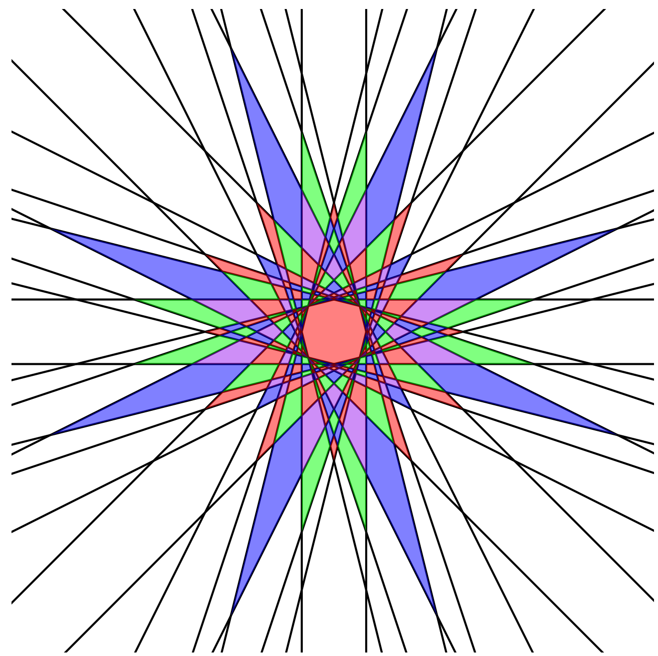
The Number of Arrangements of Pseudolines: Upper and Lower Bounds

October 17, 2024
Villenseminar FU



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contains joint work with

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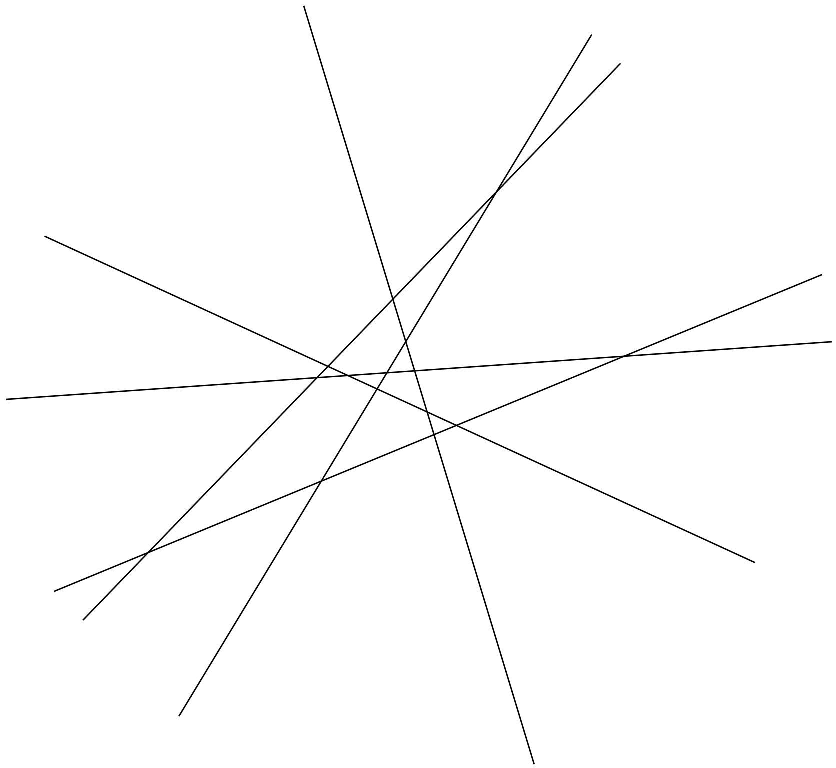
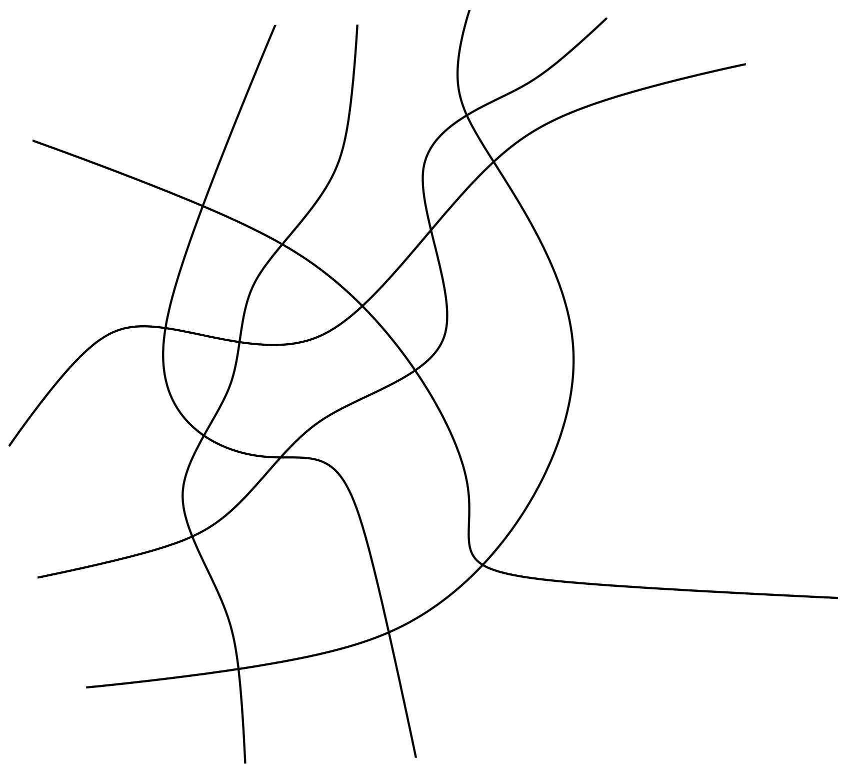
Arrangements of Lines



Pairwise crossing lines.

Arrangements of Pseudolines



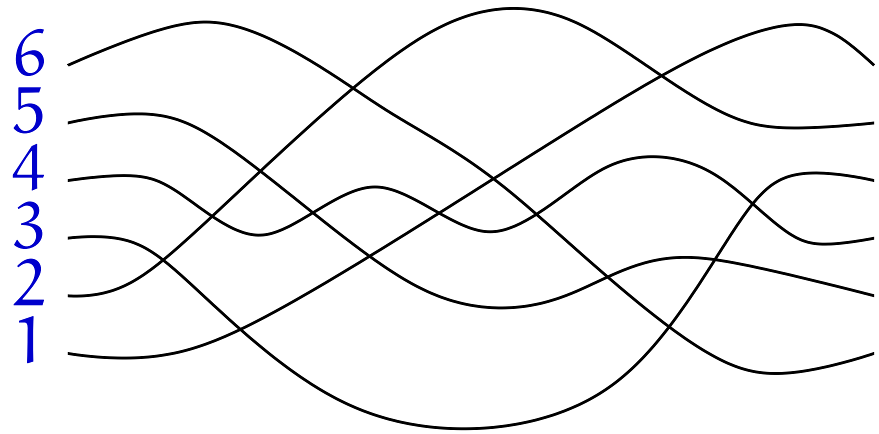
1-crossing curves extending to infinity on both sides.

Our Version of Arrangements of Pseudolines

Euclidean: arrangements in $\mathbb{R}^2$ and not in $\mathbf{P}$.

