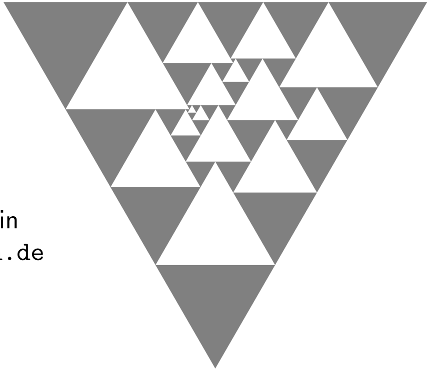
Cartograms and Contact Representations:

Applications of Schnyder Woods in Graph Drawing

Ljubljana
January 17, 2013



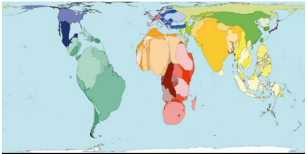
Stefan Felsner
Technische Universität Berlin
felsner@math.tu-berlin.de

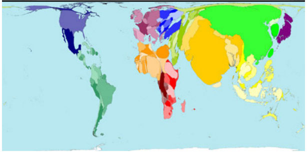
Outline

Cartograms – excerpted from Wikipedia

An area **cartogram** illustrates the **value** of some statistic in different countries **by** scaling the **area** of each country in proportion to the value; the shape and relative location of each country is retained to as large an extent as possible.

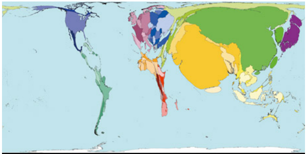
murder cartogram

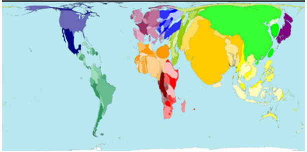




population cartogram

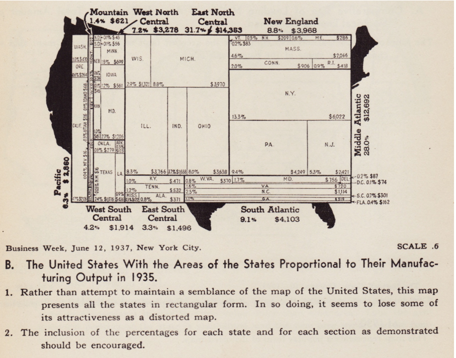
suicide cartogram





population cartogram

Cartograms and Contact Representations



by Willard C. Brinton (?)

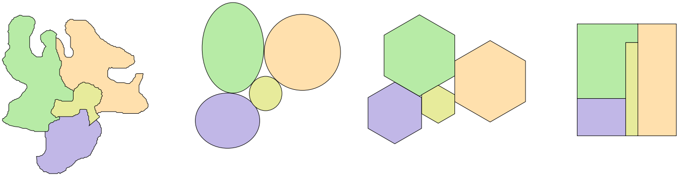
A **cartogram** for G (planar graph) and $a : V_G \to \mathbb{R}$ (prescribed area) is a collection $\{P_v\}$ of interiorly disjoint polygons such that

- $P_v \cap P_w \neq \emptyset \iff (v, w) \in E_G$ (contact representation of G)
- $\mathrm{vol}(P_v) = a(v)$ for all v (prescribed area)

Cartograms – Models and Criteria

Models

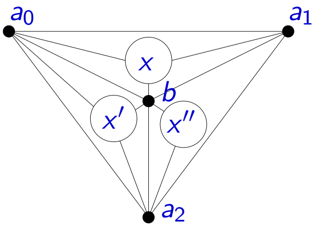
- **contacts** — (point | segment)
- **holes** — (allowed | forbidden)
- **polygons** — (arbitrary | convex | orthogonal)

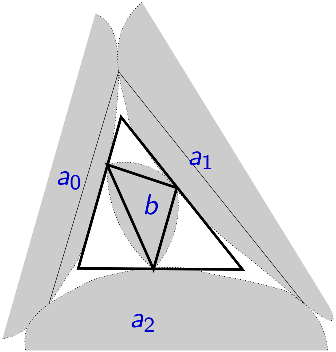


Criteria

- polygonal complexity – cartographic error – aesthetic criteria

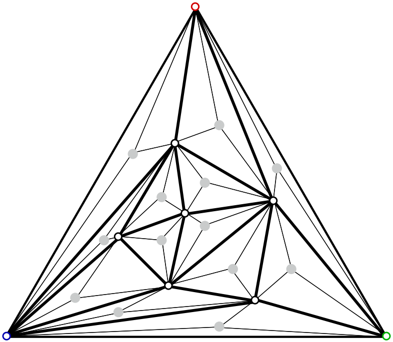
Non-convex polygons may be necessary

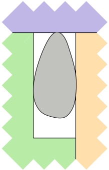




Hans Debrunner (1957) Aufgabe 260. Elemente der Mathem. 12
If a triangle DEF is inscribed in a triangle ABC with D on BC, E on CA and F on AB then the minimum of the areas of the four smaller triangles is always assumed by a corner triangle.

Orthogonal Polygons: A lower bound





- Each gray vertex is responsible for a concave corner.
- Some white vertices have ≥ 2 concave corners — complexity ≥ 8.

