Distributive Lattices from Graphs

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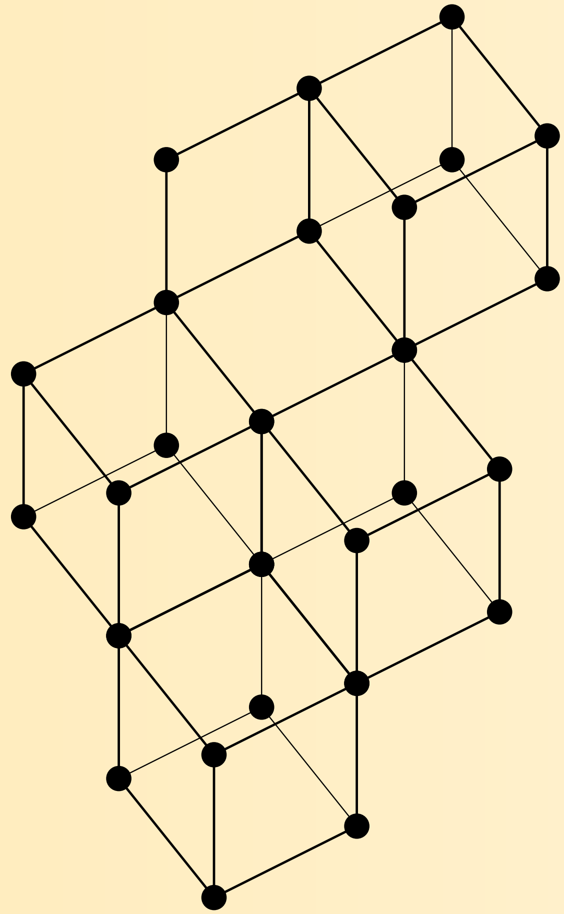

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The Talk

Lattices from Graphs

Proving Distributivity: ULD-Lattices

Embedded Lattices and D-Polytopes

Contents

Lattices from Planar Graphs

Definition. Given $G = (V, E)$ and $\alpha : V \longrightarrow \mathbb{N}$.
An α-orientation of G is an orientation with
$\text{outdeg}(v) = \alpha(v)$ for all v.

Lattices from Planar Graphs

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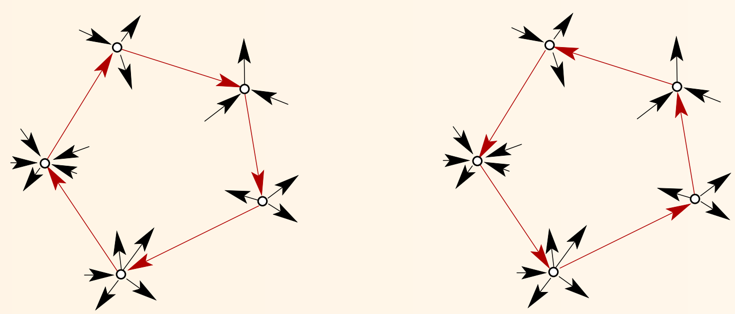
- Reverting directed cycles preserves α-orientations.



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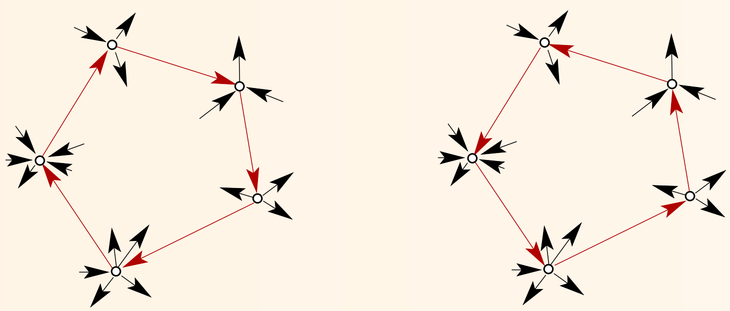


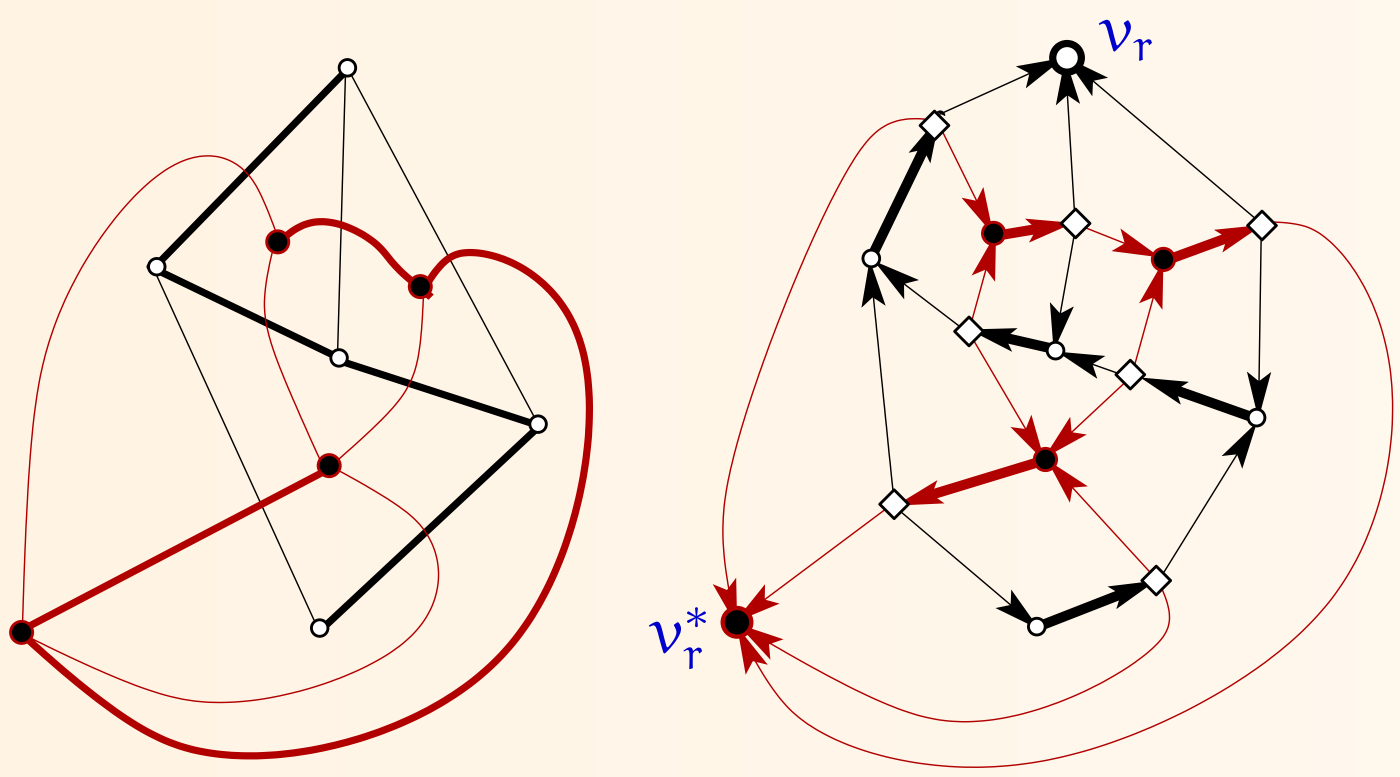
Theorem. The set of α-orientations of a planar graph G has the structure of a distributive lattice.

