

Linear size universal point sets for classes of planar graphs*

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1 Abstract

2 A finite point set $P \subseteq \mathbb{R}^2$ is n -universal with respect to a class \mathcal{G} of planar graphs if every n -vertex-
3 graph $G \in \mathcal{G}$ admits a crossing-free straight-line drawing with vertices being placed at points of P .
4 A widely studied problem in graph drawing is to identify small universal point sets.

5 For the class of all planar graphs the best known upper bound on the size of a universal point set
6 is quadratic and the best known lower bound is linear in n . One of the classical results in the area
7 is that every set of n points in general position (no three collinear) is n -universal for outerplanar
8 graphs. While some other classes are known to admit universal point sets of near linear size, we are
9 not aware of truly linear bounds for interesting classes beyond outerplanar graphs.

10 In this paper we study a specific ordered point set H (the exploding double chain) and show that
11 all planar graphs G on $n \geq 2$ vertices which are subgraphs of a planar graph admitting a one-sided
12 Hamiltonian cycle have a straight-line drawing on the initial piece H_n of size $2n - 2$ in H . Let
13 \mathcal{H}' be the class of all subgraphs of planar graphs admitting a one-sided Hamiltonian cycle. It had
14 been conjectured that all 4-connected triangulations belong to \mathcal{H}' . While the conjecture has been
15 disproved, it is still true that H_n is n -universal for a large class \mathcal{H} of planar graphs. We show that
16 all bipartite plane graphs and all cubic plane graphs belong to $\mathcal{H}' \subseteq \mathcal{H}$. Remarkably, however, not
17 all 2-trees are in \mathcal{H}' .

Lines 173

18 1 Introduction

19 Given a family \mathcal{G} of planar graphs and a positive integer n , a point set $P \subseteq \mathbb{R}^2$ is called an
20 *n -universal point set* for the class \mathcal{G} or simply *n -universal* for \mathcal{G} if for every graph $G \in \mathcal{G}$ on
21 n vertices there exists a straight-line crossing-free drawing of G such that every vertex of G
22 is placed at a point of P . It is a widely studied and fundamental open problem in geometric
23 graph theory (compare also the entry [16] in the Open Problem Garden) to determine, given
24 a class of graphs \mathcal{G} , (the asymptotics of) the minimum size $f_{\mathcal{G}}(n)$ of an n -universal point set
25 for \mathcal{G} . If \mathcal{G} is the class of all planar graphs we simply write $f(n) := f_{\mathcal{G}}(n)$.

26 Schnyder [20] showed that for $n \geq 3$ the $[n - 1] \times [n - 1]$ -grid forms an n -universal point
27 set for planar graphs, even if the combinatorial embedding of the planar graph is prescribed.
28 This shows that $f(n) < n^2 = O(n^2)$. Asymptotically, the quadratic upper bound on $f(n)$

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remains the state of the art. However, the multiplicative constant in this bound has been improved, see [4, 5]. The current upper bound is $f(n) \leq \frac{1}{4}n^2 + O(n)$ by Bannister et al. [4]. For several subclasses \mathcal{G} of planar graphs, better upper bounds are known: A classical result by Pach et al. [18] is that every outerplanar n -vertex graph embeds straight-line on *any* set of n points in general position, and hence $f_{\text{out-pl}}(n) = n$. Near-linear upper bounds of $f_{\mathcal{G}}(n) = O(n \text{ polylog}(n))$ are known for 2-outerplanar graphs, simply nested graphs, and for the classes of bounded pathwidth [3, 4]. Finally, for the class \mathcal{G} of planar 3-trees (also known as apollonian networks or stacked triangulations), an upper bound of $f_{\mathcal{G}}(n) = O(n^{3/2} \log n)$ has been proved by Fulek and Tóth [12].

As for lower bounds, the trivial bounds $n \leq f_{\mathcal{G}}(n) \leq f(n)$ hold for all $n \in \mathbb{N}$ and all planar graph classes \mathcal{G} . The best lower bound $f(n) \geq 1.293n - o(n)$ from [19] has been shown using planar 3-trees, we refer to [6, 14, 8, 9] for earlier work on lower bounds.