History of upper bounds

- 40 corners (de Berg, Mumford, Speckmann – 2005)
- 34 corners (Kawaguchi, Nagamochi – 2007)
- 12 corners (Biedl, Velázquez – 2011)

our contributions

- 10 corners
- 8 corners for Hamiltonian triangulations
- 8 corners using area universal rectangulations

ISAAC 2011 and SoCG 2012, joint work with

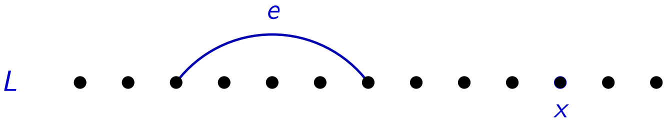
Md. Jawaherul Alam — Therese Biedl — Andreas Gerasch —
Michael Kaufmann — Stephen G. Kobourov — Torsten Ueckerdt

Outline

Dimension of Graphs

A family Γ of permutations of V is a realizer for $G = (V, E)$ provided that

* for every edge e and every $x \in V - e$ there is an $L \in \Gamma$ such that $x > e$ in L.



The dimension, $\dim(G)$, of G is the minimum t, such that there is a realizer $\Gamma = \{L_1, L_2 \ldots, L_t\}$ for G of size t.

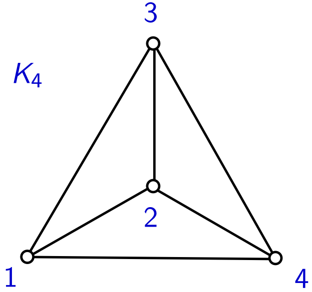
Schnyder's First Theorem

Theorem [Schnyder 1989].

A Graph G is planar

$\iff$ $\dim(G) \leq 3$.

Example.



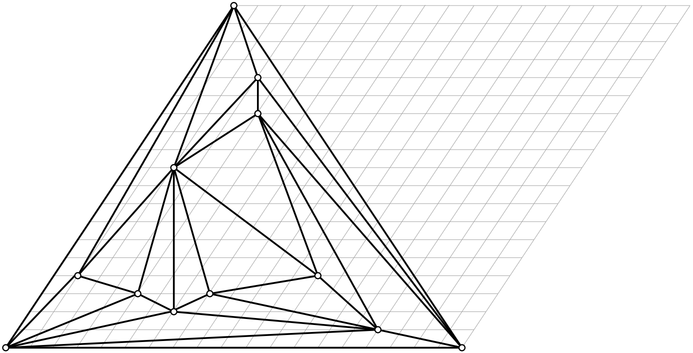
L_1 : 2 3 4 1

L_2 : 1 3 4 2

L_3 : 1 2 4 3

Schnyder's Second Theorem

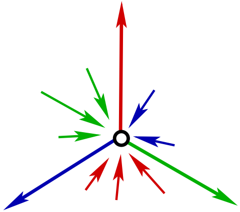
Theorem [Schnyder 1989 **].**
A planar triangulation G admit a straight line drawing on the $(2n-5) \times (2n-5)$ grid.
Example.



Schnyder Woods

$G = (V, E)$ a plane triangulation,
$F = \{a_1, a_2, a_3\}$ the outer triangle.
A coloring and orientation of the interior edges of G with colors 1,2,3 is a Schnyder wood of G iff

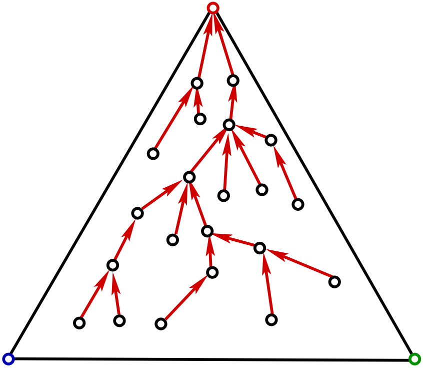
- Inner vertex condition:



- Edges $\{v, a_i\}$ are oriented $v \to a_i$ in color i.

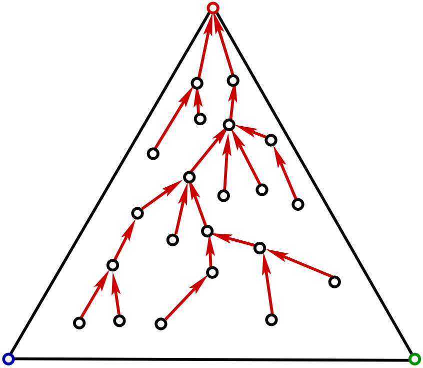
Schnyder Woods - Trees

- The set T_i of edges colored i is a tree rooted at a_i.



Schnyder Woods - Trees

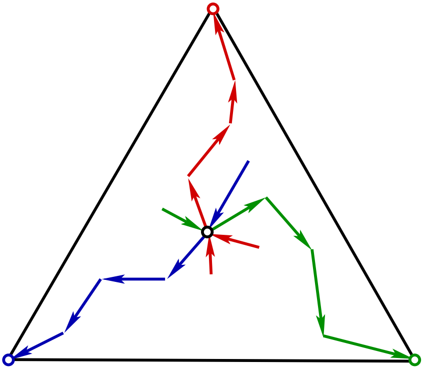
- The set T_i of edges colored i is a tree rooted at a_i.