- Diagram edge $\sim$ revert a directed essential/facial cycle.

Example 1: Spanning Trees

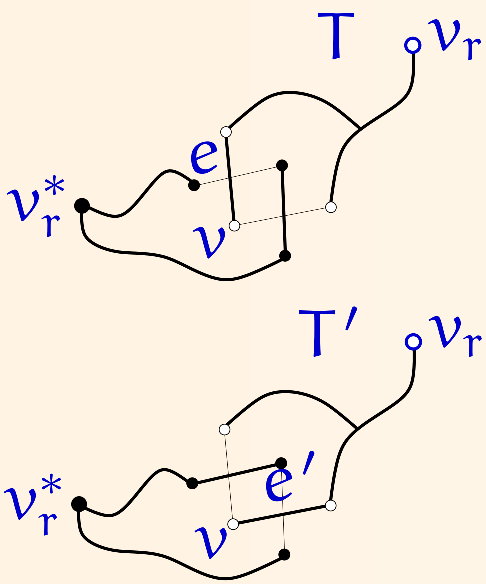
Spanning trees are in bijection with α_T orientations of a rooted primal-dual completion $\widetilde{G}$ of G

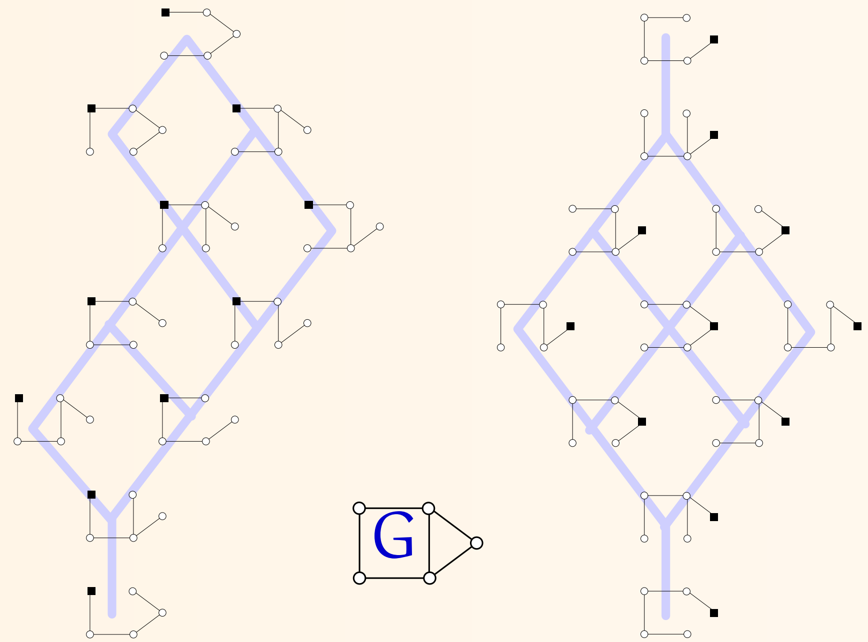
- $\alpha_T(v) = 1$ for a non-root vertex v and $\alpha_T(v_e) = 3$ for an edge-vertex v_e and $\alpha_T(v_r) = 0$ and $\alpha_T(v_r^*) = 0$.



Lattice of Spanning Trees

Gilmer and Litheland 1986, Propp 1993





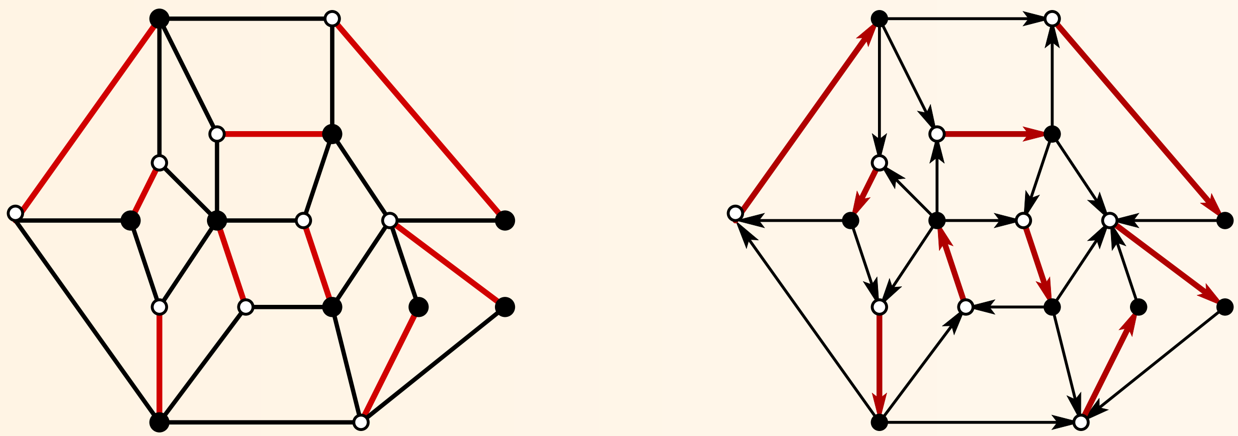
Question. How does a change of roots affect the lattice?