It seems that in order to improve the quadratic upper bound on $f(n)$ to $o(n^2)$, the considered point sets should be not too uniformly distributed. Indeed, Choi, Chrobak and Costello [7] recently proved that point sets chosen uniformly at random from the unit square must have size $\Omega(n^2)$ in order to be universal for n -vertex planar graphs, with high probability.

In this paper we study a specific ordered point set H (the exploding double chain) and let H_n be the initial piece of size $2n - 2$ in H (for $n \geq 2$). Throughout the paper, let \mathcal{H} be the class of all planar graphs G which have a plane straight line drawing on the point set H_n where $n = |V(G)|$. That is, H_n forms an n -universal point set for \mathcal{H} .

A graph is a POSH (partial one-sided Hamiltonian) if it is a spanning subgraph of a planar graph admitting a one-sided Hamiltonian cycle. Our main result (Theorem 2.1) is that every POSH is in \mathcal{H} . We let $\mathcal{H}' := \{\mathcal{G} : \mathcal{G} \text{ is POSH}\}$.

Theorem 2.1 motivates the study of \mathcal{H}' . This class of planar graphs seems to be quite large, e.g., the smallest 4-connected triangulation which is known not to be POSH has 113 vertices [2]. On the positive side we show that every bipartite plane graph is POSH (Section A, sketched in Section 3). In Section B we use the construction for bipartite graphs to show that cubic plane graphs are POSH. Section 4 gives an overview of the proof method. The appendix also contains a negative result in Section C, namely that not all 2-trees are POSH. We conclude with some conjectures and open problems in Section 5.

2 The point set and the embedding strategy

In this section we introduce an ordered point set H and a class \mathcal{H}' of planar graphs and show that for every $n \geq 2$ the initial part H_n of size $2n - 2$ is n -universal for the class \mathcal{H}' .

A sequence $Y = (y_i)_{i \geq 0}$ of real numbers satisfying $y_1 = 0$, $y_2 = 0$, and $y_{i+1} > 2y_i + y_{i-1}$ for all $i \geq 2$ is called *exploding*. Note that if $\alpha > 1 + \sqrt{2}$, then $y_1 = y_2 = 0$ and $y_i = \alpha^{i-3}$ for $i \geq 3$ is an exploding sequence. Given an exploding sequence Y let $P(Y) = (p_i)_{i \geq 0}$ be the set of points with $p_i = (i, y_i)$ and let $\bar{P}(Y) = (q_i)_{i \geq 0}$ be the set of points with $q_i = (i, -y_i)$, i.e., the reflected point set, and note that $p_1 = q_1$ and $p_2 = q_2$. Let $H(Y) = P(Y) \cup \bar{P}(Y)$ and $H_n(Y) = \{p_i, q_i | 1 \leq i \leq n\}$ so that $|H_n(Y)| = 2n - 2$. Figure 1 illustrates $H_6(Y)$.

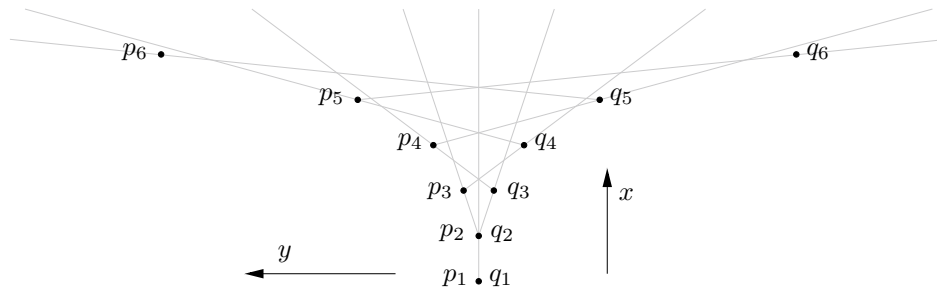
Let $H = H(Y)$ for some exploding sequence Y . For two points p and q let $H(p, q)$ be the

set of points of H in the open right half-plane of the directed line \vec{pq} . Note that¹

$$H(p_i, q_j) = \begin{cases} (p_k)_{k \leq j} \cup (p_k)_{k > i} \cup (q_\ell)_{\ell < j} & \text{if } i > j \\ (p_k)_{k < i} \cup (q_\ell)_{\ell < i} & \text{if } i = j \\ (p_k)_{k < i} \cup (q_\ell)_{\ell \leq i} \cup (q_\ell)_{\ell > j} & \text{if } i < j \end{cases}$$