Proof. Count edges in a cycle — Euler 

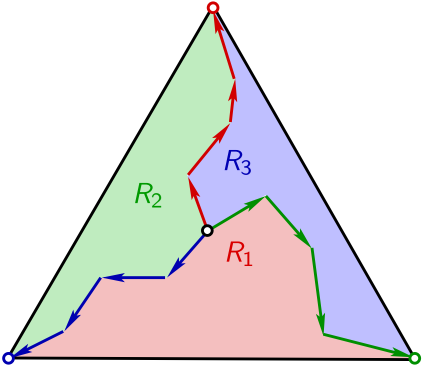
Schnyder Woods - Paths

- Paths of different color have at most one vertex in common.



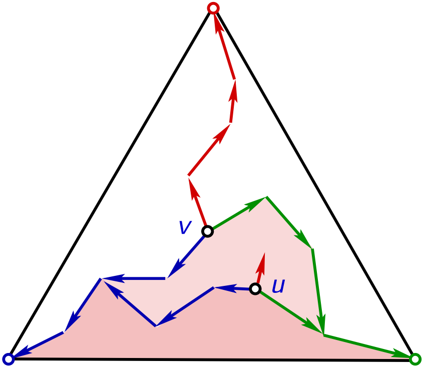
Schnyder Woods - Regions

- Every vertex has three distinguished regions.



Schnyder Woods - Regions

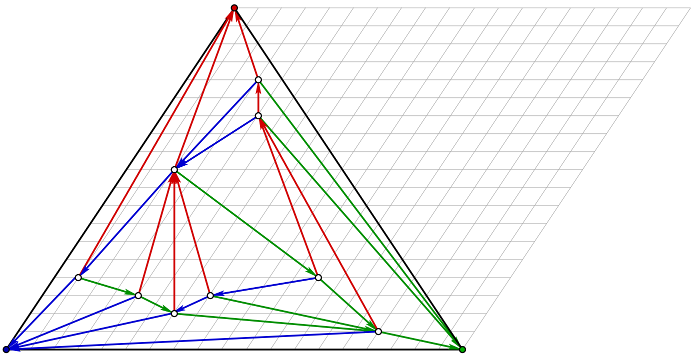
- If $u \in R_i(v)$ then $R_i(u) \subset R_i(v)$.



Grid Embeddings

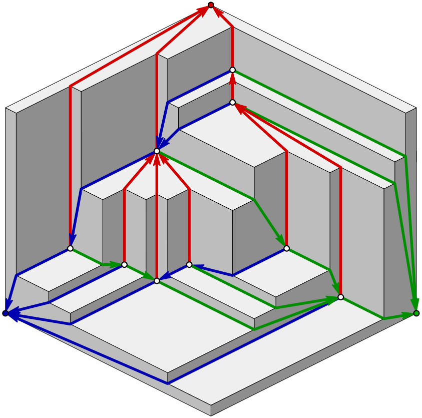
The count of faces in the green and red region yields two coordinates (v_g, v_r) for vertex v.

Theorem. Planar triangulations admit a straight line drawing on the $(f-1) \times (f-1)$ grid.



Embeddings in Three Dimensions

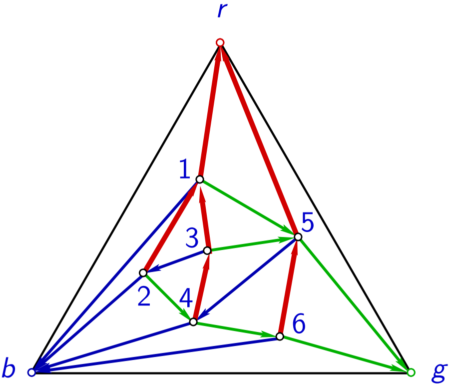
Using all three face count coordinates we obtain an embedding of T on an orthogonal surface.

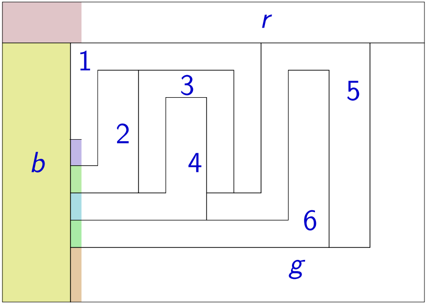


This implies Schnyder's Dimension Theorem.

Outline

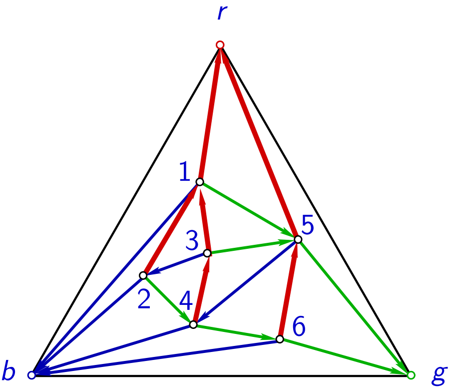
Cartograms of triangulations

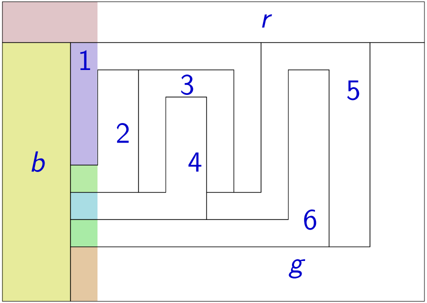




- Draw *b* and initialize polygons of all adjacent vertices.