Example2: Matchings and f-Factors

Let G be planar and bipartite with parts (U, W). There is bijection between f-factors of G and α_f orientations:

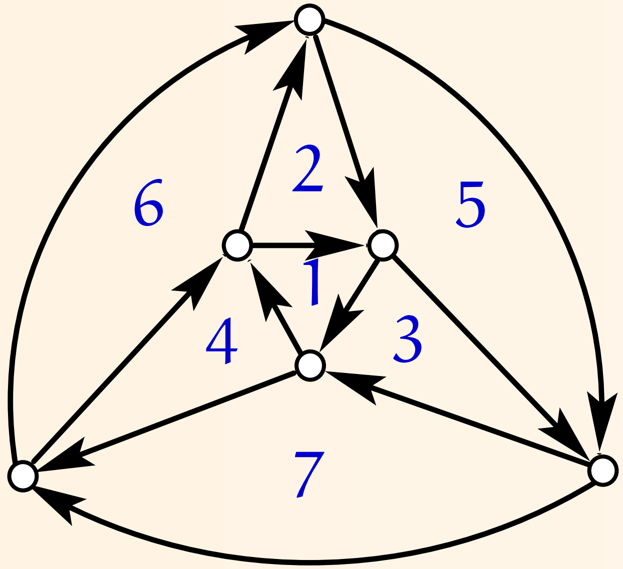
- Define α_f such that $\text{indeg}(u) = f(u)$ for all $u \in U$ and $\text{outdeg}(w) = f(w)$ for all $w \in W$.

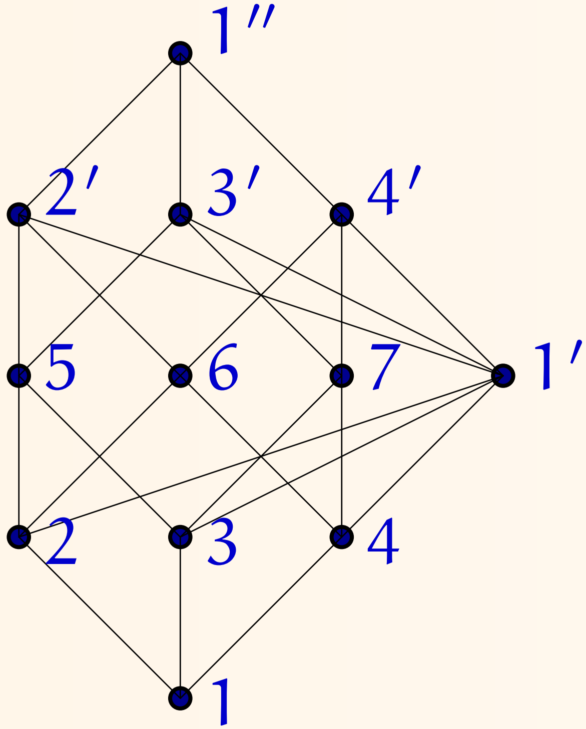
Example. A matching and the corresponding orientation.



Example 3: Eulerian Orientations

- Orientations with $\text{outdeg}(v) = \text{indeg}(v)$ for all v, i.e. $\alpha(v) = \frac{d(v)}{2}$



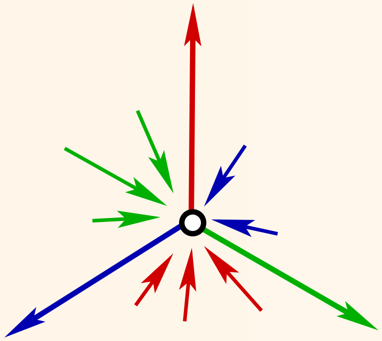


Example 4: Schnyder Woods

G a plane triangulation with outer triangle $F = \{a_1, a_2, a_3\}$.

A coloring and orientation of the interior edges of G with colors $1, 2, 3$ is a Schnyder wood of G iff

- Inner vertex condition:

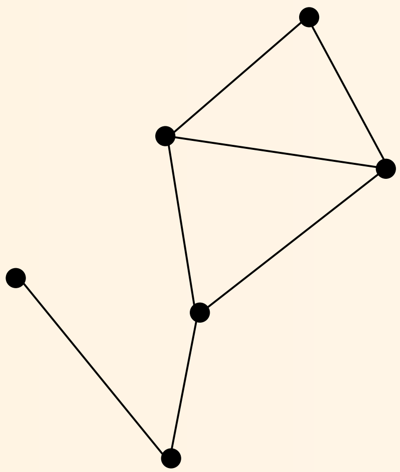


- Edges $\{v, a_i\}$ are oriented $v \to a_i$ in color i.

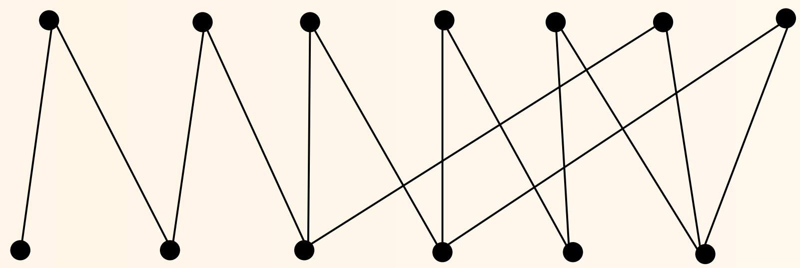
Digression: Schnyder's Theorem

The incidence order P_G of a graph G

G



P_G



Theorem [Schnyder 1989].

A Graph G is planar $\iff$ $\dim(P_G) \leq 3$.