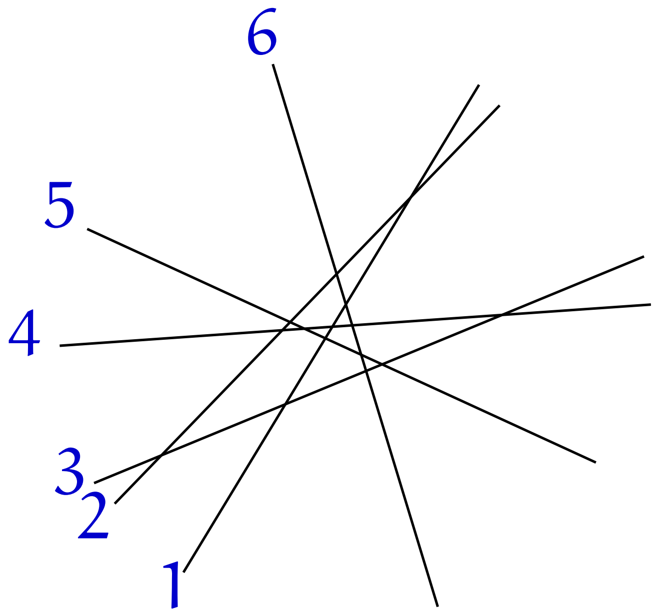
simple: no multiple crossings.

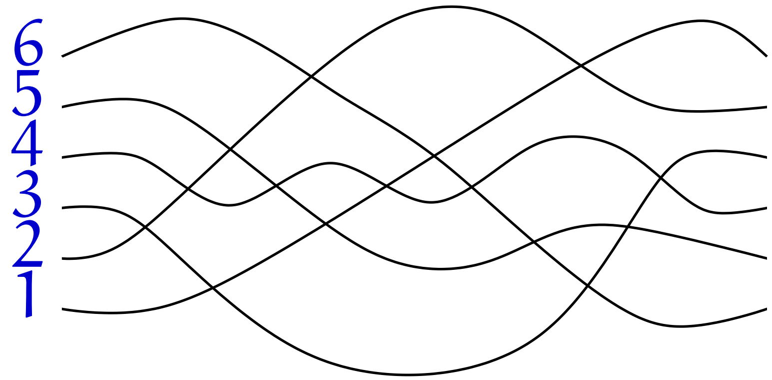
marked: a special unbounded cell is the north-cell.



1-crossing x-monotone curves.

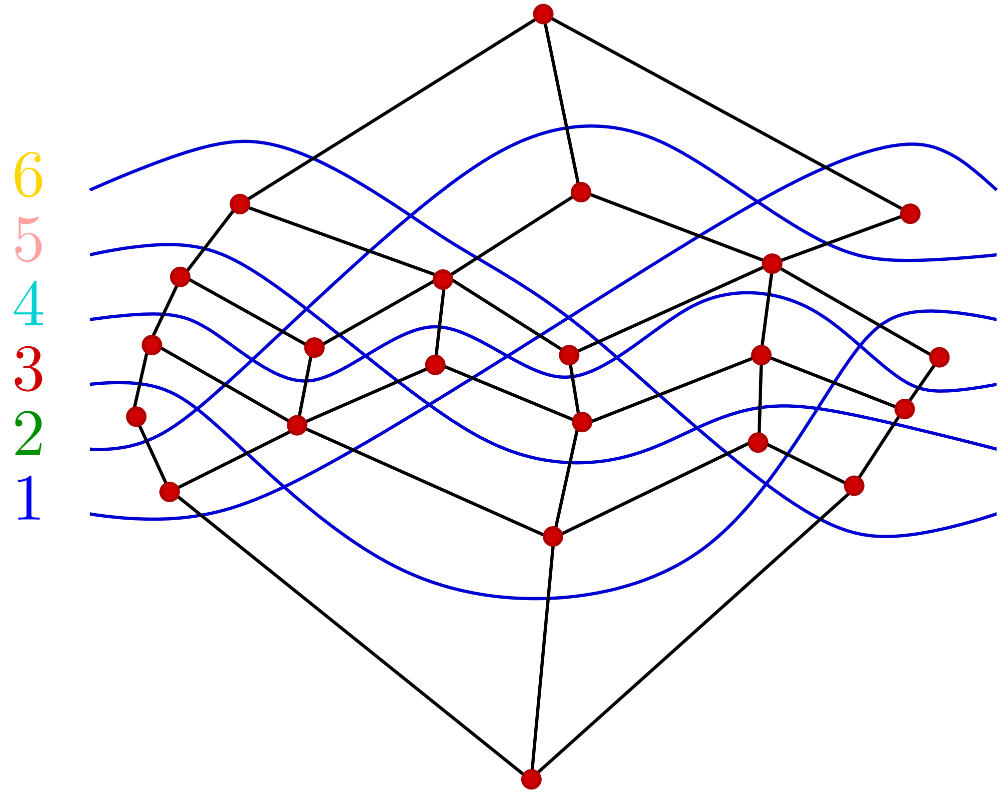
Isomorphism



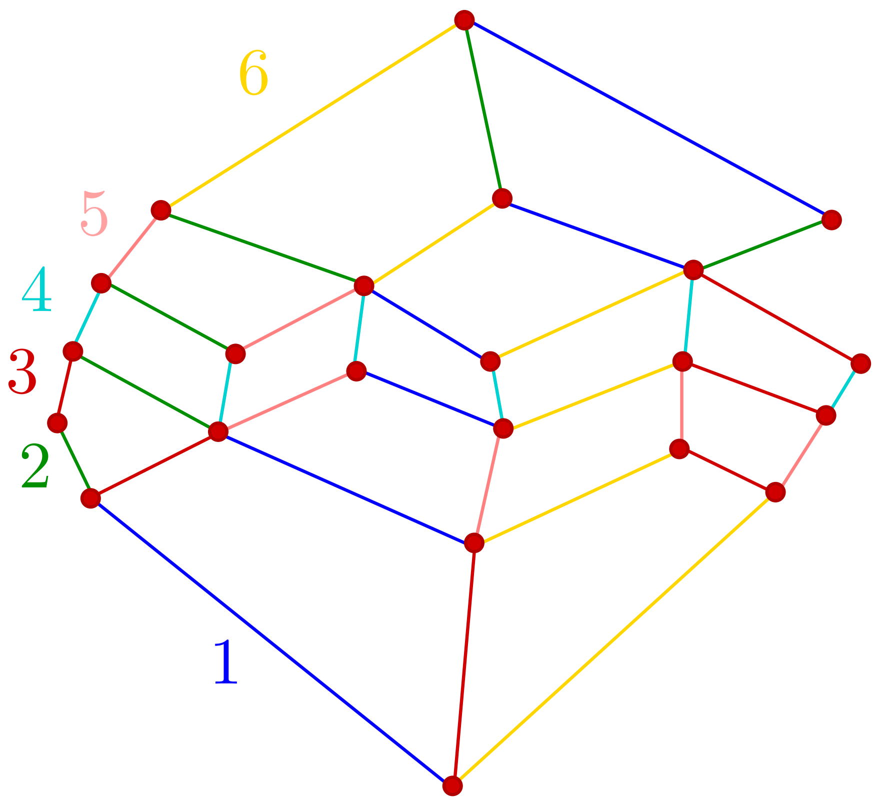


Two isomorphic arrangements.