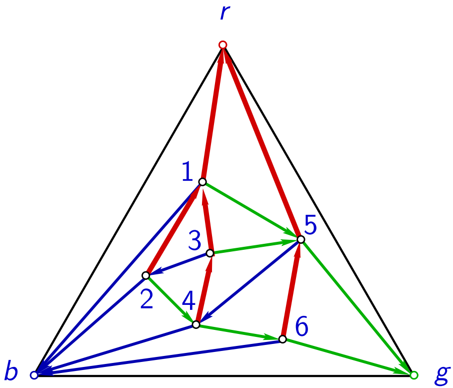
Cartograms of triangulations

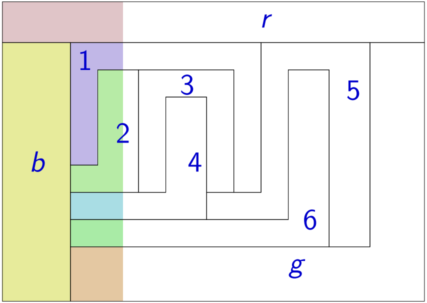




- Raise 1.

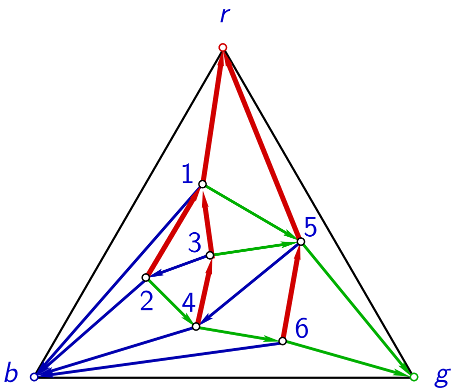
Cartograms of triangulations

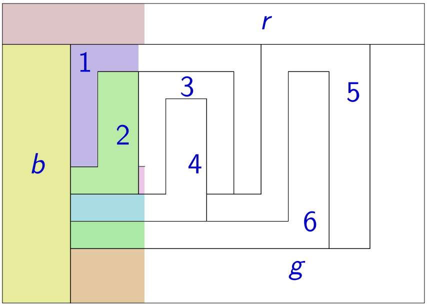




- Raise 2.

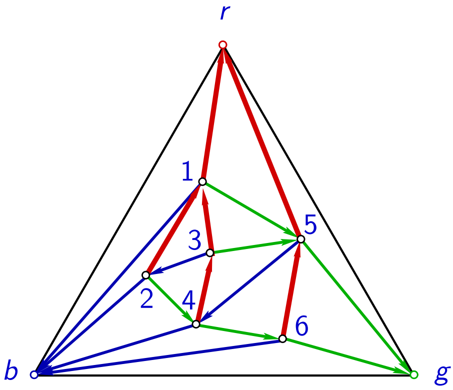
Cartograms of triangulations





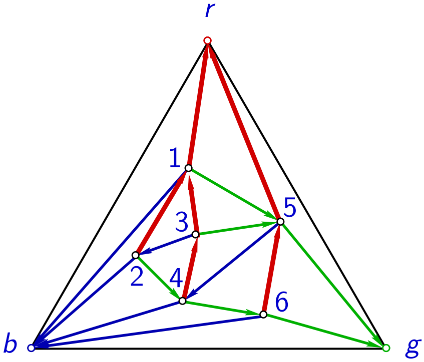
- Complete 2 and initialize polygons of blue descendents.

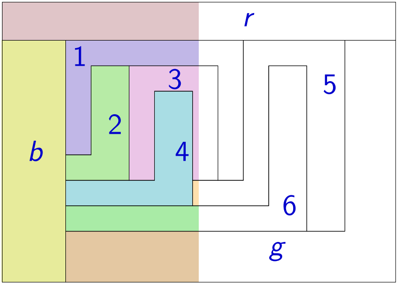
Cartograms of triangulations



- Raise 3.

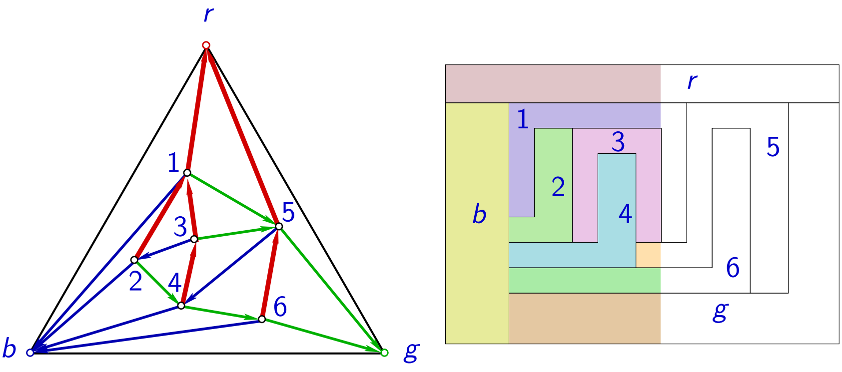
Cartograms of triangulations





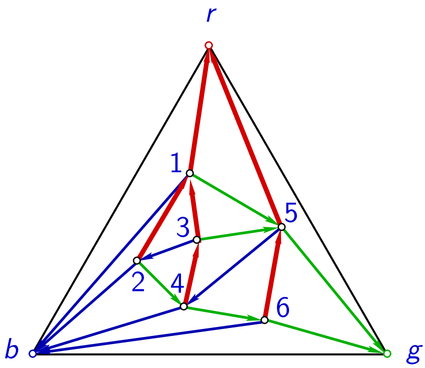
- Raise and complete 4 and initialize polygons of blue descendents.