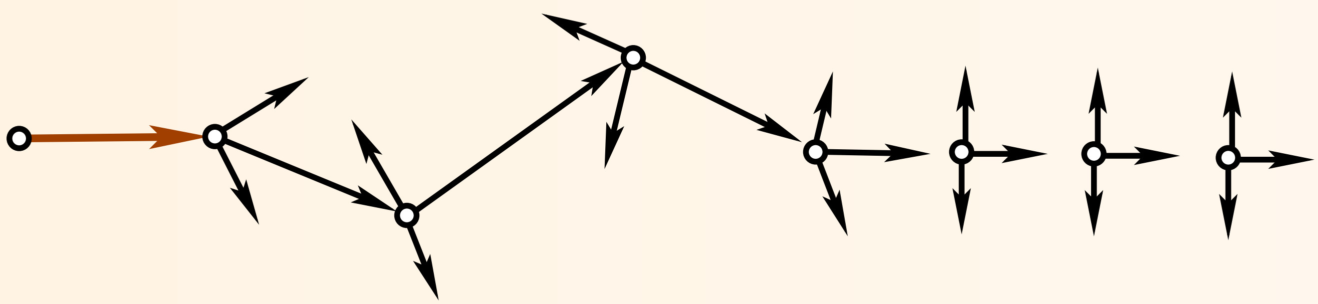
Schnyder Woods and 3-Orientations

Theorem. Schnyder wood and 3-orientation are in bijection.

Proof.

- All edges incident to a_i are oriented $\rightarrow a_i$.
  Prf: G has $3n-9$ interior edges and $n-3$ interior vertices.

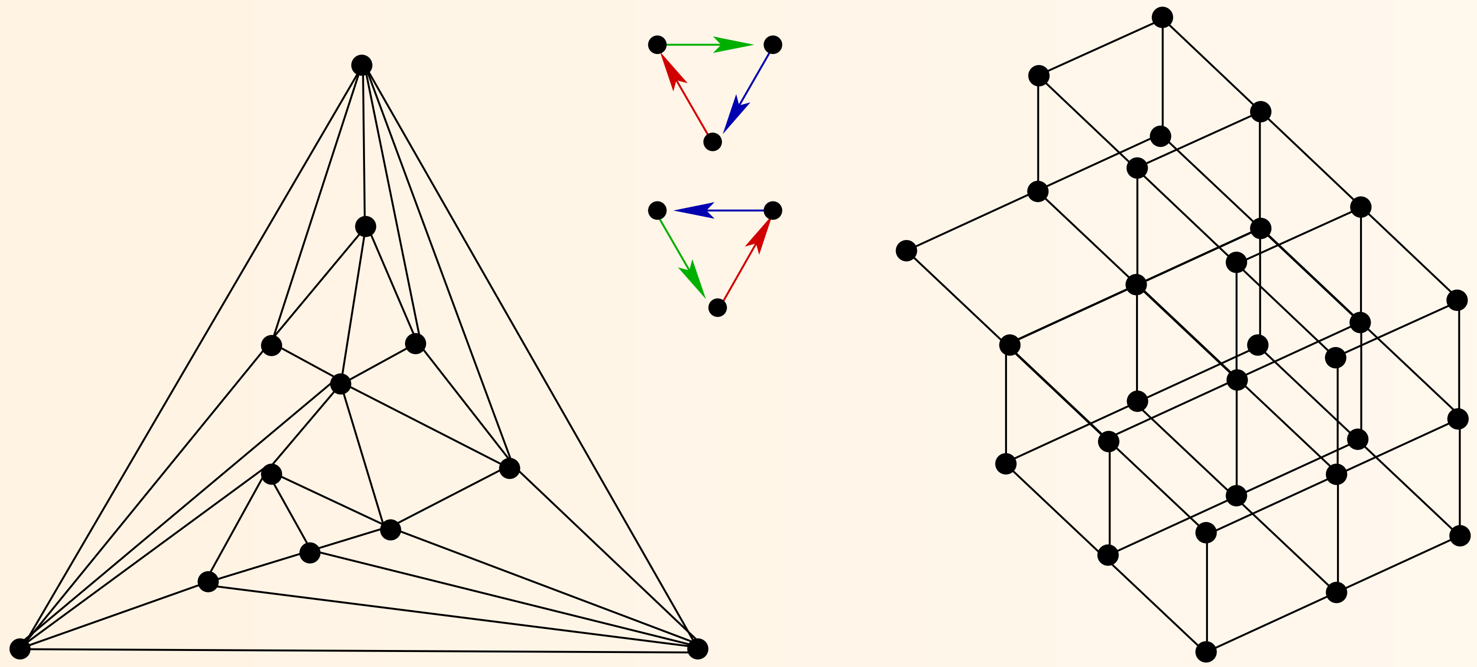
- Define the path of an edge:



- The path is simple (Euler), hence, ends at some a_i.

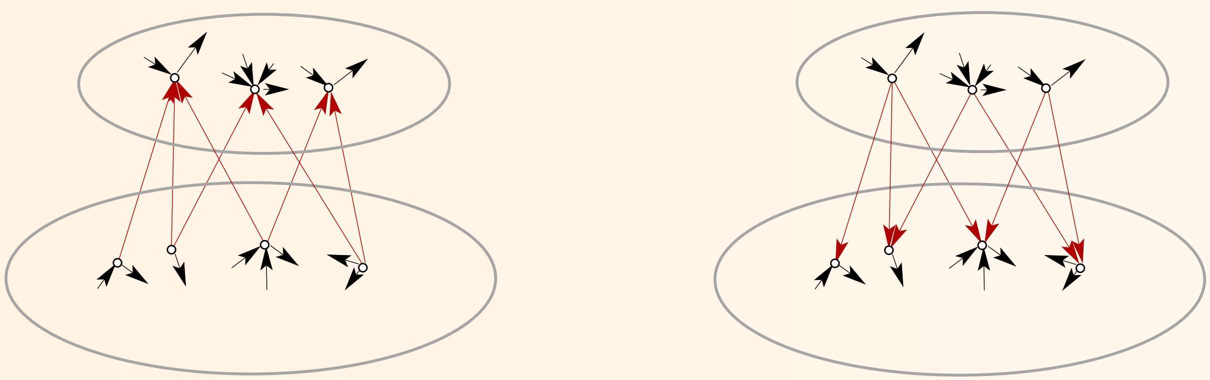
The Lattice of Schnyder Woods

Theorem. The set of Schnyder woods of a plane triangulation G has the structure of a distributive lattice.