70 Moreover, if $i < j$ then $H(q_i, q_j) = H(p_i, q_j) \setminus \{q_i\}$ and if $i > j$ then $H(p_i, p_j) = H(p_i, q_j) \setminus \{p_j\}$.
 71 These sidedness informations characterize the order type of H . A point set $A = \{p_i, q_i | i \geq 1\}$
 72 is an *exploding double chain* if it has the order type of H .

73 In the context of graph drawing an exploding double chain was previously used by Löffler
 74 and Tóth [15]. They show that every planar graph with n vertices has a 1-bend drawing on
 75 a subset S_n of H with $|S_n| = 6n - 10$. Note that our result about bipartite graphs implies a
 76 better bound. Since subdivision of $n - 2$ edges is enough to make any planar graph bipartite,
 77 a subset of size $4n - 6 = 2(n + n - 2) - 2$ is large enough. Universality for 1-bend and 2-bend
 78 drawings has been studied by Kaufmann and Wiese [13], they show that every n element
 79 point set is universal for 2-bend drawings of planar graphs.



80 ■ **Figure 1** An example of a point set H_6 in a rotated coordinate system ($p_i = q_i$ for $i = 1, 2$).

81 A plane graph G has a *one-sided Hamiltonian cycle* with reverse edge vu if it has a
 82 Hamiltonian cycle (v_1, v_2, \dots, v_n) such that $u = v_1, v = v_n$ and uv is incident to the outer
 83 face. Moreover, for every $j = 2, \dots, n$ in the restriction $G[v_1, \dots, v_j, v_{j+1}]$ of G the edges
 84 $v_{j-1}v_j$ and $v_{j+1}v_j$ are consecutive in the rotation of v_j . A more pictorial reformulation of
 85 the second condition is that in the embedding of G for every j either all the back-edges $v_i v_j$
 86 with $i < j$ are drawn inside the closed bounded region D whose boundary is the Hamiltonian
 87 cycle or they are all drawn in the closed region outside of the cycle. We let V_I be the set of
 88 vertices v_j which have a back-edge $v_i v_j$ with $i < j - 1$ drawn inside D and $V_O = V \setminus V_I$.

89 In the context of cartograms Alam et al. [1] conjectured that every plane 4-connected
 90 triangulation has a one-sided Hamiltonian cycle. Later Alam and Kobourov [2] found a plane
 91 4-connected triangulation on 113 vertices which has no one-sided Hamiltonian cycle.

92 Recall that \mathcal{H}' is the class of POSH graphs, i.e., of plane graphs which are spanning
 93 subgraphs of plane graphs admitting a one-sided Hamiltonian cycle. Our interest in this
 94 class is motivated by the following theorem.

95 ► **Theorem 2.1.** *Let G' be a POSH and let v_1, \dots, v_n be a one-sided Hamiltonian cycle of a*
 96 *plane supergraph G of G' on the same vertex set. Then there is a crossing-free embedding*
 97 *of G' on H_n with the property that v_i is placed on either p_i or q_i .*

68 ¹ In cases where i or j are in $\{1, 2\}$ the following may list one of the two points defining the halfspace
 69 with its second name as member of the halfspace. For correctness such listings have to be ignored.

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98 **Proof.** It is sufficient to describe the embedding of the supergraph G on H_n . For the proof
 99 we assume that in the plane drawing of G the sequence v_1, \dots, v_n traverses the boundary
 100 of D in counter-clockwise direction. For each i vertex v_i is embedded at $\bar{v}_i = p_i$ if $v_i \in V_I$
 101 and at $\bar{v}_i = q_i$ if $v_i \in V_O$.

102 Let $G_i = G[v_1, \dots, v_i]$ be the subgraph of G induced by $\{v_1, \dots, v_i\}$. The path $\Lambda_i =$
 103 v_1, \dots, v_i separates G_i , the *left part* GL_i consists of the intersection of G_i with D , the *right*
 104 *part* GR_i is G_i minus all edges which are interior to D . The intersection of GL_i and GR_i
 105 is Λ_i and their union is G_i . The counter-clockwise boundary walk of G_i consists of a path
 106 ∂R_i from v_1 to v_i which is contained in GR_i and a path from v_i to v_1 which is contained in
 107 GL_i , let ∂L_i be the reverse of this path.