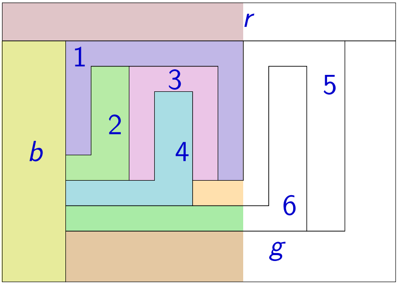
Cartograms of triangulations



- Complete 3 and initialize polygons of blue descendents.

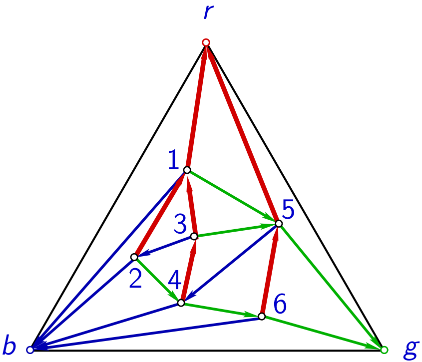
Cartograms of triangulations

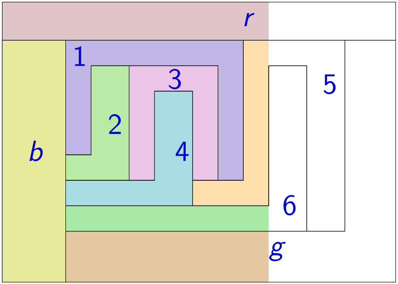




- Complete 1 and initialize polygons of blue descendents.

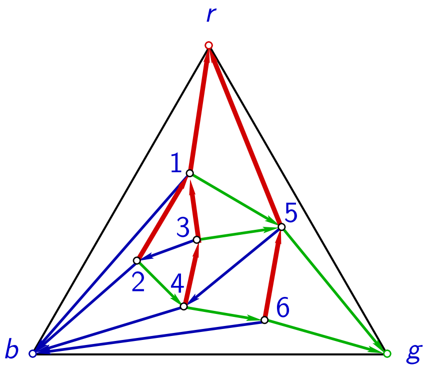
Cartograms of triangulations

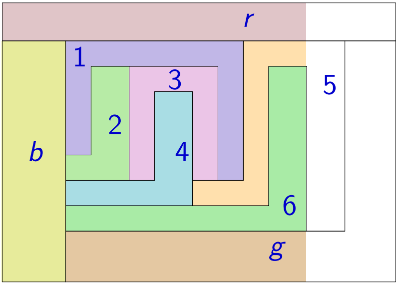




- Raise 5.

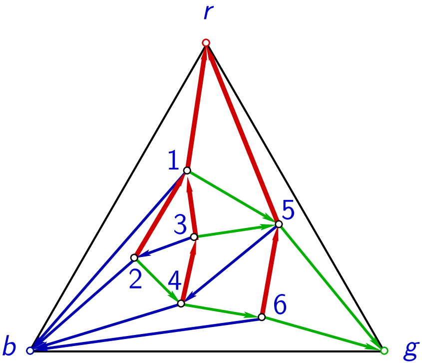
Cartograms of triangulations

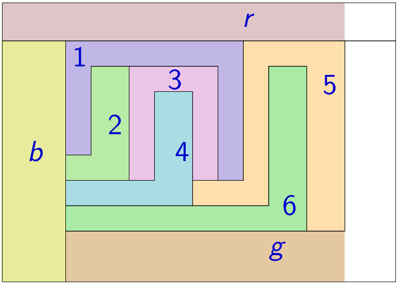




- Raise 6 and complete 6.

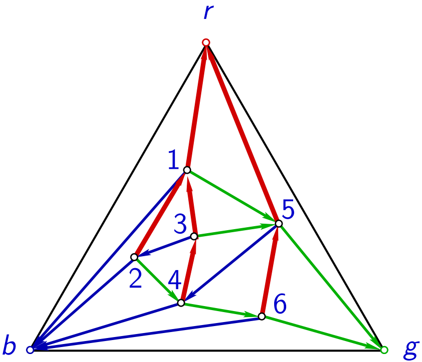
Cartograms of triangulations

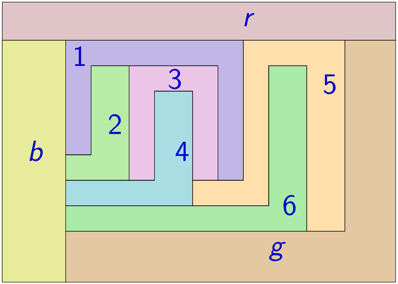




- Complete 5.