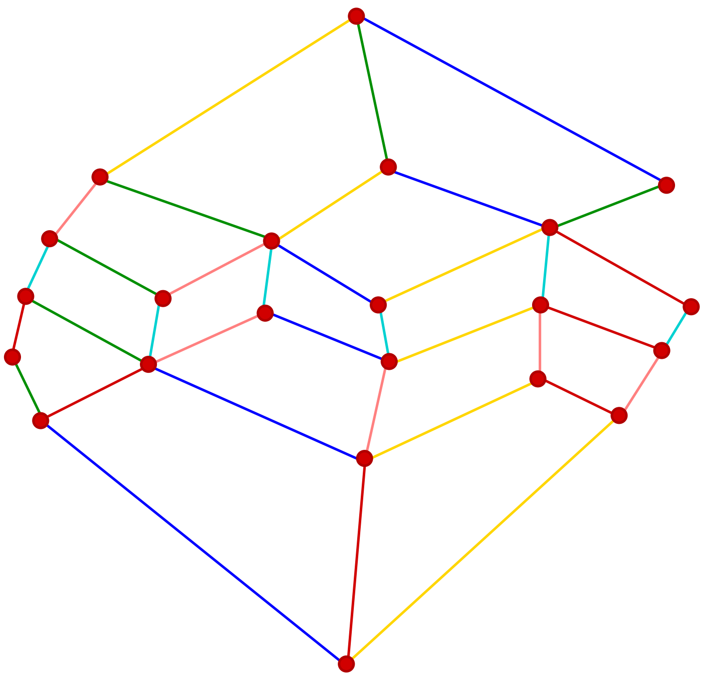
Dual of an Arrangement

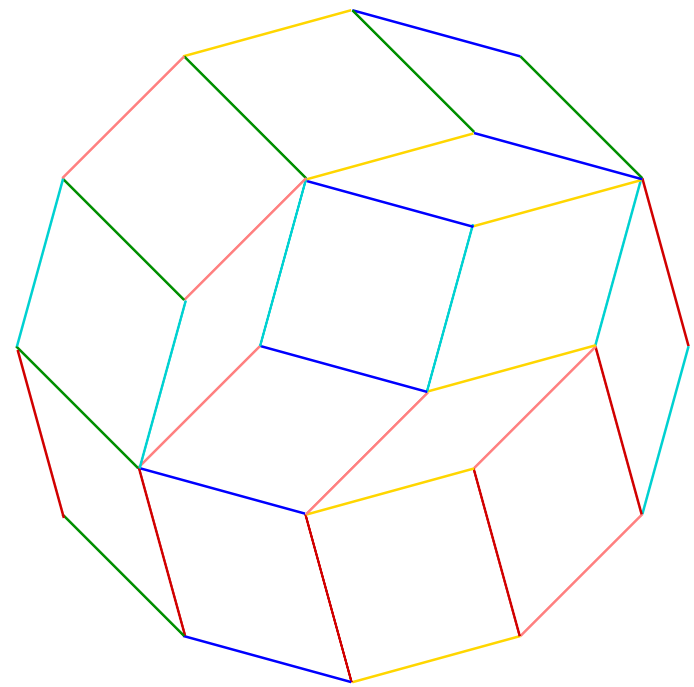


Dual of an Arrangement



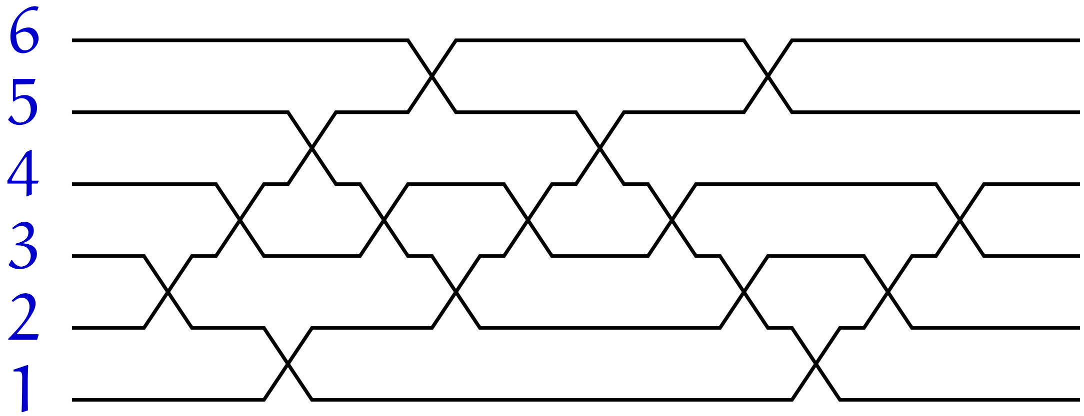
Zonotopal Tiling





A tiling of an $2n$-gon with rhombic tiles.

Wiring Diagram



Confine the n pseudolines to n horizontal wires and add crossings as Xs. (Goodman 1980)

Remark.

Relates to sorting networks and reduced decompositions.

Counting Arrangements – Values

B_n number of isomorphism classes of simple arrangements of n pseudolines.

$n = 3$	2	
4	8	
5	62	
6	908	
7	24698	
8	1232944	
9	112018190	Knuth '92
10	18410581880	Felsner '96
11	5449192389984	Yamanaka et al. '10
12	2894710651370536	Samuel '11
13	2752596959306389652	Kawahara et al. '11
14	4675651520558571537540	
15	14163808995580022218786390	Tanaka '13
16	7641307372577259323046193673 6	Rote '21

Counting Arrangements – Asymptotics

It is known that $B_n \approx 2^{bn^2}$.
We are interested in the value* of b.

History

Goodman & Pollak '83	$1/8 = 0.125 < b$
Knuth '92	$1/6 = 0.166 < b < 0,7924$
Felsner '97	$b < 0.6974$
Felsner & Valtr '11	$0.1887 < b < 0.6571$
Dumitrescu & Mandal '20	$0.2083 < b$
CK & D & F & S '24	$0.2721 < b$

For $n \leq 16$ the $\frac{\log_2 B_n}{n^2}$ is increasing up to 0.3748.

*existence: $\liminf \frac{\log_2 B_n}{n^2} \stackrel{?}{=} \limsup \frac{\log_2 B_n}{n^2}$.

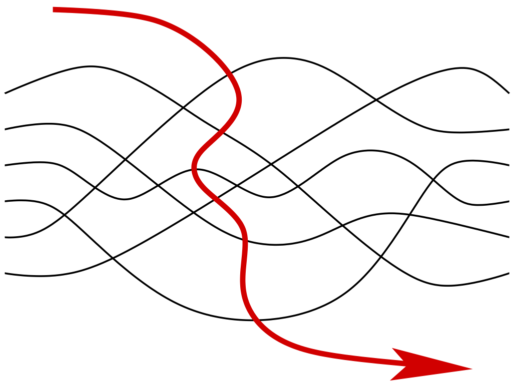
Counting Arrangements

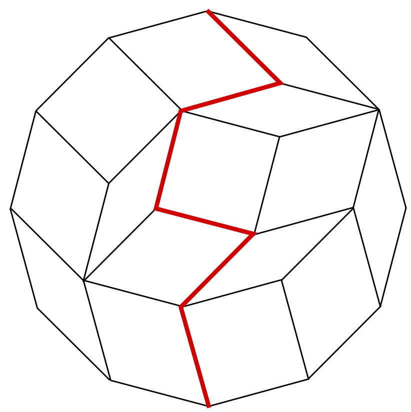
Early lower and upper bounds

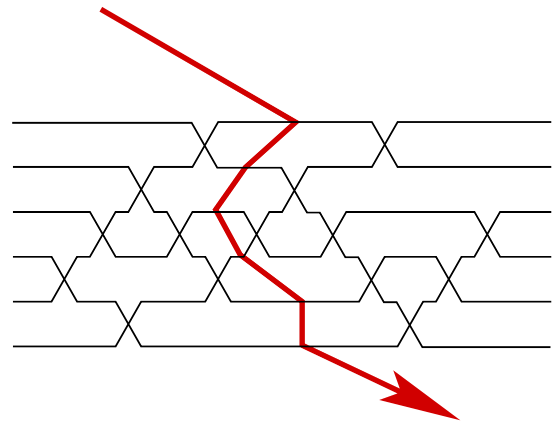
- Triangle flips and triangular grid spanned by 3 bundles.
- Encoding of local sequences as binary sequences.

Cut-Paths

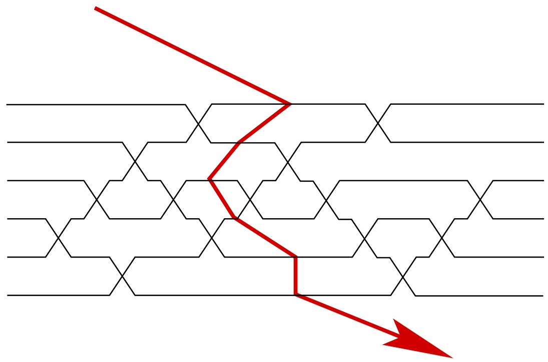
A curve from the north-cell to the south-cell crossing each pseudoline in a single edge.

