On the proof of Artin's homology criterion

Lemma 1. *Let* $c : [0,1] \to G$ *and* $\phi : [0,1] \to [0,1]$ *be continuous with* $\phi(0) = 0, \phi(1) = 1$. *Then*

(i) For $c^{inv}(t) := c(1-t)$
$$c^{inv} \underset{G}{\sim} \ominus c.$$

(ii) For $c^{\phi} := c(\phi(t))$
$$c^{\phi} \underset{G}{\sim} c.$$

(iii) For $\alpha \in]0,1[$ *define* $c_1(t) := c(\alpha t)$ *and* $c_2(t)((1-t)\alpha + t))$. *Then*
$$c \underset{G}{\sim} c_1 \oplus c_2.$$

(iv) For $c_1, c_2 : [0,1] \to G$ *with* $c_1(1) = c_2(0)$
$$c_1 c_2 \underset{G}{\sim} c_1 \oplus c_2.$$

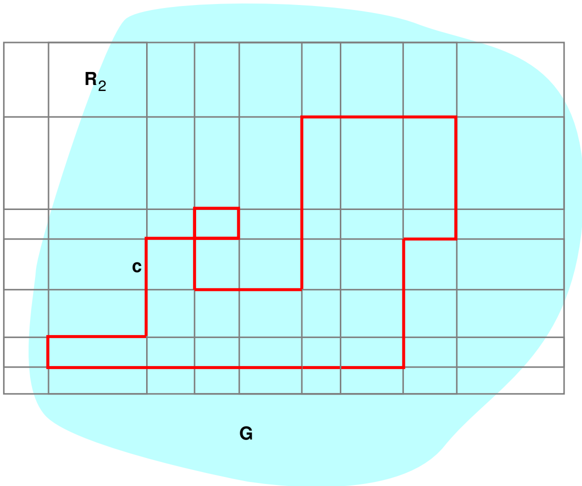
Lemma 2. *Any 1-chain* c *in a region* G *is homologous in* G *to a 1-chain* $\tilde{c} = \sum_{j=1}^{n} m_j \gamma_j$ *with "edge path" curves*
$$\gamma_j(t) = (1-t)a_j + tb_j, \quad 0 \le t \le 1, \tag{1}$$
where $\operatorname{Re} a_j = \operatorname{Re} b_j$ *or* $\operatorname{Im} a_j = \operatorname{Im} b_j$. *If* c *is a 1-cycle, so is* $\tilde{c}$.

Theorem 3 (Artin's criterion). *A 1-cycle* c *is homologous to 0 in a region* G*, if and only if*
$$n(c,a) = 0 \text{ for any } a \notin G.$$

Proof. If $c \underset{G}{\sim} 0$, then $n(c,a) = 0$ for all $a \notin G$ by Cauchy's integral theorem. We prove the converse.

Step 1. By the lemmas we may assume that c is a formal linear combination of edge pathes.

We choose a compact rectangle R that contains $|c|$ in its interior, and subdivide this rectangle into rectangles $R_j, 1 \leq j \leq n$, by drawing horizontal and vertical lines through each initial- or endpoint of each of the edges in c. The R_j are considered compact, so adjacent rectangles overlap on their boundaries. We denote by R_j also the patch $[0,1]^2 \to \mathbb{C}$ that maps the unit square in the obvious way onto the rectangle R_j, and we denote by $\mathcal{E}$ the set of edges of the R_j, i.e. the sides of the $R_j : [0,1]^2 \to \mathbb{C}$.



Then, again by Lemma 1, we have

$$c \underset{G}{\sim} \sum_{\gamma \in \mathcal{E}} m(\gamma)\gamma.$$

To save notation we assume

$$c = \sum_{\gamma \in \mathcal{E}} m(\gamma)\gamma. \tag{2}$$

Step 2. We choose a_j in the interior of R_j for any $j \in \{1, \dots, n\}$. We then define a 2-chain

$$C^* = \sum_{j=1}^{n} n(c, a_j) R_j.$$

If $a \in R_j \setminus G$, then by assumption $n(c, a) = 0$, and the same is true for all $\tilde{a}$ in the same connected component of $\mathbb{C} \setminus |c|$, in particular for $\tilde{a} = a_j$. Therefore R_j has coefficient 0 in C^*, and C^* is a 2-chain in G.

Step 3. We have

$$c^* := \partial C^* = \sum_{j=1}^{n} n(c, a_j) \partial R_j =: \sum_{\gamma \in \mathcal{E}} m^*(\gamma)\gamma,$$

and we claim that $m^*(\gamma) = m(\gamma)$ for all γ, see (2). Then

$$\partial C^* = c,$$

and we proved $c \underset{G}{\sim} 0$ as desired. First we have

$$n(c^*, a_k) = \sum_{j=1}^{n} n(c, a_j) n(\partial R_j, a_k) = n(c, a_k) \tag{3}$$

for any $k \in \{1, \ldots, n\}$. The same is true for $k = 0$, if we choose $a_0 \in \mathbb{C} \setminus R$, which we do.

<u>Assume $m^*(\gamma_0) - m(\gamma_0) = m_0 \neq 0$ for some $\gamma_0 \in \mathcal{E}$.</u> Then there is a j_0 (in general we have the choice of two) such that R_{j_0} contains $\oplus\gamma$ or $\ominus\gamma$ as one of its sides. Assume the sign is $\oplus$, the other case is similar. Put

$$c^\# := c^* \ominus c \ominus m_0 \, \partial R_{j_0}.$$

This is a 1-cycle, and from (3) we see

$$n(c^\#, a_k) = -m_0 \, n(\partial R_{j_0}, a_k) = -m_0 \, \delta_{j_0 k}$$

for $k \in \{0, \ldots, n\}$. Moreover <u>$c^\#$ does not contain γ_0</u>.

There are two possible cases: γ_0 lies on the boundary of R. Then a_{j_0} and a_0 lie in the same connected component of $\mathbb{C} \setminus |c^\#|$, and

$$0 = n(c^\#, a_0) = n(c^\#, a_{j_0}) = -m_0.$$

Contradiction.

The other possibility is that γ_0 is a common side of R_{j_0} and some R_{j_1}. Then the same argument holds with a_{j_1} instead of a_0, and we get again a contradiction.
$\square$