Cartograms of triangulations

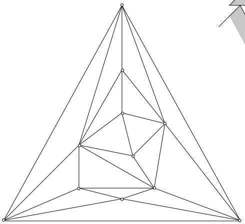


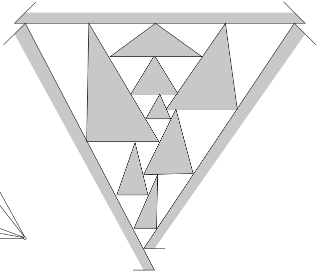


- Done.

Digression: Triangle Contact Representation

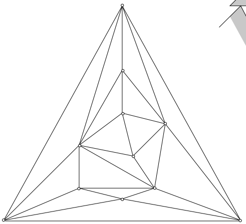
de Fraysseix, de Mendez and Rosenstiehl construct triangle contact representations of triangulations.

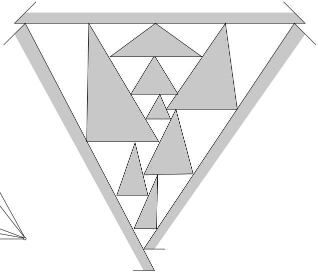




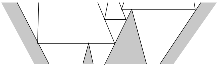
Digression: Triangle Contact Representation

de Fraysseix, de Mendez and Rosenstiehl construct triangle contact representations of triangulations.

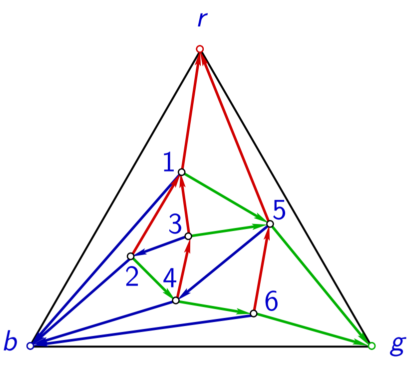




Construct along a good ordering of vertices
$T_1 + T_2{}^{-1} + T_1{}^{-1}$

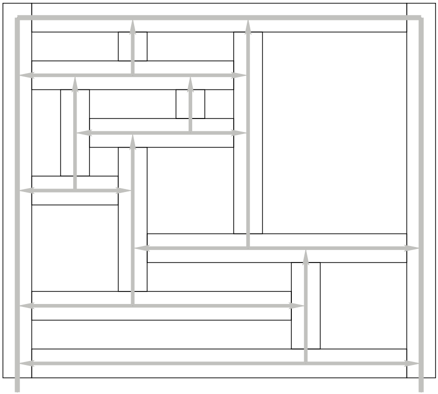


Cartograms of triangulations – Method 2



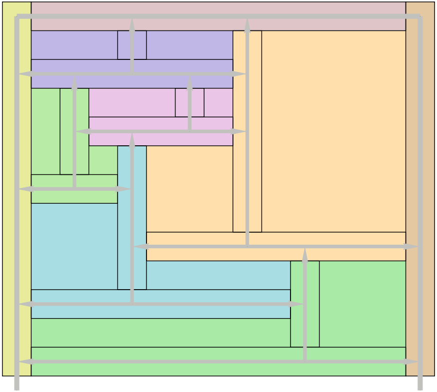
- A ⊥ representation.

Cartograms of triangulations



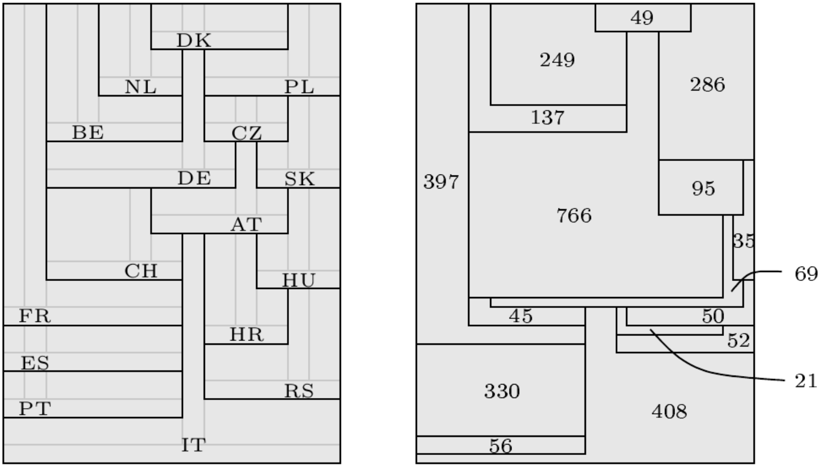
- Yields a rectangular dissections.
- The dissection is one-sided.

Cartograms of triangulations



- One-sided rectangular dissections are area universal.
  (Wimer-Koren-Cederbaum '88 / Eppstein-Mumford-Speckmann-Verbeek '09)
- $\implies$ Cartograms with ≤ 8 gons in the orthogonal model.