108 Let \bar{G}_i be the straight line drawing of the plane graph G_i induced by placing each
 109 vertex v_j at the corresponding \bar{v}_j . A vertex \bar{v} of \bar{G}_i is said to *see a point* p if there is no
 110 crossing between the segment $\bar{v}p$ and an edge of \bar{G}_i . With induction we show:

- 111 1. The drawing of \bar{G}_i is plane, i.e., non-crossing.
- 112 2. \bar{G}_i and G_i have the same outer boundary walks.
- 113 3. Every vertex of ∂L_i in \bar{G}_i sees all the points p_j with $j > i$ and every vertex of ∂R_i in \bar{G}_i
 114 sees all the points q_j with $j > i$.

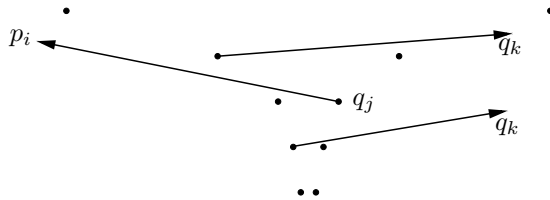
115 For $i = 2$ the graph G_i is just an edge and the three claims are immediate, for property 3
 116 just recall that the line spanned by p_1 and p_2 separates the p -side and the q -side of H_n .

117 Now assume that $i \in \{3, \dots, n\}$, the properties are true for \bar{G}_{i-1} and suppose that $v_i \in V_I$
 118 (the argument in the case $v_i \in V_O$ works symmetrically). This implies that all the back-edges
 119 of v_i are in the interior of D whence all the neighbors of v_i belong to ∂L_{i-1} . Since $v_i \in V_I$
 120 we have $\bar{v}_i = p_i$ and property 3 of \bar{G}_{i-1} implies that the edges connecting to \bar{v}_i can be added
 121 to \bar{G}_{i-1} without introducing a crossing. This is property 1 of \bar{G}_i .

122 Since G_{i-1} and \bar{G}_{i-1} have the same boundary walks and v_i respectively \bar{v}_i belong to the
 123 outer faces of G_i and \bar{G}_i and since v_i has the same incident edges in G_i as \bar{v}_i in \bar{G}_i , the outer
 124 walks of G_i and \bar{G}_i again equal each other, i.e., property 2.

125 Let j be minimal such that $v_j v_i$ is an edge and note that ∂L_i is obtained by taking the
 126 prefix of ∂L_{i-1} whose last vertex is v_j and append v_i . The line spanned by \bar{v}_j and $\bar{v}_i = p_i$
 127 separates all the edges incident to \bar{v}_i in \bar{G}_i from all the segments $\bar{v}_\ell p_k$ with $\ell < j$ and $\bar{v}_\ell \in \partial L_i$
 128 and $k > i$. This shows that every vertex of ∂L_i in \bar{G}_i sees all the points p_k with $k > i$. For
 129 the proof of the second part of property 3 we refer to Figure 2, it shows that the new edges
 130 $\bar{v}_j p_i$ do not obstruct the visibility between vertices of ∂R_i and any q_k with $k > i$. Of course
 131 this can also be derived formally by translating the condition for a crossing between two
 132 segments into sidedness conditions and then compare with the sidedness conditions given for
 133 the order type of H . This completes the proof of property 3 and thus the inductive step.