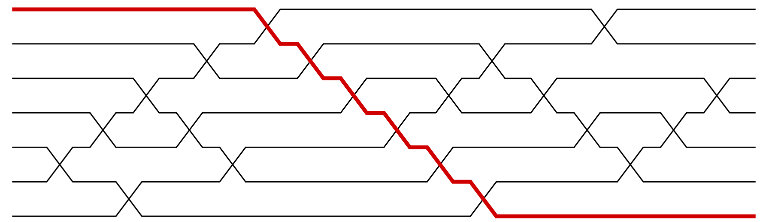






Cut-Paths





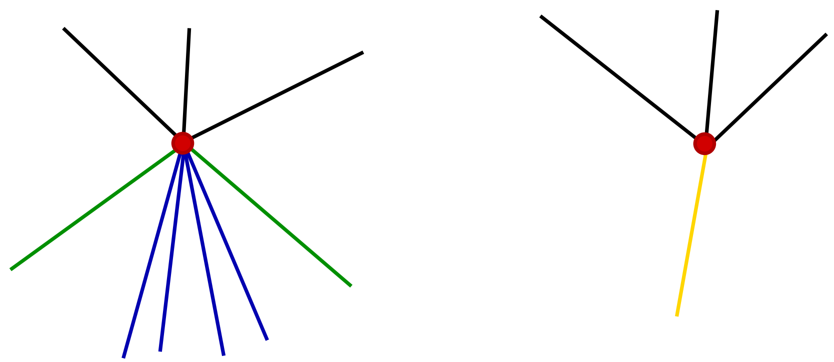
If γ_n is the maximal number of cut-paths of an arrangement of n pseudolines, then

$$B_n \leq \gamma_{n-1} \cdot B_{n-1} \leq \gamma_{n-1} \cdot \gamma_{n-2} \cdot \ldots \cdot \gamma_2 \cdot \gamma_1.$$

Task: Find good bounds on γ_n.

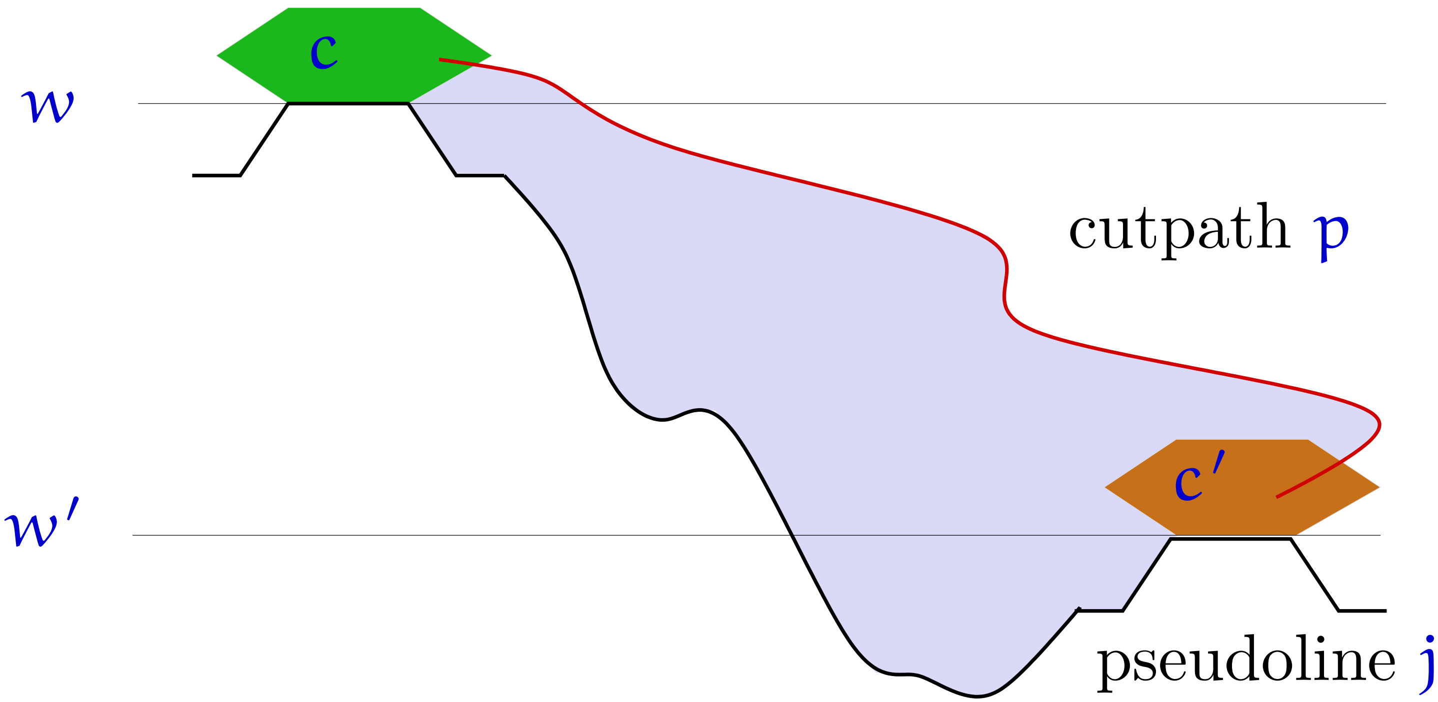
Edges of a Cut-Path

- We distinguish left, middle, right and unique edges on a cut-paths



The Key Lemma

Lemma. [Knuth '92]
For every pseudoline j and every cutpath p: p sees a middle of color j at most once.



Encoding Cut-Paths I

With a cutpath p we associate two combinatorial objects:

- A set $M_p \subset [n]$ consisting of all j such that pseudoline j is crossed by p as a middle.
- A binary vector $\beta_p = (b_0, b_1, \ldots, b_{n-1})$ such that $b_i = 1 \iff p$ takes a right when crossing wire i.

Fact. The mapping $p \to (M_p, \beta_p)$ is injective.

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$$\gamma_n \leq 2^n \, 2^n = 4^n.$$

Encoding Cut-Paths II

If $|M_p| = k$, then we only need $n - k$ entries of β_p.

Redefine β_p so that b_i encodes the left/right step at the ith lookup.

$$\gamma_n \leq \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} = 2^n \left(1 + \frac{1}{2}\right)^n = 3^n.$$

Reversed Cut-Paths

We don't need an entry of β_p when taking a unique.