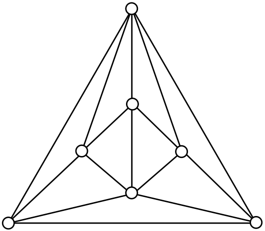
Example: CO_2 emissions 2009

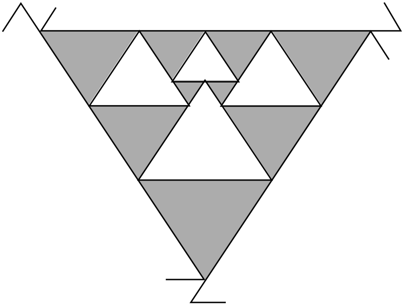


Outline

Homothetic Triangle Contact Representations

Theorem [Gonçalves, Lévêque, Pinlou (GD 2010)].
Every 4-connected triangulation has a triangle contact representation with homothetic triangles.





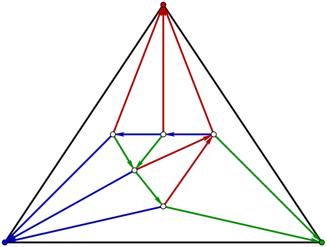
Triangle Contact Representations

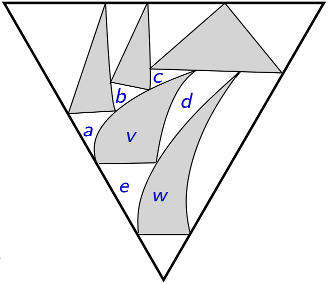
G-L-P observe that the conjecture follows from a corollary of Schramm's “Monster Packing Theorem”.

Theorem. Let T be a planar triangulation with outer face $\{a, b, c\}$ and let C be a simple closed curve partitioned into arcs $\{P_a, P_b, P_c\}$. For each interior vertex v of T prescribe a convex set Q_v containing more than one point. Then there is a contact representation of T with homothetic copies.

Remark. In general homothetic copies of the Q_v can degenerate to a point. Gonçalves et al. show that this is impossible if T is 4-connected.

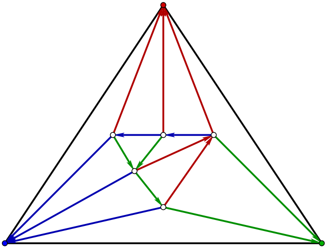
Schnyder Woods and Triangle Contacts

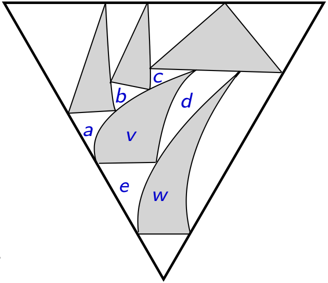




A Schnyder wood induces an *abstract triangle contact representation*.

Triangle Contacts and Equations





The abstract triangle contact representation implies equations for the sidelength:

$x_a + x_b + x_c = x_v$ and $x_d = x_v$ and $x_e = x_v$ and $x_d + x_e = x_w$ and $\ldots$

Solving the Equations

Theorem. The system of equations has a unique solution.

Solving the Equations

Theorem. The system of equations has a unique solution.

The proof is based on counting matchings.

Solving the Equations

Theorem. The system of equations has a unique solution.

The proof is based on counting matchings.

In the solution some variables may be negative.

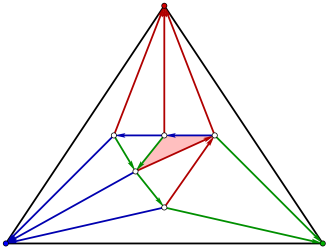
Solving the Equations

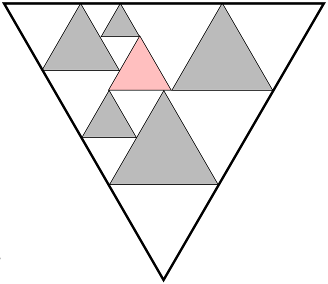
Theorem. The system of equations has a unique solution.

The proof is based on counting matchings.

In the solution some variables may be negative.

Still the solution yields a triangle contact representation.





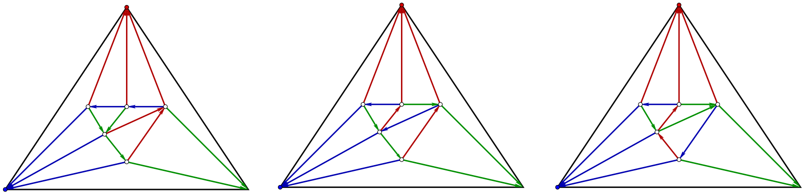
Flipping Cycles

Proposition. The boundary of a negative area is a directed cycle in the underlying Schnyder wood.

From the bijection

$$\text{Schnyder woods} \iff \text{3-orientations}$$

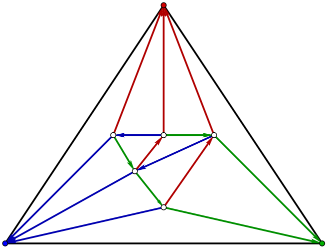
it follows that cycles can be reverted (flipped).

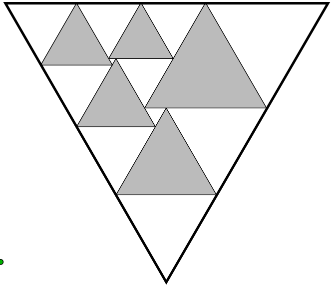


Resolving

A new Schnyder wood yields new equations and a new solution.