134 Finally, property 1 for G_n implies the theorem. ◀



135 ■ **Figure 2** Vertices from ∂R_i see q_k

3 Plane bipartite graphs

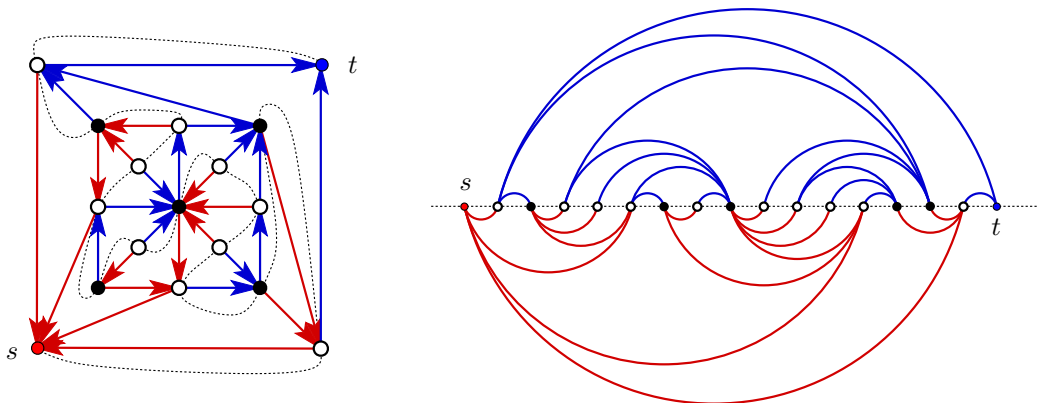
In this section we consider bipartite plane graphs and sketch a proof that they are POSH. The full proof can be found in Section A.

► **Theorem 3.1.** *Every bipartite plane graph $G = (V, E)$ is a spanning subgraph of a plane graph G' on the same vertex set V which has a one-sided Hamiltonian cycle, i.e., G is POSH.*

Quadrangulations are the plane graphs with all faces of degree four. Equivalently they are the maximal plane bipartite graphs. Every connected bipartite plane graph with at least two vertices in each color class is a spanning subgraph of a plane quadrangulation. Therefore it suffices to prove the theorem for plane quadrangulations.

A *separating decomposition* of a quadrangulation is an orientation and 2-coloring of the edges, such that s and t only have incoming edges in red and blue respectively, while each other vertex has outgoing edges in both colors as shown in Figure ??.

Every quadrangulation admits a separating decomposition [11, 17]. The equatorial line of the separating decomposition separates the red and blue edges which form trees rooted in s and t respectively, see [10]. Figure 3 shows that the equatorial line is a one-sided hamiltonian path, white vertices have red and black vertices have blue backward edges. This shows that quadrangulations and hence bipartite graphs are POSH.



■ **Figure 3** A quadrangulation with a separating decomposition, the equatorial line (dotted), and the induced drawing with a one-sided hamiltonian path.

4 Plane cubic graphs

This section is devoted to sketching a proof (full proof in section B) for the following theorem:

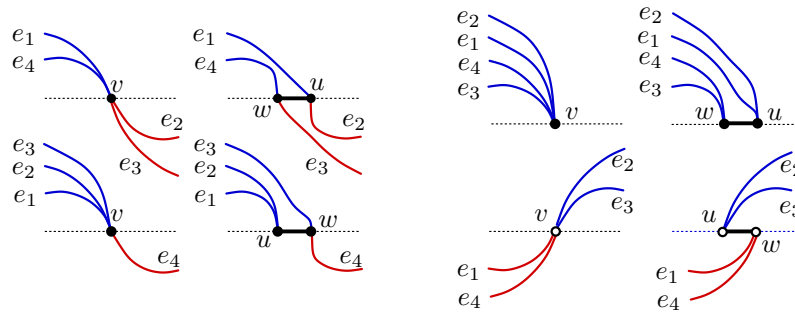
► **Theorem 4.1.** *Every plane cubic graph G is a spanning subgraph of a plane graph G' on the same vertex set V which has a one-sided Hamiltonian cycle, i.e., G is POSH.*

To prove this, we use Theorem 3.1 and the following lemma:

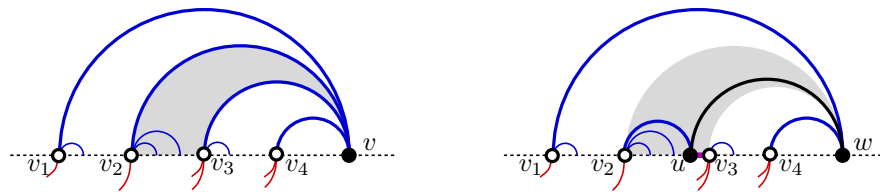
► **Lemma 4.2.** *Let G be a cubic graph. Then G admits a matching M such that contracting all the edges of M results in a bipartite multi-graph.*

The technique used to prove the theorem using the lemma is to do vertex splits carefully. We distinguish between *local* and *far* splits, some of which are illustrated in figures 4 and 5.