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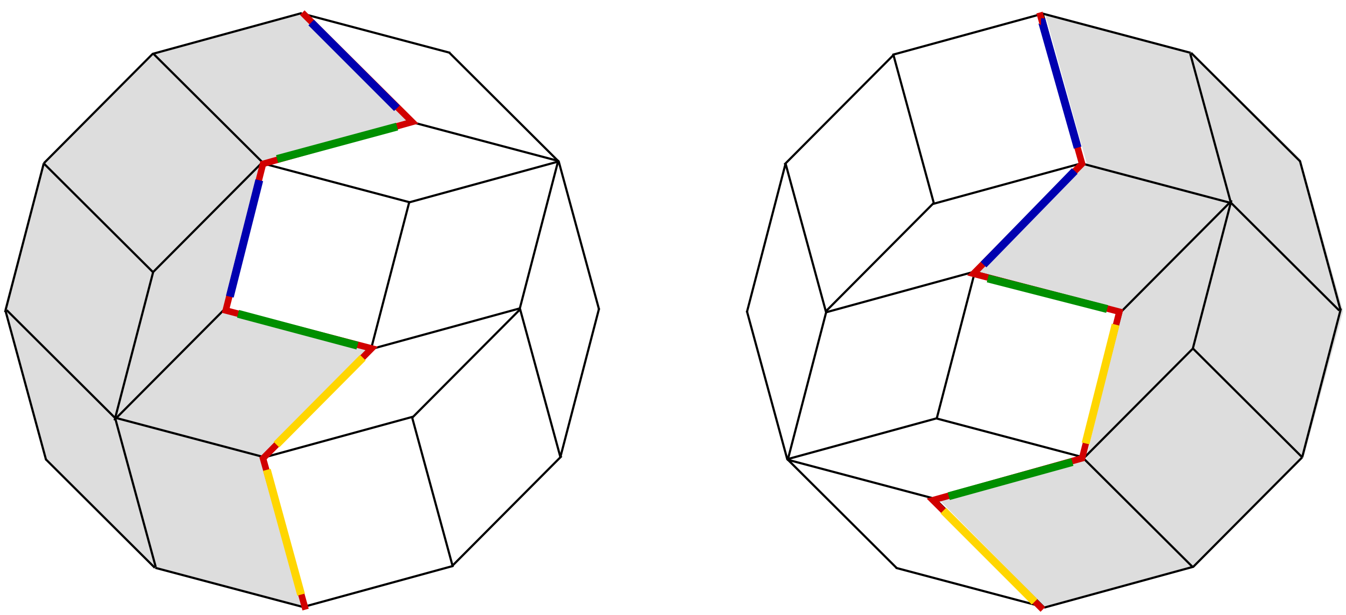
Lemma. A middle of p is a unique of the reversed cut path.



Encoding Cut-Paths III

If $\Gamma(k, r)$ is the number of cutpaths that take k middles and r unique edges, then $\Gamma(k, r) \le \binom{n}{k} 2^{n-k-r}$ and by the reversal symmetry $\Gamma(k, r) \le \binom{n}{r} 2^{n-k-r}$.

Lemma. $\Gamma(k, r) \le \min \left\{ \binom{n}{k}, \binom{n}{r} \right\} 2^{n-k-r}$.

Encoding Cut-Paths III

$$\gamma_n \le \sum_{k,r} \Gamma(k,r) \le \sum_{k,r} \min\left\{ \binom{n}{k}, \binom{n}{r} \right\} 2^{n-k-r}$$

$$\le 2 \cdot 2^n \sum_{k=0}^{n} \binom{n}{k} 2^{-k} \sum_{r \ge k} 2^{-r}$$

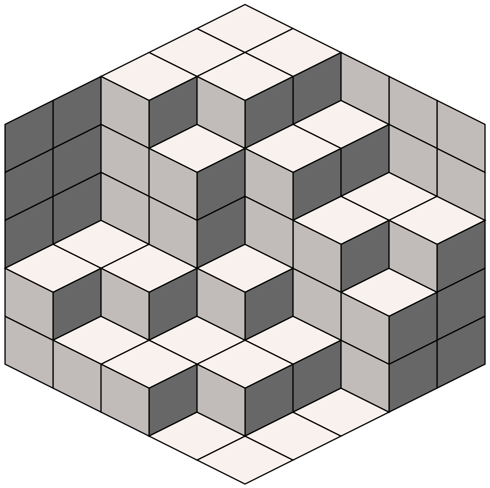
$$= 2^{n+1} \sum_{k=0}^{n} \binom{n}{k} 2^{-2k} \sum_{j \ge 0} 2^{-j}$$

$$= 2^{n+2} \left(1 + \frac{1}{4}\right)^n = 4 \left(\frac{5}{2}\right)^n$$

Corollary. $\log_2(B_n) \le 0.6609n^2$ for n large.

Further improvement from 0.6609 to 0.6571 with more technical arguments.

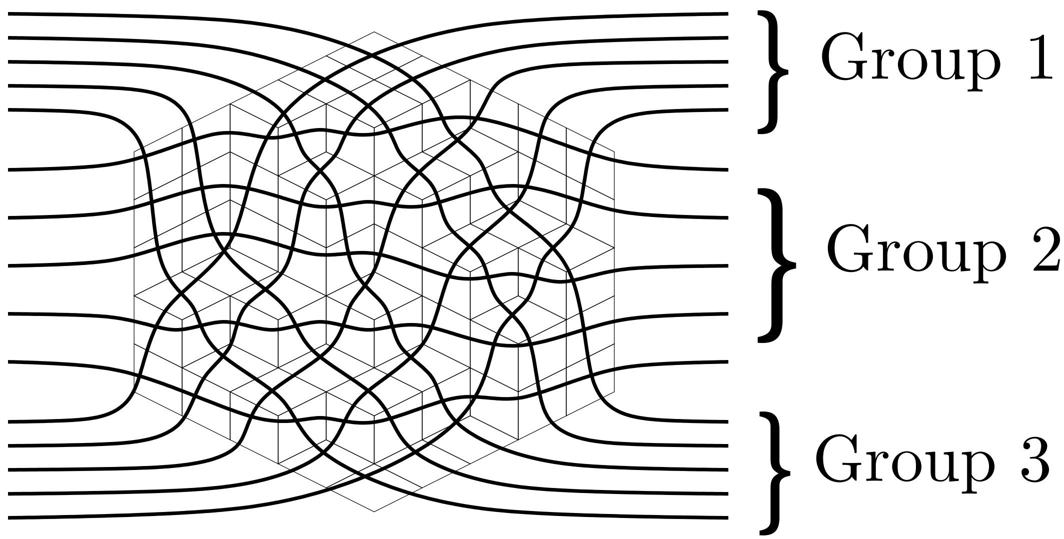
The FV Lower Bound



The MacMahon formula for the number of plane partitions in $n \times n \times n$, i.e., rhombic tilings of a hexagon with all sides of length n is

$$\mathbf{P}(n) = \prod_{a=0}^{n-1} \prod_{b=0}^{n-1} \prod_{c=0}^{n-1} \frac{a+b+c+2}{a+b+c+1}.$$

The FV Lower Bound



The construction implies $B_{3n} \geq \mathbf{P}(n)\, {B_n}^3$.

The FV Lower Bound

The rest is a Maple supported computation:

$$\ln \prod_{a=0}^{n-1} \prod_{b=0}^{n-1} (a+b+k+1) \approx \int_{x=0}^{n} \int_{y=0}^{n} \ln(x+y+k+1)\, dy\, dx$$

