|_{\partial G} = u_0\right\}$, where the regularity of the $\tilde{u}$ is assumed to suffice for the following arguments. On $\mathcal{F}$ we consider the functional

$$E(\tilde{u}) := \int_{\bar{G}} \|\operatorname{grad} \tilde{u}\|^2 \, dxdy$$

and take $u \in \mathcal{F}$ which minimizes this functional:

$$E(u) := \min_{\tilde{u} \in \mathcal{F}} E(\tilde{u}).$$

Then for any function $v : \bar{G} \to \mathbb{C}$ with compact support in G

$$\begin{aligned} E(u + tv) &= \int_{\bar{G}} \|\operatorname{grad} u + t \operatorname{grad} v\|^2 dxdy \\ &= \int_{\bar{G}} \|\operatorname{grad} u\|^2 + 2t \int_{\bar{G}} \langle \operatorname{grad} u, \operatorname{grad} v \rangle \, dxdy + t^2 \int_{\bar{G}} \|\operatorname{grad} v\|^2 dxdy. \end{aligned}$$

This is minimal for $t = 0$ if and only if $\int_{\bar{G}} \langle \operatorname{grad} u, \operatorname{grad} v \rangle \, dxdy = 0$.

But

$$\begin{aligned}
\int_{\bar{G}} \langle \operatorname{grad} u, \operatorname{grad} v \rangle \, dxdy &= \int_{\bar{G}} \operatorname{div}(v \operatorname{grad} u) dxdy - \int_{\bar{G}} v \Delta u \, dxdy \\
&= \int_{\partial G} \left\langle \underbrace{v \operatorname{grad} u}_{=0}, \text{unit normal } \nu \right\rangle dO - \int_{\bar{G}} v \Delta u \, dxdy.
\end{aligned}$$

Besides the product rule for $\operatorname{div} v \operatorname{grad} u$ we used Stokes and the information $v|_{\partial G} = 0$. Now

$$\int_{\bar{G}} v \Delta u \, dxdy = 0$$

for all compactly supported v implies $\Delta u = 0$, i.e. u is harmonic. This is referred to as the *Dirichlet principle*.

Locally, u is the real part of a holomorphic function g with derivative

$$g' = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}.$$

But this is defined on all of G, and therefore g can be analytically continued along any curve in G. Now G is **simply connected**, and the monodromy theorem applies to give a holomorphic function g with real part u globally defined on G. Let us assume it to be continuous on $\bar{G}$. Then (2) defines f with the required properties.

This "proof" has several unclear points: The regularity requirements for the boundary, the regularity of the functions involved, but overall the *existence of the minimizer* $u \in \mathcal{F}$ for the energy functional E.