BEETHOVEN Classic 3 – Polish-German Project

Order & Geometry

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Project duration: 2020-2022 (3 years)

DESCRIPTION OF THE RESEARCH PROJECT

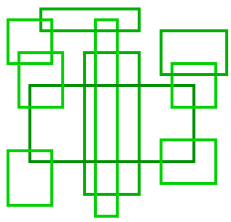
Graphs and orders defined by means of geometric objects provide a rich class of examples in combinatorics and graph theory. The geometric intuition often guides through constructions that are complex and complicated otherwise. Moreover, graphs and orders defined in terms of geometric objects model dependencies in optimization problems and theoretical computer science. Within this project we focus on the combinatorial side of this realm. The research is grouped into three lines and each line will be motivated by some notoriously open, long-standing problems.

The most commonly studied geometrically defined graphs are containment graphs, intersection graphs, and contact graphs. Containment graphs naturally come with a partial order relation but posets also capture relevant structural aspects in many types of intersection and contact graphs. We aim at exploiting these connections between geometrically defined graphs and orders. This is a rather broad program, in the project description we focus on three more specific problem areas where the interplay of order and geometry is relevant, these are:

- Chi-bounded classes of geometrically defined graphs
- Geometrically defined classes of graphs with linear structure
- Geometry and encodings

Chi–bounded classes of geometrically defined graphs

Recall that $\chi(G)$ is the chromatic number of graph G and $\omega(G)$ is the clique number of G that is the size of the largest clique in G. A class of graphs is *χ-bounded*, if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $\chi(G) \leqslant f(\omega(G))$ holds for any graph G from the class. Since there are triangle-free graphs with arbitrarily large chromatic number χ-boundedness is a non-trivial property. In fact, it constitutes a very lively field of research. E.g. Scott, Seymour et al. published a whole series of papers (see e.g. [CSSS17]) containing, in particular, proofs of three long-standing conjectures of Gyárfás [Gya87].



The study of χ-boundedness for geometric intersection graphs was initiated by Asplund and Grünbaum [AG60]. They proved that every family $\mathcal{F}$ of axis-aligned rectangles in the plane satisfies $\chi(\mathcal{F}) \leqslant 4\omega(\mathcal{F})^2 - 3\omega(\mathcal{F})$. The proof uses a partial order on crossing rectangles and a degeneracy argument. For general families of axis-aligned rectangles, we do not know much more than the result of Asplund and Grünbaum. The lower bound is still linear and the upper bound was only modestly improved to $\chi(\mathcal{F}) \leqslant 3\omega(\mathcal{F})^2 - 2\omega(\mathcal{F}) - 1$ by Hendler [Hen98]. It is a true challenge to verify whether

- $\chi(\mathcal{F}) = o(\omega^2(\mathcal{F}))$, for every axis-aligned family $\mathcal{F}$ of rectangles.

For families $\mathcal{F}$ of rectangles with no containment between rectangles, Chalermsook [Cha11] obtained $\chi(\mathcal{F}) = O(\omega(\mathcal{F}) \log \omega(\mathcal{F}))$. A linear bound for this specific case would also improve the quality of the best known approximation algorithm for the Maximum-Independent-Set-of-Rectangles problem (see [Cha11]).

Burling [Bur65] showed that triangle-free intersection graphs of axis-aligned boxes in $\mathbb{R}^3$ can have arbitrarily large chromatic number. The graphs used for the construction are now known as Burling graphs.

In the 1970s, Paul Erdős asked whether intersection graphs of line segments in the plane are χ-bounded. A negative answer was provided by Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter, and Walczak [PKK+14]: The authors represented Burling graphs as intersection graphs of segments in the plane. This result also disproves the conjecture of Scott [Sco97] that, for every graph H, the class of graphs excluding every subdivision of H as an induced subgraph is χ-bounded. More recently Burling graphs have been used to disprove a conjecture about orthogonal tree-decompositions, see Dujmović et al. [DJM+18] and Felsner, Micek et al. [FJM+18]. Studying particular properties of Burling graphs is an exciting topic on its own and by now they have shown the potential to test long-standing conjectures in graph theory.

The construction from [PKK+14] was extended in [PKK+13] to other shapes like axis-aligned ellipses, rhombuses, L-shapes, etc. There is some evidence that unrestricted scaling in two directions is the key property, necessary to make the chromatic number large, while keeping the clique number small. For instance, Suk [Suk14] proved that for families $\mathcal{F}$ of *unit-length* segments in the plane $\chi(\mathcal{F})$ is bounded by a double exponential function of $\omega(\mathcal{F})$. Also families of curves attached to a single line (outerstrings) have χ bounded in terms of ω, see a paper by Micek et al. [LMPW14] and by Rok and Walczak [RW17].