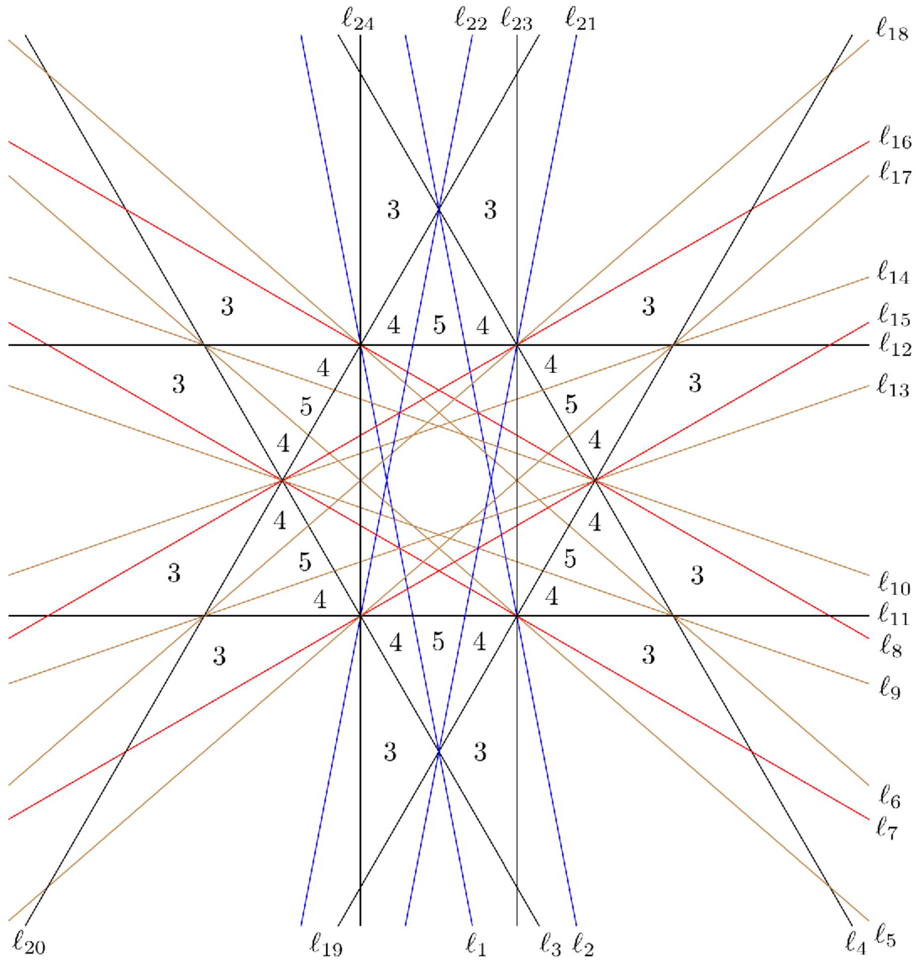
yields

$$\ln \mathbf{P}(n) \approx \left(\frac{9}{2}\ln(3) - 2\ln(2)\right) n^2$$

and finally:

Theorem. The number B_n of arrangements of n pseudo-lines is at least $2^{0.1887\, n^2}$.

The DM Lower Bound



- $\lambda_i(m)$ = # i-wise crossings when bundles have m lines.

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*Figure taken from Dumitrescu and Mandal: DOI:10.20382/jocg.v11i1a3

The DM Lower Bound

$$F(12n) \geq \prod_{i=3}^{12} B_i^{\lambda_i(n)} \geq 2^{\frac{283n^2}{35\cdot 12^2}} \cdot 8^{\frac{7n^2}{8\cdot 12^2}} \cdot 62^{\frac{7n^2}{20\cdot 12^2}} \cdot 908^{\frac{27n^2}{140\cdot 12^2}} \cdot$$

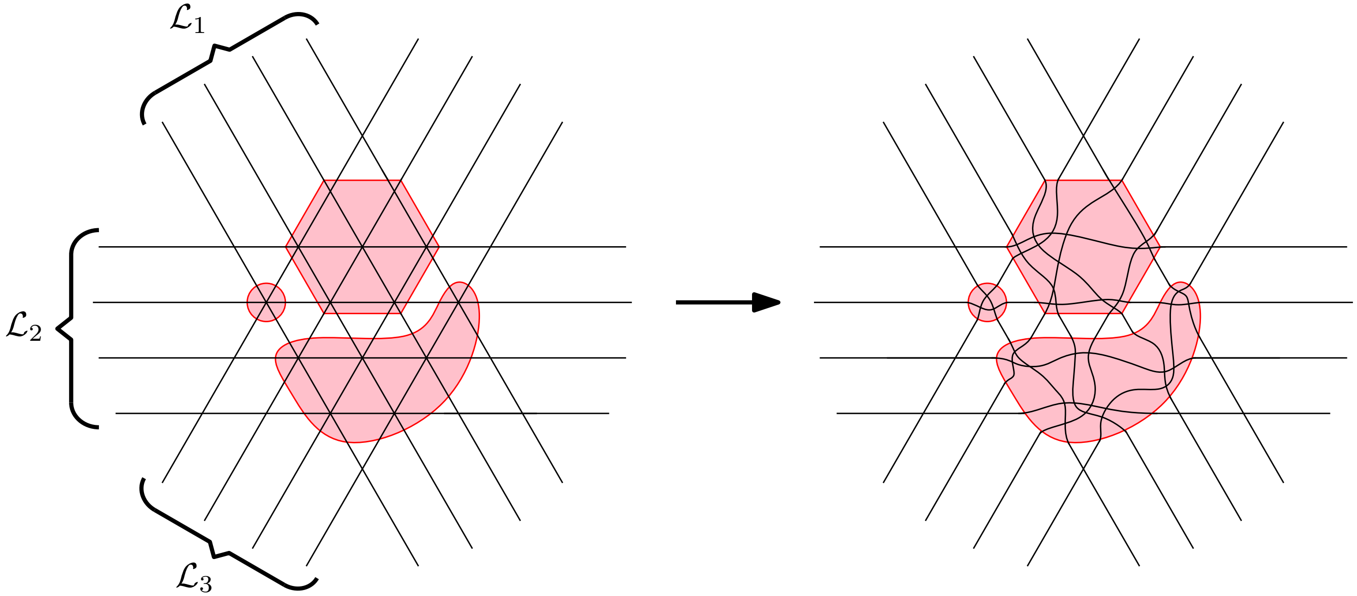
$$24698^{\frac{23n^2}{140\cdot 12^2}} \cdot 1232944^{\frac{19n^2}{280\cdot 12^2}} \cdot 112018190^{\frac{9n^2}{140\cdot 12^2}} \cdot$$

$$18410581880^{\frac{13n^2}{280\cdot 12^2}} \cdot 5449192389984^{\frac{n^2}{70\cdot 12^2}} \cdot$$