A Dual Construction: c-Orientations

- Reorientations of directed cuts preserve flow-difference (#forward arcs − #backward arcs) along cycles.



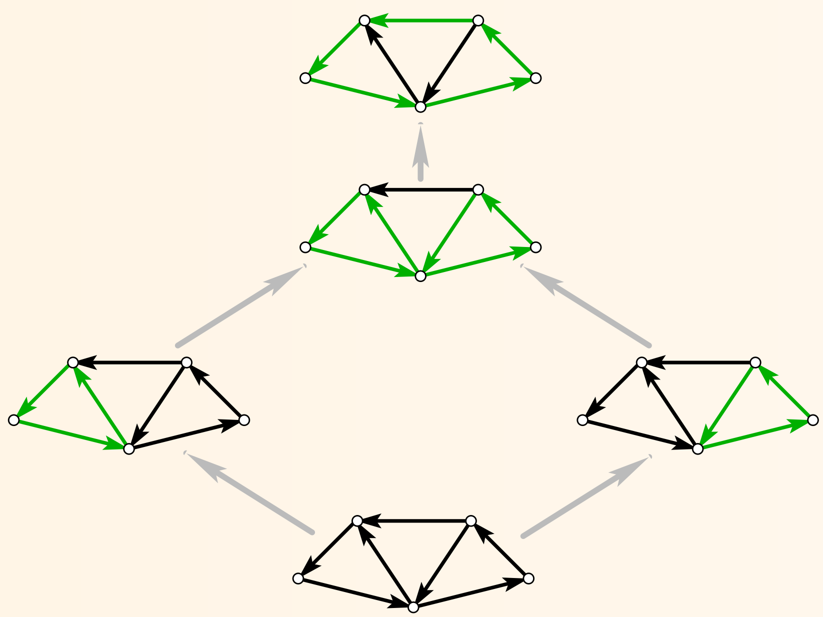
Theorem [Propp 1993]. The set of all orientations of a graph with prescribed flow-difference for all cycles has the structure of a distributive lattice.

- Diagram edge $\sim$ push a vertex ($\neq v_\dagger$).

Circulations in Planar Graphs

Theorem [Khuller, Naor and Klein 1993**].**
The set of all integral flows respecting capacity constraints ($\ell(e) \leq f(e) \leq u(e)$) of a planar graph has the structure of a distributive lattice.

$0 \leq f(e) \leq 1$



- Diagram edge $\sim$ add or subtract a unit of flow in ccw oriented facial cycle.

Δ-Bonds

$G = (V, E)$ a connected graph with a prescribed orientation.

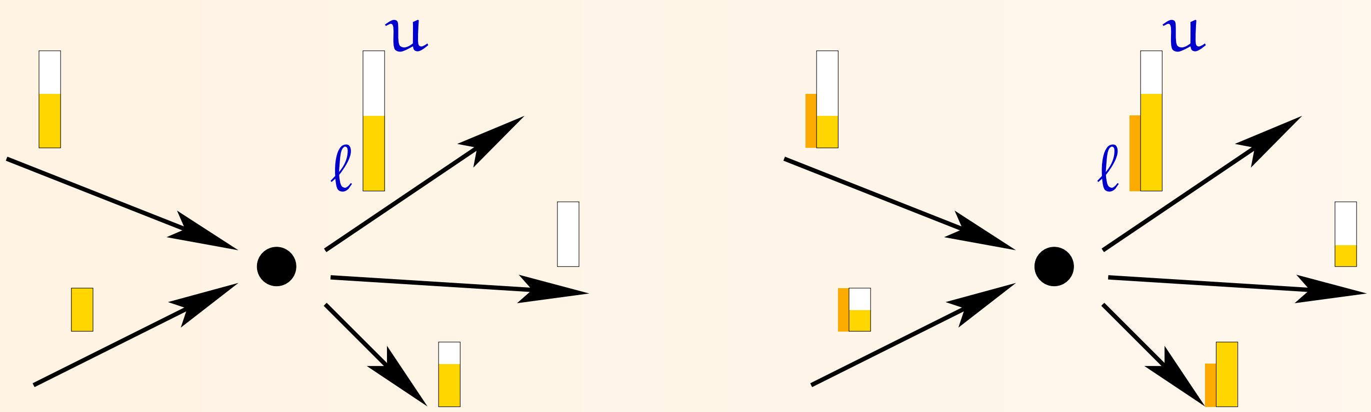
With $x \in \mathbb{Z}^E$ and C cycle we define the circular flow difference

$$\Delta_x(C) := \sum_{e \in C^+} x(e) - \sum_{e \in C^-} x(e).$$

With $\Delta \in \mathbb{Z}^{\mathcal{C}}$ and $\ell, u \in \mathbb{Z}^E$ let $\mathcal{B}_G(\Delta, \ell, u)$ be the set of $x \in \mathbb{Z}^E$ such that $\Delta_x = \Delta$ and $\ell \le x \le u$.

The Lattice of Δ-Bonds

Theorem [Felsner, Knauer 2007].
$\mathcal{B}_G(\Delta, \ell, u)$ is a distributive lattice.
The cover relation is vertex pushing.



Δ-Bonds as Generalization

$\mathcal{B}_G(\Delta, \ell, u)$ is the set of $x \in \mathbb{R}^E$ such that

- $\Delta_x = \Delta$ (circular flow difference)
- $\ell \leq x \leq u$ (capacity constraints).