Interestingly, the complements of intersection graphs of segments in the plane are χ-bounded. In that case, the chromatic number χ has been shown to be $O(\omega^4)$ by Pach, Törőcsik et al. [LJMT94]. The proof for this bound is a beautiful example of the interplay of orders and geometry: four partial orders are defined on the family of segments, a four times repeated application of Dilworth's Theorem then yields the result. Pach, Tardos and Tóth [PTT17] show that the same bound holds for the disjointness graph of segments in arbitrary dimension. Disjointness graphs of curves in the plane have also been studied, they are not χ-bounded in general but as soon as the curves are x-monotone and intersect the y-axis the precise χ-bounding function $\frac{\omega+1}{2}\binom{\omega+2}{3}$ has been established by Pach and Tomon [PT18].

Kim, Kostochka and Nakprasit [KKN04] showed that families $\mathcal{F}$, of homothetic copies of a fixed convex compact set in the plane, have $\chi(\mathcal{F}) \leqslant 6\omega(\mathcal{F})$. The result was generalized (with a very simple counting argument) to pseudo-discs in [MP13]. A family $\mathcal{F}$ of simply connected sets in the plane, is a family of *pseudo-discs*, if the boundaries of every two sets from $\mathcal{F}$ intersect in at most two points. Micek and Pinchasi show that $\chi(\mathcal{F}) \leqslant 19\omega(\mathcal{F})$ for families of pseudo-discs. It is an annoying open problem to improve this bound for at least a tiny bit.

Geometrically defined classes of graphs and linear patterns

Interval graphs may be considered to be the best understood class of graphs with linear structure. This class was introduced by Benzer in 1959 [Ben59] and helped to understand the linear structure of the DNA. A second classical example are permutation graphs which are readily described by a permutation (linear order) of the appropriately labelled vertices. Alternatively permutation graphs can be described as intersection graphs of segments with endpoints on two parallel lines. The modern theory of geometric intersection graphs was established in the 1990s by Kratochvíl [Kra91a, Kra91b] and Matoušek [Kra91a, Kra91b, KM91, KM94]. By now geometric intersection graphs are ubiquitous in discrete and computational geometry, and deep connections to other fields such as complexity theory [SSv03, Sch09, Mat14] and order dimension theory [CHO⁺14, Fel14, CFHW18] have been established.

A family $\mathcal{F}$ of geometric objects is called *grounded* if every element of $\mathcal{F}$ is contained in a half-plane H and touches the boundary line ∂H of H in an *anchor point*. Listing the objects according to their anchor point gives a (canonical) linear ordering of the vertices. We have already mentioned intersection graphs of grounded strings (outer strings) in the context of χ-binding.

Outer segment graphs form a natural subclass of outer string graphs. They also generalize the class of circle graphs, which are intersection graphs of chords of a circle. Outerplanar graphs form a proper subclass of circle graphs [WP85], hence of outer segment graphs. Cabello and Jejčič [CJ17] proved that a graph is outerplanar if and only if its 1-subdivision is an outer segment graph. Intersection graphs of rays in two directions (a subclass of outer segment graphs, see [CFHW18]) have been studied by Soto and Telha [ST11], they show connections with the jump number of some orders and hitting sets of rectangles. The class has been further studied by Shrestha et al. [STU10], and Mustaţă et al. [MNT⁺16].

Intersection graphs of L-shapes anchored at their bend has been investigated as *hook-graphs* and as *max point-tolerance graphs*, see [Hix13], [CCF⁺17], and [ST15]. They generalize interval graphs and have various geometric representations and characterizations, e.g., they can be seen as intersection graphs of the rectangles spanned by the L's. A direct proof of χ-boundedness for this class (hopefully with a linear bound) would be of great interest. The recognition problem for hook-graphs and even for the still simpler intersection graphs of grounded vertical and horizontal segments (*stick graphs*) is also open. For the case where the order of anchor points on the grounding line is prescribed there is a polynomial recognition algorithm due to De Luca et al. [LHK⁺18].

Stick graphs and many related bipartite intersection graphs have been studied by Felsner, Hoffmann et al. [CFHW18]. The aim of that paper was to identify the inclusion order on the graph classes and the order dimension turned out to be a very effective tool. Cardinal, Felsner et al. [CFM⁺18] introduced the 'Cycle Lemma' which allows to prescribe the order of anchor points for certain graphs. The lemma was used in [CFM⁺18] and [JT18] to separate further grounded classes of graphs. Jelínek and Töpfer [JT18] also studied forbidden patterns. It had been observed by several groups that hook-graphs can be characterized by a forbidden pattern on the anchor sequence of four vertices, see Figure 1.