$$28947106513705536^{\frac{n^2}{10\cdot 12^2}} \cdot$$

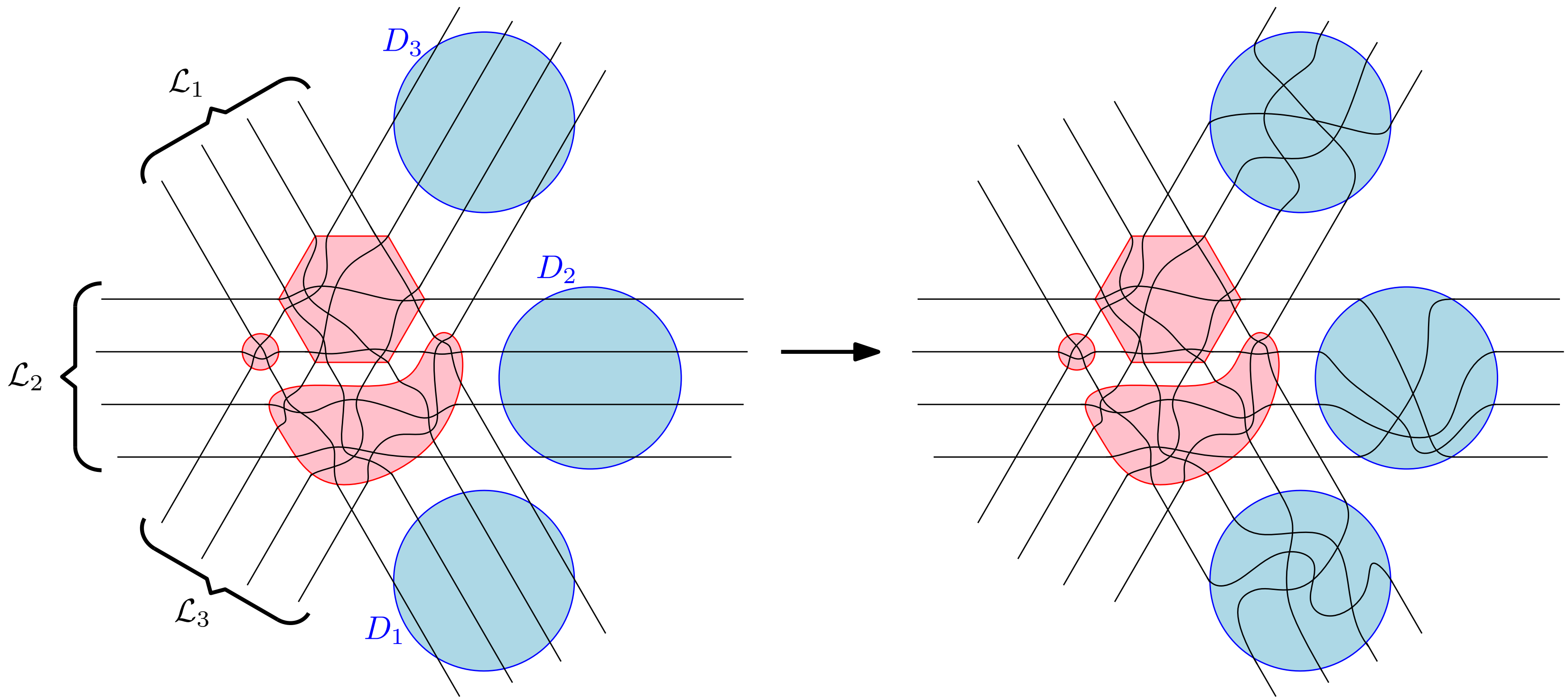
Lower Bound: Step 1

k bundles of lines



$F_k(n)$ a lower bound on the number of consistent partial arrangements

Lower Bound: Step 2



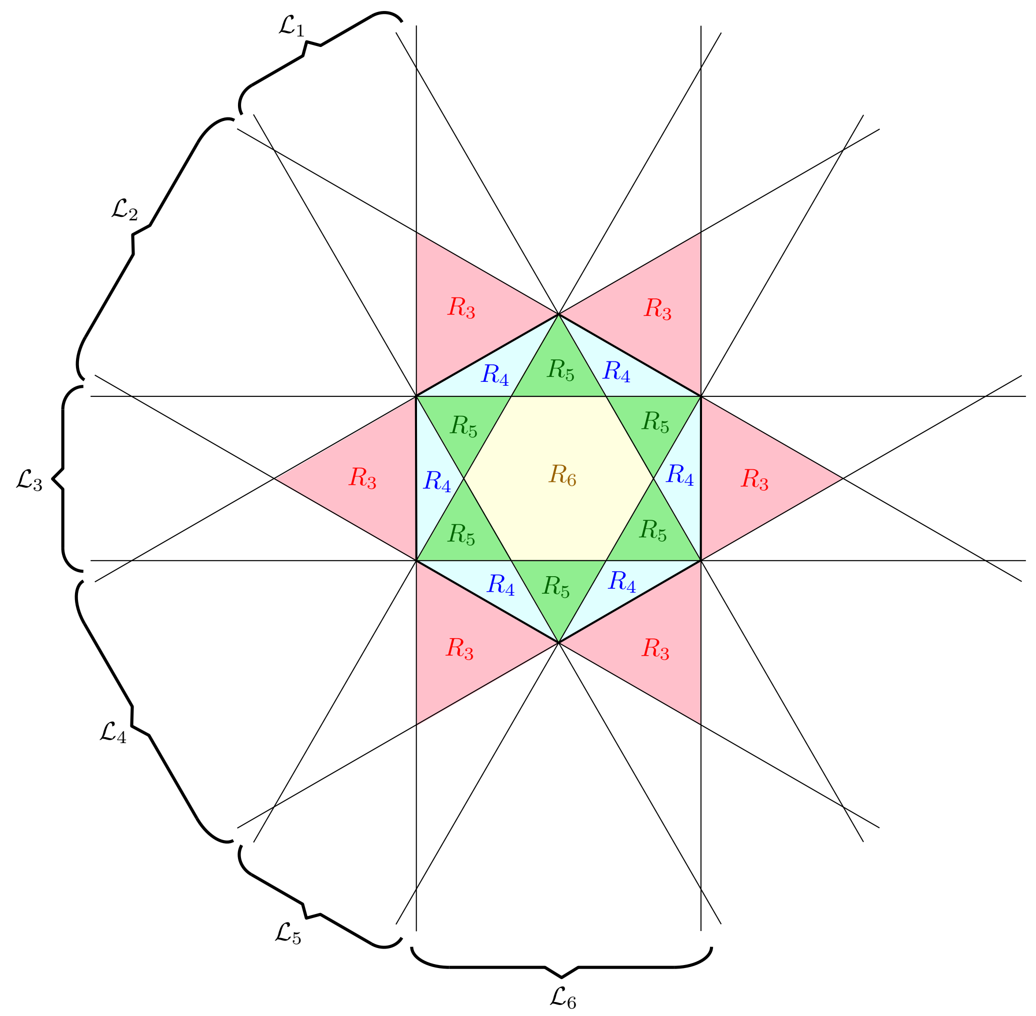
$$B_n \geq \underbrace{F_k(n)}_{\text{Step 1}} \cdot \underbrace{(B_{\lfloor \frac{n}{k} \rfloor})^k}_{\text{Step 2}}$$

A Lemma for Step 2

Lemma. If $F_k(n) \geq 2^{cn^2 - O(n)}$ for some $c > 0$, then

$$B_n \geq 2^{\frac{ck}{k-1}n^2 - O(n \log n)}.$$

Six Bundles



$\mathcal{L}_1$
$\mathcal{L}_2$
$\mathcal{L}_3$
$\mathcal{L}_4$
$\mathcal{L}_5$
$\mathcal{L}_6$
R_3
R_3
R_3
R_3
R_3
R_3
R_4
R_4
R_4
R_4
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R_5
R_5
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R_5
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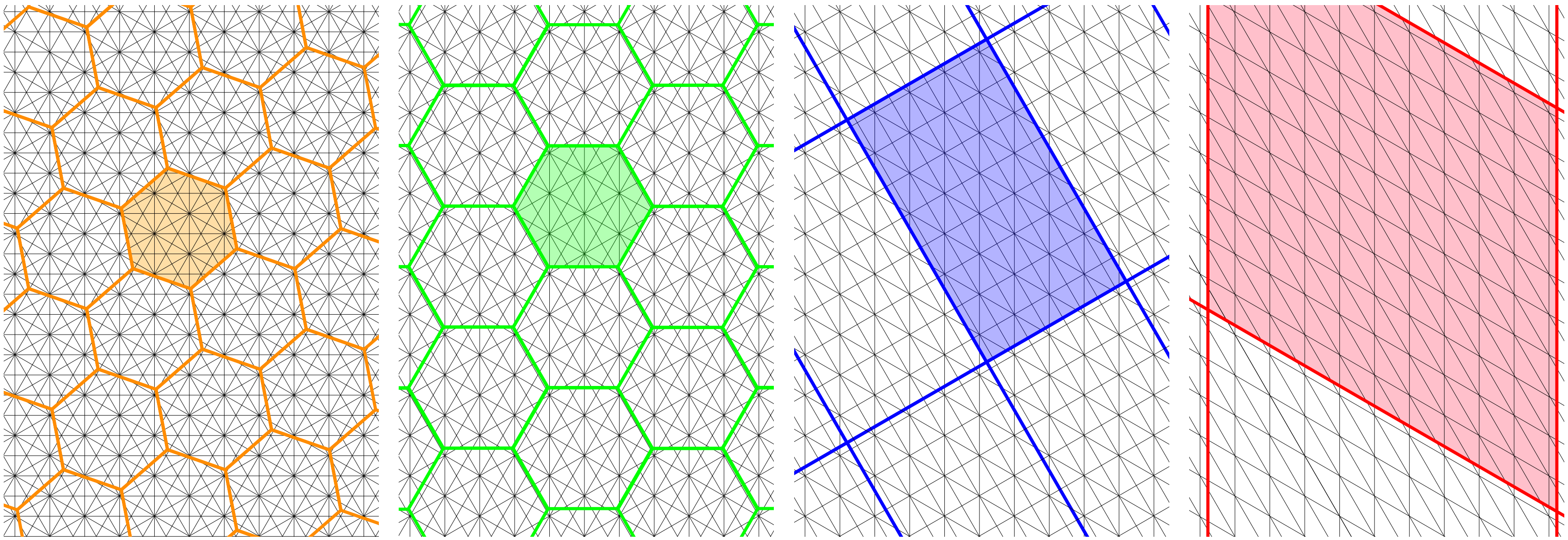
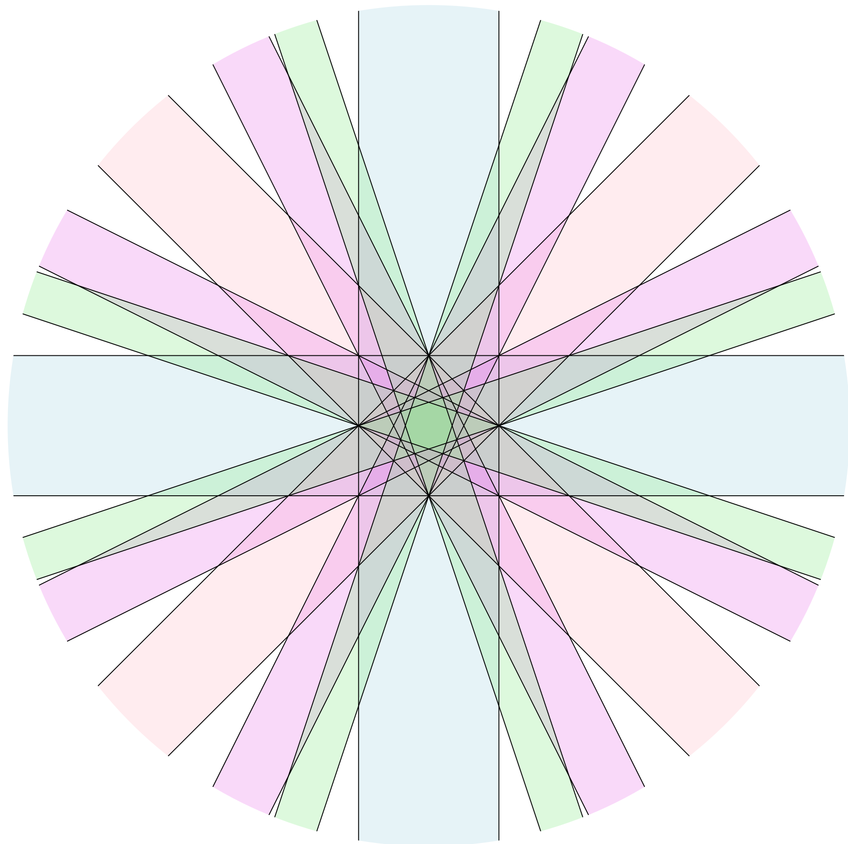
R_5
R_6