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164 ■ **Figure 4** Four cases for the local split of a vertex v .



165 ■ **Figure 5** Far split within the gray region of vertex v with edges to the left in the upper half-plane.

5 Concluding remarks

167 We have introduced the exploding double chain as a special point set (order type) and shown
 168 that the initial part H_n of size $2n - 2$ is n -universal for graphs on n vertices which are POSH.

169 ► **Conjecture 1.** *Every triangle-free plane graph is POSH.*

170 ► **Conjecture 2.** *Every 5-connected planar triangulation is POSH.*

171 We have shown that 2-trees and their superclasses series-parallel and planar Laman
 172 graphs are not contained in the class \mathcal{H}' of POSH graphs. The question whether these classes
 173 admit universal point sets of linear size remains intriguing.

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219 **A** Plane bipartite graphs: Full version

220 In this section we consider bipartite plane graphs and show that they are POSH.

221 ► **Theorem 3.1.** *Every bipartite plane graph $G = (V, E)$ is a spanning subgraph of a plane*
222 *graph G' on the same vertex set V which has a one-sided Hamiltonian cycle, i.e., G is POSH.*

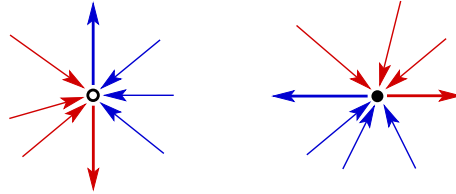
223 Quadrangulations are the plane graphs with all faces of degree four. Equivalently they
224 are the maximal plane bipartite graphs, i.e., any edge-addition either breaks bipartiteness
225 or planarity. Every bipartite plane graph with at least two vertices in each color class is a
226 spanning subgraph of a plane quadrangulation. Since every spanning subgraph of a POSH
227 is a POSH it suffices to prove the theorem for plane quadrangulations (the case where one
228 class is of size one (stars) is trivial).

229 Let Q be a quadrangulation and let V_B and V_W be the color classes of a 2-coloring.
230 Label the two black vertices of the outer face as s and t . Henceforth, when talking about
231 a quadrangulation we think of an embedded quadrangulation endowed with s and t . A
232 *separating decomposition* is a pair $D = (Q, Y)$ where Q is a quadrangulation and Y is an
233 orientation and coloring of the edges of Q with colors red and blue such that:

- 234 1. The edges incident to s and t are incoming in color red and blue, respectively.

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- 235 2. Every vertex $v \notin \{s, t\}$ is incident to a non-empty interval of red edges and a non-empty
 236 interval of blue edges. If v is white, then, in clockwise order, the first edge in the interval
 237 of a color is outgoing and all the other edges of the interval are incoming. If v is black,
 238 the outgoing edge is the clockwise last in its color (see Figure 6).

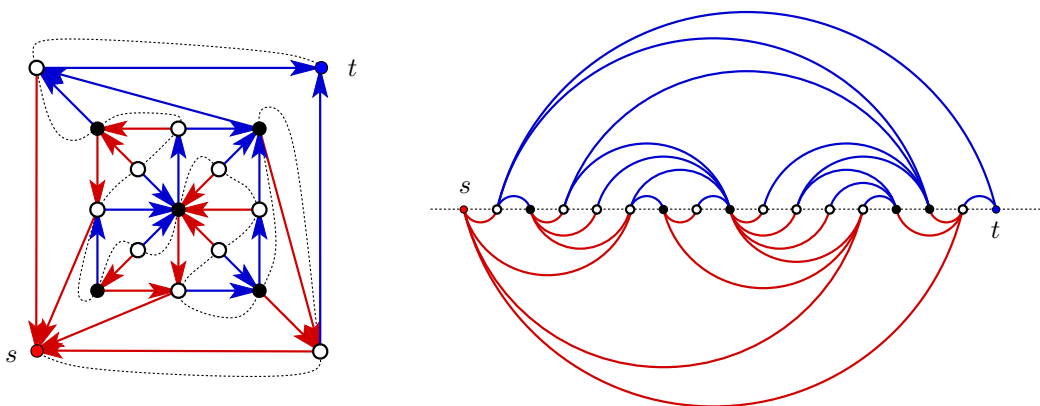


239 ■ **Figure 6** Edge orientations and colors at white and black vertices.