Jelínek and Töpfer show that *grounded-L graphs*, i.e., graphs admitting an intersection representation by L's anchored with the upper end of the vertical bar at a horizontal line, admit a forbidden pattern characterization with two patterns on four vertices. The characterization of a class of graphs by forbidden vertex order patterns might conceivably lay the grounds for efficient recognition algorithms. Note, however, that a graph class characterized by a forbidden vertex order pattern may have NP-hard recognition [DGR95]. On the other hand Hell et al. [HMR14] unified many previous results by giving a general polynomial time recognition algorithm for all classes described by a set of forbidden patterns of order at most three.

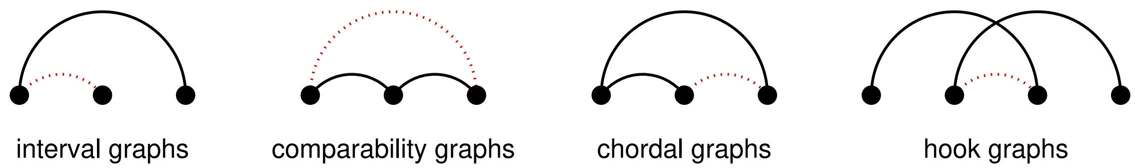


Figure 1: Forbidden order patterns for graph classes. Solid arcs denote compulsory edges and dotted arcs denote compulsory non-edges.

Forbidden patterns are also at the core of important parameters studied for graphs and posets. In a total order of the vertices of a graph, an independent pair of edges can be crossing, nested, or disjoint. A *k-stack* layout (respectively, *k-queue* layout) of a graph consists of a total order of the vertices, and a partition of the edges into k sets of pairwise non-crossing (respectively, non-nested) edges. Motivated by numerous applications, stack layouts (also called book embeddings) and queue layouts are widely studied, see e.g. a survey on stack and queue layouts by Dujmović and Wood [DW04] and a survey on vertex orderings in a broader context by Diaz et al. [DPS02].

When the vertex order is fixed, the minimum number of queues required for a queue layout equals the maximum size of a nesting family of edges. Computing the minimum number of stacks, however, is NP-hard even when the vertex order is prescribed. Despite intense research on the parameters some of the problems and conjectures from the seminal paper by Heath, Lipton and Rosenberg [HLR92] are still unresolved. Two central questions in the field are:

- Is the queue-number of planar graphs bounded?
- Is the stack-number or the queue-number bounded by a function of the other?

Nowakowski and Parker [NP89b] defined the stack-number of a poset as the stack-number of its Hasse diagram viewed as a dag, i.e., the vertex ordering has to be a linear extension. They derive a general lower bound on the stack-number of a planar poset and an upper bound on the stack-number of a lattice. They conclude by asking

- whether the stack-number of the class of planar posets is unbounded.

There have been some recent attacks to the problem, see e.g. Frati et al. [FFR13], still the question in its general form remains open.

Heath and Pemmaraju [HP97] initiated the study of queue layouts of posets. Again, this is the queue-number of the diagram of the poset, whence the vertex ordering has to be a linear extension. They observe that the queue-number of the class of planar posets is unbounded, and bound the queue-number of a planar poset in terms of its width. They conjecture that a poset of width w has queue-number at most w. Knauer, Micek and Ueckerdt [KMU18] continue the study of the queue-number of posets. They have shown that a planar poset of width w has queue-number at most $3w-2$, while the bound on the queue-number for general posets remains $O(w^2)$.

Geometry and encodings

The most important measure for the complexity of a poset is its dimension. The *dimension* $\dim(P)$ of a poset P is the least integer d such that points of P can be embedded into $\mathbb{R}^d$ in such a way that $x \leqslant y$ in P if and only if the point of x is below the point of y with respect to the product order of $\mathbb{R}^d$. Though this definition justifies the geometric intuition behind the notion of dimension, usually we work with the following equivalent. A *realizer* of a poset P is a set $\{L_1,\dots,L_d\}$ of linear extensions of P such that for every $x,y \in P$

$$x \leqslant y \text{ in } P \iff (x \leqslant y \text{ in } L_1) \wedge \cdots \wedge (x \leqslant y \text{ in } L_d),$$

and the dimension of P is the minimum size of its realizer.