Special cases:

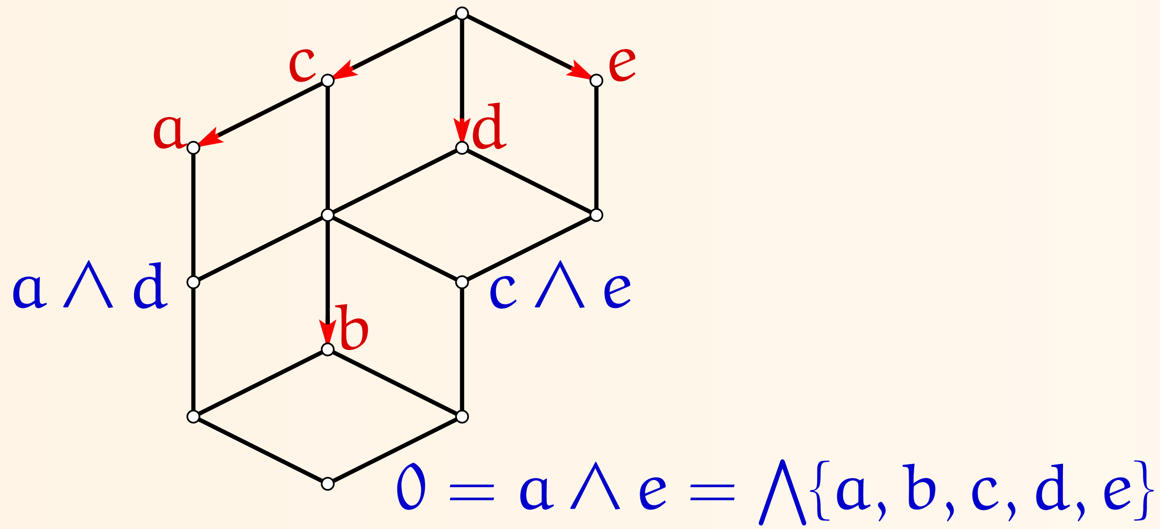
- c-orientations are $\mathcal{B}_G(\Delta, 0, 1)$ ($\Delta(C) = |C^+| - c(C)$).
- Circular flows on planar G are $\mathcal{B}_{G^*}(0, \ell, u)$ (G^* the dual of G).
- α-orientations.

Contents

ULD Lattices

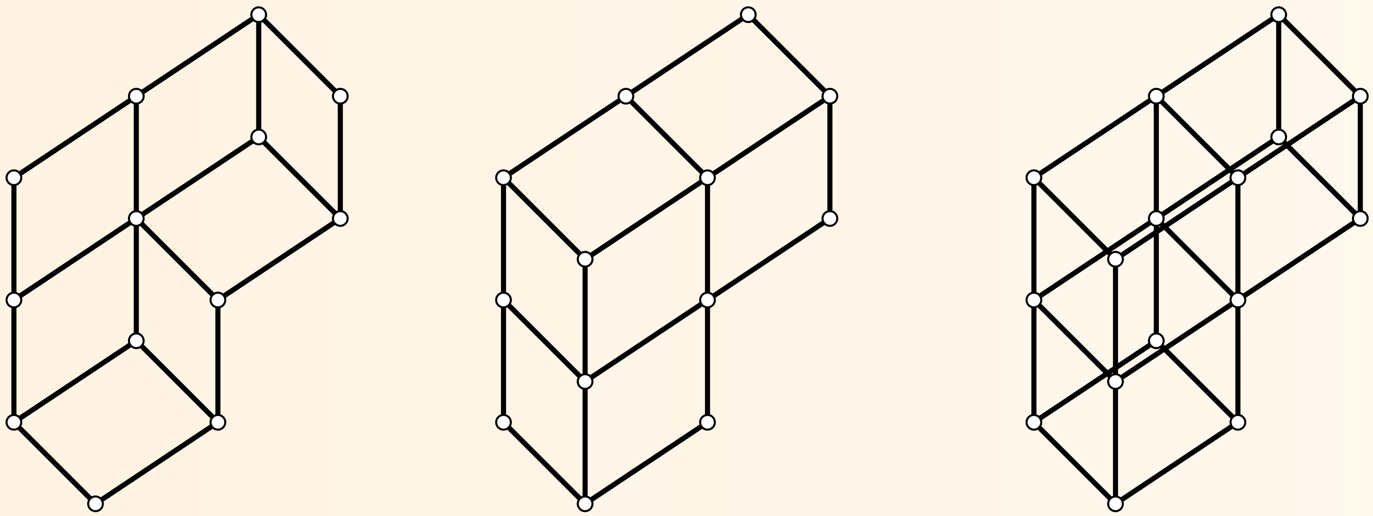
Definition. [Dilworth]
A lattice is an upper locally distributive lattice (ULD) if each element has a unique minimal representation as meet of meet-irreducibles, i.e., there is a unique mapping $x \to M_x$ such that

- $x = \bigwedge M_x$ (representation.) and
- $x \neq \bigwedge A$ for all $A \subsetneq M_x$ (minimal).



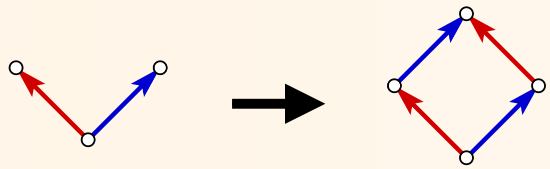
ULD vs. Distributive

Proposition.
A lattice it is ULD and LLD $\iff$ it is distributive.



Diagrams of ULD lattices: A Characterization

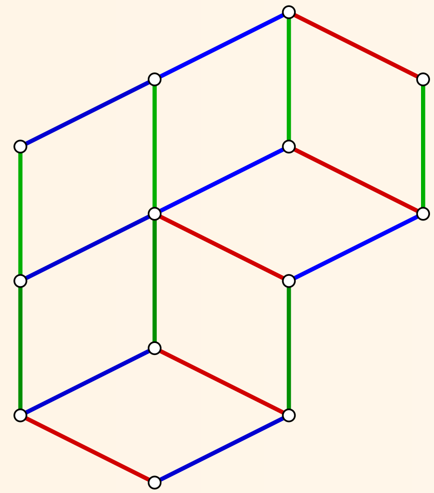
A coloring of the edges of a digraph is a $\mathfrak{U}$-coloring iff

- arcs leaving a vertex have different colors.
- completion property: 

Theorem.
A digraph D is acyclic, has a unique source and admits a $\mathfrak{U}$-coloring $\iff$ D is the diagram of an ULD lattice.

$\hookrightarrow$ Unique $\mathbf{1}$.