Six Bundles - Patches in Regions



This yields $b > 0.2541$.

Twelve Bundles



12 bundles and 19 different regions. This yields $b > 0.2721$.

Twelve Bundles – Technicalities

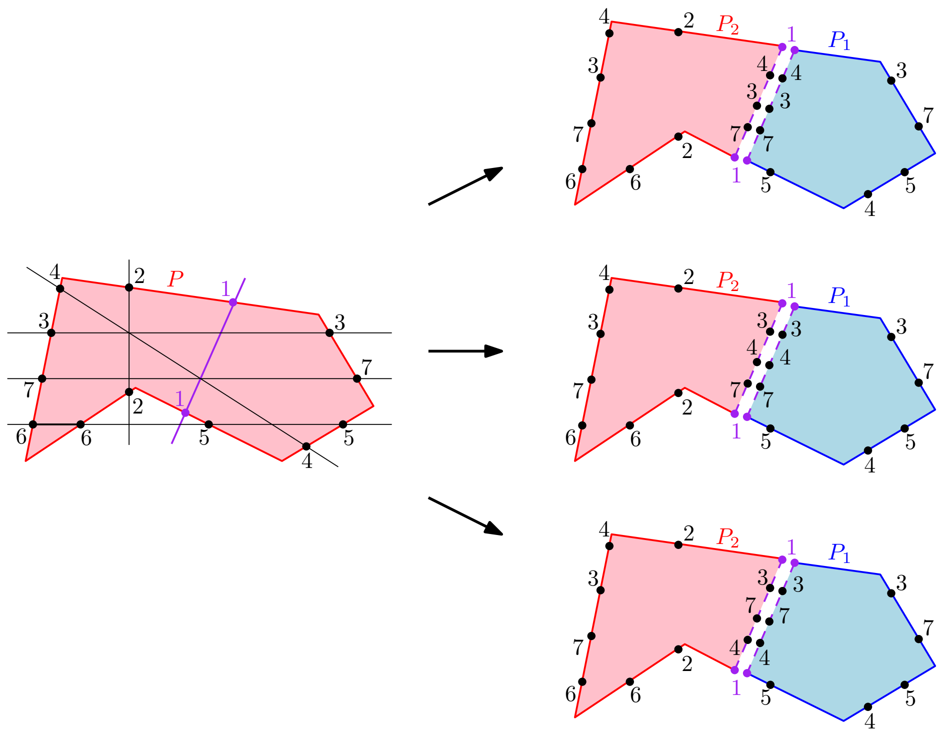
- For regions with 3 slopes we use Lindström–Gessel–Viennot on 1000×1000 patches.

- Regions with more slopes are subdivides in patches.

  - We use dynamic programming to compute the number of partial arrangements consistent with a given boundary bipermutation.

  - Results for bipermutations are stored for reuse.
    $\triangleright$ Minimal representative of a bipermutation.

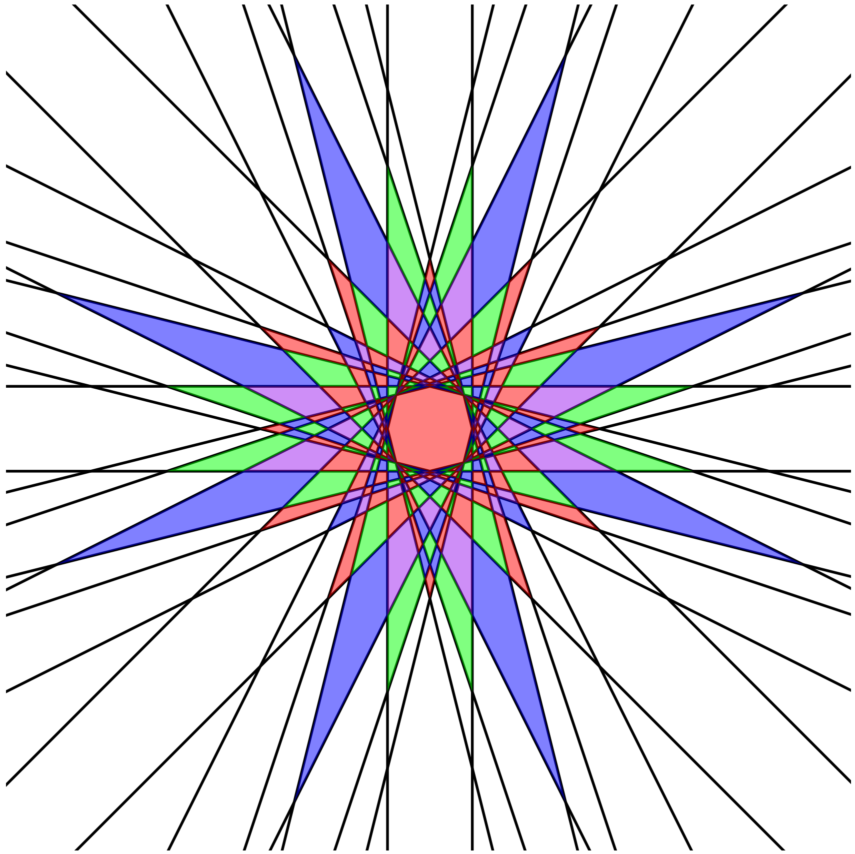
Splitting a patch



$$F(P) = \sum F(P_1 L) \cdot F(P_2 \overline{L})$$

The sum is over linear extensions L on consistency poset of curves crossed by ℓ_1 in the patch.

In Preparation 16 Slopes



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