This reveals the second nature of the dimension: Realizers provide a way to succinctly encode posets. Indeed if a poset is given with a realizer witnessing dimension d, then a query of the form "is $x \leqslant y$?" can be answered by looking at the relative position of x and y in each of the d linear extensions of the realizer. This application motivates the following more powerful encoding of posets proposed by Nešetřil and Pudlák [NP89a] in 1989. The *Boolean realizer* of a poset P is a set of permutations $\{L_1,\dots,L_d\}$ of elements of P for which there exists a d-ary Boolean formula ϕ such that

$$x \leqslant y \text{ in } P \iff \phi((x \leqslant y \text{ in } L_1),\dots,(x \leqslant y \text{ in } L_d)) = 1,$$

and the *Boolean dimension* of P, denoted $\operatorname{bdim}(P)$, is the minimum size of its Boolean realizer. Clearly, for every poset P we have $\operatorname{bdim}(P) \leqslant \dim(P)$.

The usual dimension of a poset on n elements may be linear in n. Nešetřil and Pudlák showed that Boolean dimension of posets on n elements is $O(\log n)$. They also provide an easy counting argument showing that there are posets on n elements with Boolean dimension at least $c \log n$ for some constant c.

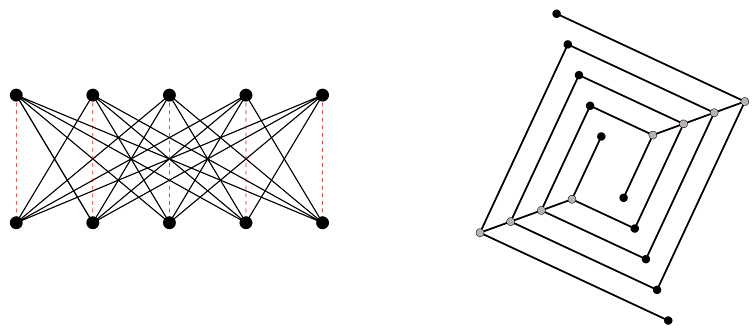


Figure 2: The standard example S_5 (left). Kelly's planar poset containing an induced S_5 (right).

A poset is *planar* if it has a planar diagram. Somewhat unexpectedly planar posets have arbitrarily large dimension. Kelly [Kel81] gave a construction that embeds the standard example S_n of an n-dimensional poset as a subposet into a planar poset (see Figure 2). This shows that the dimension of planar posets is unbounded. Still, the Boolean dimension of standard examples and Kelly's construction is at most 4. There is a beautiful open problem posed by Nešetřil and Pudlák in [NP89a] that remains a challenge with essentially no progress over the years:

- Is the Boolean dimension of planar posets bounded?

We believe to have made an important step towards a resolution of the problem by proving that posets with cover graphs of bounded treewidth have bounded Boolean dimension [FMM17]. This stays in contrast to the ordinary dimension as Kelly's examples have treewidth 3.

Recently, Trotter and Walczak [TW17] studied the interplay of Boolean dimension with yet another concept, the *local dimension*. They propose constructions of families of posets where one of the parameters stays bounded while the other goes to infinity.

The usual dimension is known to be at most 3 for posets with cover graphs being forests (Trotter, Moore [TM77]) and at most 1276 for posets with cover graphs of tree-width 2 (Joret et al. [JMT+17]). As mentioned before, Kelly's examples have tree-width 3 and arbitrarily large dimension. This certifies that Boolean realizers are capable to represent natural classes of posets that are out of reach in the default setting.

It is tempting to speculate, whether the result from [FMM17] generalizes for broader classes of sparse posets. Besides planar posets, it might be true even for posets whose cover graphs exclude a fixed graph as a minor. On the other hand, we have an example (a subdivision of universal interval orders) that this result does not hold for posets whose cover graphs exclude a fixed topological minor. This line of research resembles the series of papers where poset dimension is bounded in terms of the height for posets whose cover graphs are planar (Streib and Trotter [ST14]), or have bounded tree-width (Micek et al. [JMM+16]), or exclude a fixed graph as a minor (Walczak [Wal17] and Micek, Wiechert [MW15], or belong to a fixed class with bounded expansion (Micek et al. [JMW17]).

By now Boolean dimension is not yet well understood. In particular we lack lower bound techniques. We even got stuck on the following easy looking question

- Is the Boolean dimension of a Boolean lattice of order n equal n?

If it is true we expect that there is a beautiful combinatorial argument behind it. See [KMM+18] for similar considerations concerning local dimension.

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