Examples of U-colorings



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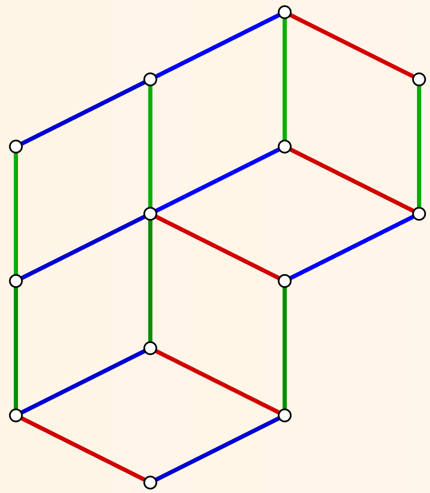


- Chip firing game with a fixed starting position (the source), colors are the names of fired vertices.

- Δ-bond lattices, colors are the names of pushed vertices. (Connected, unique $\mathbf{0}$).

More Examples

Some LLD lattices with respect to inclusion order:

- Subtrees of a tree (Boulaye '67).
- Convex subsets of posets (Birkhoff and Bennett '85).
- Convex subgraphs of acyclic digraphs (Pfaltz '71).
  (C is convex if with x, y all directed (x, y)-paths are in C).
- Convex sets of an abstract convex geometry, this is an universal family of examples (Edelman '80).

Contents

Embedded Lattices

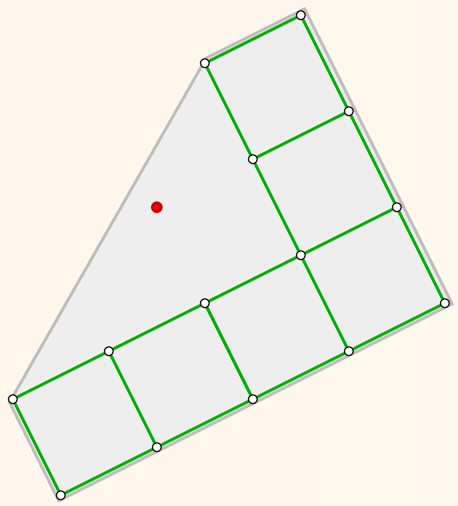
A $\mathcal{U}$-coloring of a distributive lattice L yields a cover preserving embedding $\phi : L \longrightarrow \mathbb{Z}^{\#\text{colors}}$.

Embedded Lattices

A U-coloring of a distributive lattice L yields a cover preserving embedding $\phi : L \to \mathbb{Z}^{\#\text{colors}}$.

In the case of Δ-bond lattices there is a polytope $P = conv(\phi(L)$ in $\mathbb{R}^{n-1}$ such that

$$\phi(L) = P \cap \mathbb{Z}^{n-1}$$

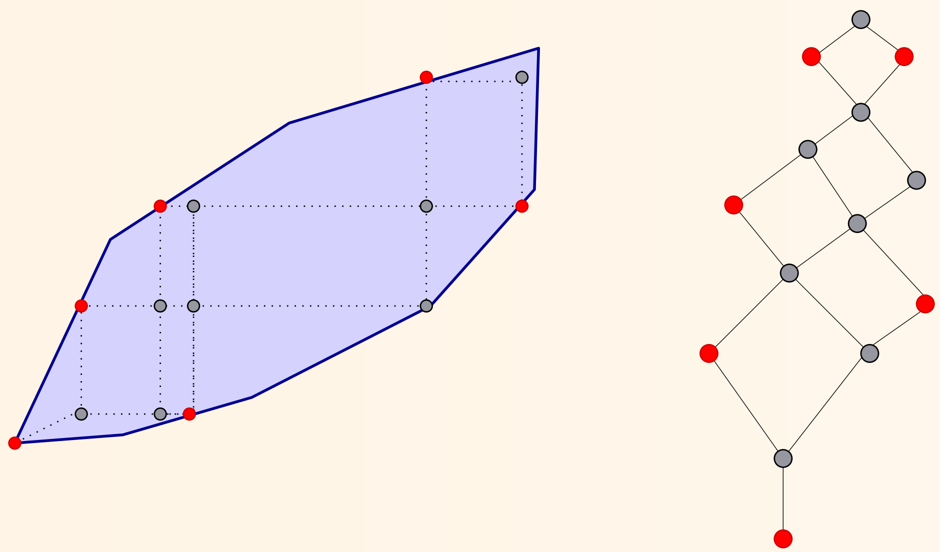


- This is a special property:

D-Polytopes

Definition. A polytope P is a D-polytope if with $x, y \in P$ also $\max(x, y), \min(x, y) \in P$.

- A D-polytope is a (infinite!) distributive lattice.
- Every subset of a D-polytope generates a distributive lattice in P. E.g. Integral points in a D-polytope are a distributive lattice.



D-Polytopes

Remark. Distributivity is preserved under

- scaling
- translation
- intersection

Theorem. A polytope P is a D-polytope iff every facet inducing hyperplane of P is a D-hyperplane, i.e., closed under $\max$ and $\min$.

D-Hyperplanes

Theorem. An hyperplane is a D-hyperplane iff it has a normal $\mathbf{e}_i - \lambda_{ij}\mathbf{e}_j$ with $\lambda_{ij} \geq 0$.

$(\Leftarrow)$ $\lambda_{ij}\mathbf{e}_i + \mathbf{e}_j$ together with $\mathbf{e}_k$ with $k \neq i, j$ is a basis. The coefficient of $\max(x, y)$ is the $\max$ of the coefficients of x and y.

$(\Rightarrow)$ Let $n = \sum_i a_i\mathbf{e}_i$ be the normal vector. If $a_i > 0$ and $a_j > 0$, then $x = a_j\mathbf{e}_i - a_i\mathbf{e}_j$ and $y = -x$ are in $n^\perp$ but $\max(x, y)$ is not.

A First Graph Model for D-Polytopes

Consider $\ell, u \in \mathbb{Z}^m$ and a Λ-weighted network matrix N_Λ of a connected graph. (Rows of N_Λ are of type $\mathbf{e}_i - \lambda_{ij}\mathbf{e}_j$ with $\lambda_{ij} \geq 0$.)