240 Separating decompositions of a quadrangulation Q have been defined by de Fraysseix
 241 and Ossona de Mendez [17]. They show a bijection between separating decompositions and
 242 2-orientations (orientations of the edges of Q such that every vertex $v \notin \{s, t\}$ has out-degree
 243 2) and show the existence of a 2-orientation of Q with an argument related to flows and
 244 matchings. An inductive proof for the existence of separating decompositions was given by
 245 Felsner et al. [11], this proof is based on identifying pairs of opposite vertices on faces.

246 It is known that in a separating decomposition the red edges form a tree directed towards
 247 s , and the blue edges form a tree directed towards t . Each of the trees connects all the
 248 vertices $v \notin \{s, t\}$ to the respective root. Proofs for this can be found in either of [17, 11, 10].
 249 The latter two of these sources continue to show that the edges of the two trees can be
 250 separated by a curve which starts in s , ends in t , and traverses every vertex and every inner
 251 face of Q . This curve is called the *equatorial line*.

252 If Q is redrawn such that the equatorial line is mapped to the x -axis with s being the
 253 left end and t being the right end of the line, then the red tree and the blue tree become
 254 *alternating trees* (defined below) drawn in the upper respectively lower half-plane defined by
 255 the x -axis. Note that such a drawing of Q is a 2-book embedding, we call it an *alternating*
 256 *2-book embedding* to emphasize that the graphs drawn on the two pages of the book are
 257 alternating trees.



258 ■ **Figure 7** A quadrangulation Q with a separating decomposition S , and the alternating 2-book
 259 embedding induced by the equatorial line of S .

260 An *alternating tree* is a plane tree T with a plane drawing such that the vertices of T are
 261 placed at different points of the x -axis and all edges are embedded in the half-plane above

262 the x -axis (or all below). Moreover, for every vertex v it holds that all its neighbors are on
 263 one side, either they are all left of v or all right of v . In these cases we call the vertex v
 264 respectively a *right* or a *left vertex* of the alternating layout.

265 Let Q be a plane quadrangulation on n vertices and let S be a separating decomposition
 266 of Q . Let $s = v_1, v_2, \dots, v_n = t$ be the spine of the alternating 2-book embedding of Q based
 267 on S . Let Q^+ be obtained from Q by adding $v_n v_1$ and all the edges $v_i v_{i+1}$ which do not yet
 268 belong to the edge set of Q . By construction v_1, v_2, \dots, v_n is a Hamiltonian cycle of Q^+ and
 269 due to the alternating property of the embedding this Hamiltonian cycle is one-sided with
 270 reverse edge $v_n v_1 = ts$. Hence Q is POSH.

271 It is worth noting that the Hamiltonian cycle read in the reverse direction, i.e., as
 272 v_n, v_{n-1}, \dots, v_1 , is again one-sided, now the reverse edge is $v_1 v_n = st$.

273 **B Plane cubic graphs: Full version**

274 In this section we identify another large subclass of the class \mathcal{H} and more precisely of the
 275 subclass \mathcal{H}' of graphs which are POSH.

276 ► **Theorem 4.1.** *Every plane cubic graph G is a spanning subgraph of a plane graph G' on*
 277 *the same vertex set V which has a one-sided Hamiltonian cycle, i.e., G is POSH.*

278 We will use Theorem 3.1, therefore, we need to associate a plane quadrangulation with
 279 G . To this end, we find a matching whose contraction results in a bipartite graph.