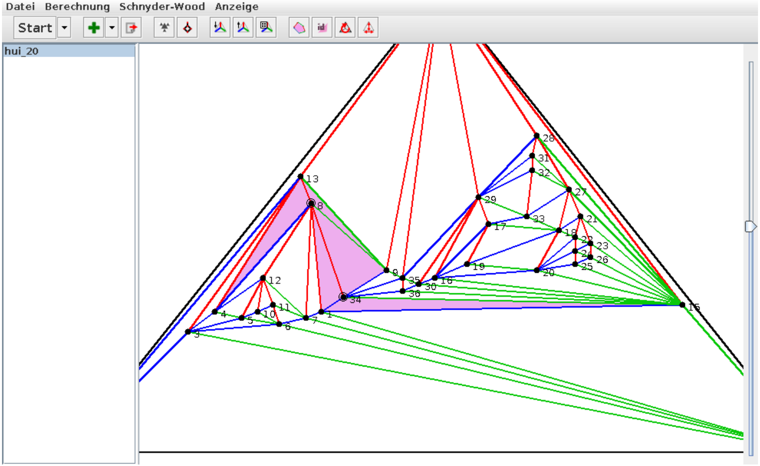
Theorem. A negative triangle becomes positive by flipping.





More Complications

It may be necessary to flip longer cycles.



The Status

- We have no proof that the process always ends with a homothetic triangle representation.
- From a program written by my student Julia Rucker we have strong experimental evidence that it does.

Outline

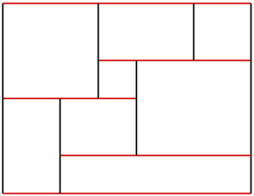
Thomassen's Theorem

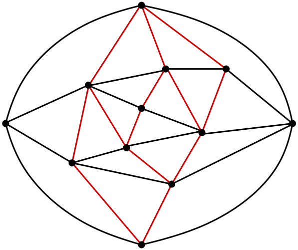
Theorem. Every planar graph has a contact representation of axis aligned boxes in three dimensions.

Thomassen's Theorem

Theorem. Every planar graph has a contact representation of axis aligned boxes in three dimensions.

- 4-connected planar graphs can be represented as rectangular duals (Ungar '53, He '93).





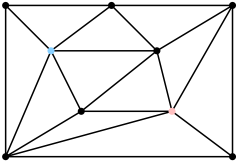
The Cube Theorem

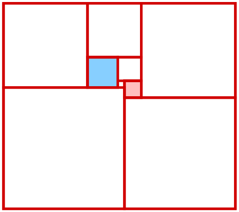
Theorem [Felsner, Francis].
Every planar graph has a contact representation of axis aligned cubes in three dimensions.

The Cube Theorem

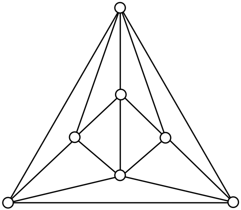
Theorem [Felsner, Francis].
Every planar graph has a contact representation of axis aligned cubes in three dimensions.

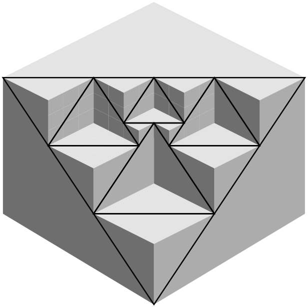
- 5-connected planar graphs have square-contact representation (Schramm '93).



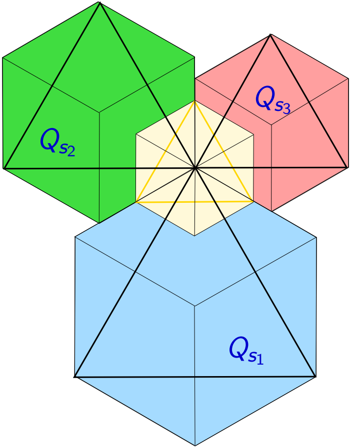


Edge-Coplanar Orthogonal Surfaces





Dealing with Separating Triangles



Open Problems

Cube Representations

Face to face contact representations of planar graphs with cubes.

Open Problems

Cube Representations

Face to face contact representations of planar graphs with cubes.

Homothetic Triangle Representations

Prove that the iterative algorithms stops with the intended result.

Open Problems

Cube Representations

Face to face contact representations of planar graphs with cubes.

Homothetic Triangle Representations

Prove that the iterative algorithms stops with the intended result.

Schnyder Woods: Applications

Combinatorial algorithm for the 8-gon representation.

Open Problems

Cube Representations

Face to face contact representations of planar graphs with cubes.

Homothetic Triangle Representations

Prove that the iterative algorithms stops with the intended result.

Schnyder Woods: Applications

Combinatorial algorithm for the 8-gon representation.

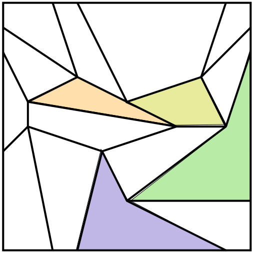
Cartograms

Do 6-gon cartograms exist for all 4-connected triangulations?

Do 4-connected triangulations have convex cartograms?

Table cartograms

4.5	4.5	16	2.5
4	3	4.5	3
2.5	6	4.5	10.5
7	9	9	6



The End.

Thank you