- [Strong case, $\mathrm{rank}(N_\Lambda) = n$]
  The set of $p \in \mathbb{Z}^n$ with $\ell \leq N_\Lambda^\top p \leq u$ is a distributive lattice.

- [Weak case, $\mathrm{rank}(N_\Lambda) = n - 1$]
  The set of $p \in \mathbb{Z}^{n-1}$ with $\ell \leq N_\Lambda^\top (0, p) \leq u$ is a distributive lattice.

A Second Graph Model for D-Polytopes

(Rows of N_Λ are of type $\mathbf{e}_i - \lambda_{ij}\mathbf{e}_j$ with $\lambda_{ij} \geq 0$.)

Theorem [Felsner, Knauer 2008].
Let $Z = \ker(N_\Lambda)$ be the space of Λ-circulations. The set of $x \in \mathbb{Z}^m$ with

- $\ell \leq x \leq u$ (capacity constraints)
- $\langle x, z \rangle = 0$ for all $z \in Z$
(weighted circular flow difference).

is a distributive lattice $\mathcal{D}_G(\Lambda, \ell, u)$.

- Lattices of Δ-bonds are covered by the case $\lambda_{ij} = 1$.

The Strong Case

For a cycle C let

$$\gamma(C) := \prod_{e \in C^+} \lambda_e \prod_{e \in C^-} \lambda_e^{-1}.$$

A cycle with $\gamma(C) \neq 1$ is strong.

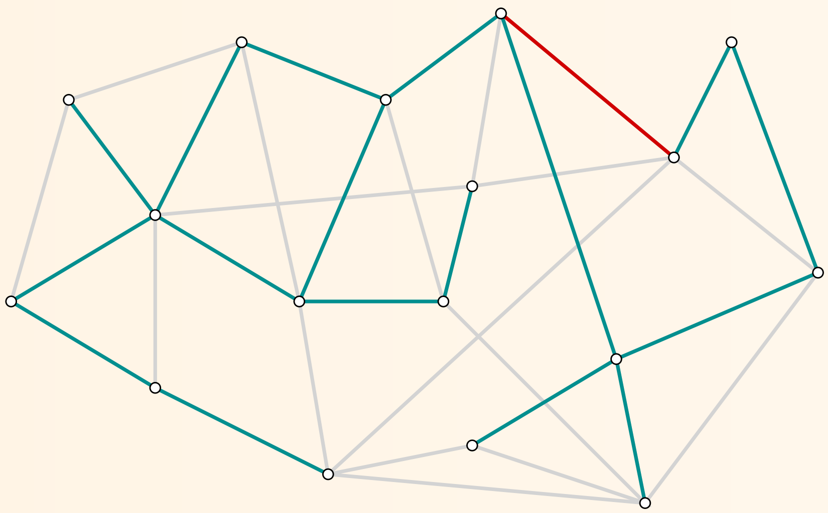
Proposition. $\mathrm{rank}(N_\Lambda) = n$ iff it contains a strong cycle.

Remark.
C strong $\implies$ there is no circulation with support C.

Fundamental Basis

A fundamental basis for the space of Λ-circulations:

- Fix a 1-tree T, i.e, a unicyclic set of n edges. With $e \notin T$ there is a circulation in $T + e$



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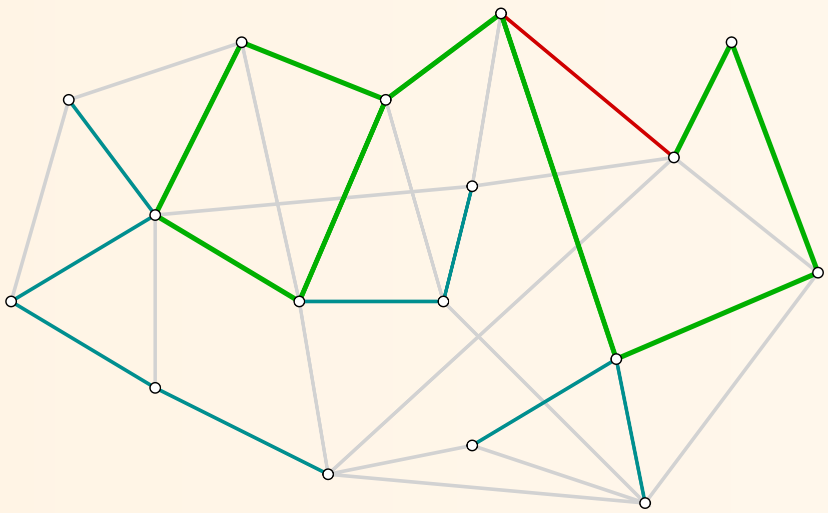

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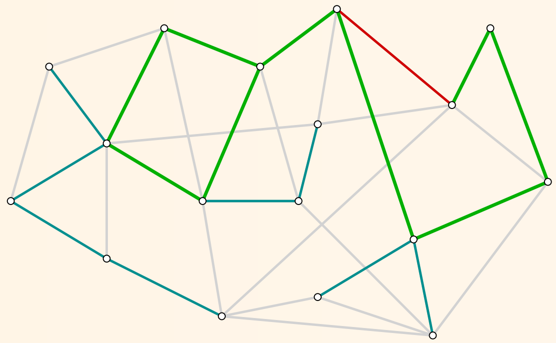

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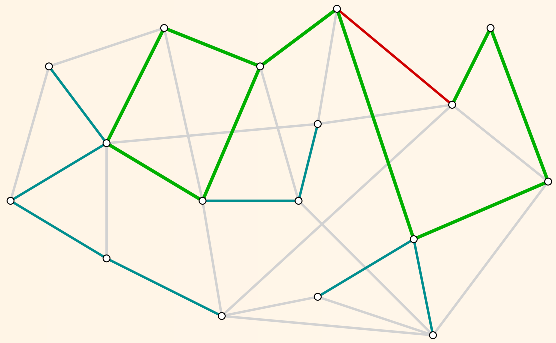

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$\Longrightarrow$ Further topic: D-polytopes and optimization.

Conclusion

- Δ-bond lattices generalize previously known distributive lattices from graphs.

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The End