280 ► **Lemma 2.1.** *Let G be a cubic graph. Then G admits a matching M such that contracting*
 281 *all the edges of M results in a bipartite multi-graph.*

282 **Proof.** Pick a partition $X \cup Y = V(G)$ of the vertex-set of G such that the size of the cut,
 283 i.e., the number of edges in G with exactly one endpoint in X and exactly one endpoint in
 284 Y , is maximized. We claim that $G[X]$ and $G[Y]$ are graphs of maximum degree at most
 285 1 (and thus, a disjoint union of isolated vertices and a matching). Suppose that a vertex
 286 $v \in X$ has at least two neighbors in $G[X]$. Then v has at most one neighbor in Y , and hence
 287 moving v from X to Y increases the size of the cut by at least one, a contradiction. The
 288 same argument works to show that $G[Y]$ has maximum degree at most one.

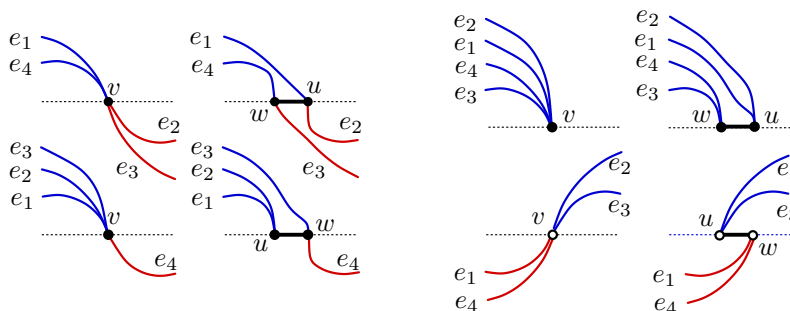
289 Let M be the matching in G consisting of all the edges in $G[X]$ and $G[Y]$. Contracting
 290 all the edges in M transforms $G[X]$ and $G[Y]$ into independent sets, and hence results in a
 291 bipartite multi-graph G/M . ◀

292 Given a cubic plane graph G we find a matching M as in Lemma 4.2 and let B be the
 293 plane bipartite multi-graph obtained from G by contracting the edges in M . Let B' be the
 294 underlying simple graph of B and let Q be a quadrangulation which has B' as a spanning
 295 subgraph. The proof of Theorem 3.1 shows that there is a left to right placement v_1, \dots, v_n
 296 of the vertices of Q on the x -axis such that for each $i \in [n]$ all the edges $v_j v_i$ with $j < i - 1$
 297 are in one half-plane and all edges $v_i v_j$ with $j > i + 1$ are in the other half-plane. Delete all
 298 the edges from Q which do not belong to B' , and duplicate the double-edges of B in the
 299 drawing. This yields an embedding Γ of B .

300 Vertices of degree four in B are exactly the vertices which have been obtained by
 301 contracting edges of M . We now show how to undo the contractions, i.e., *split* vertices of
 302 degree four, in the drawing Γ in such a way that at the end we arrive at a one-sided 2-book
 303 drawing Γ^* of G , that is, a 2-book embedding of G with vertex-sequence v_1, \dots, v_n such that
 304 for every $j \in \{1, \dots, n\}$ the incident back-edges $v_i v_j$ with $1 \leq i < j$ are all drawn either on

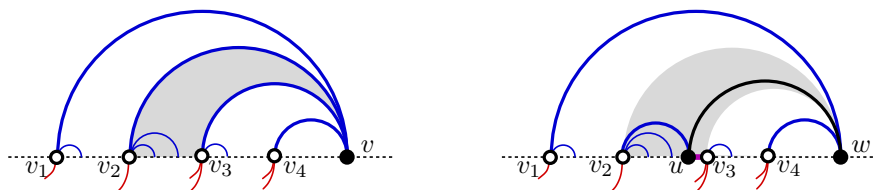
305 the spine or on the same page of the book embedding (all above or all below the spine).
 306 Once we have obtained such a book embedding, we can add the spine edges to G to obtain a
 307 supergraph G^+ of G which has a one-sided Hamiltonian path. This then shows that G is
 308 POSH.

309 Let v be a vertex of degree four of B . Vertex v was obtained by contracting an edge
 310 $uw \in M$. Label the edges of v in clockwise order as e_1, e_2, e_3, e_4 such that in G the edges
 311 e_1, e_2 are incident to u and e_3, e_4 are incident to w . If the two angles $\angle e_2e_3$ and $\angle e_4e_1$
 312 together take part of both half-planes defined by the x -axis, then it is possible to select two
 313 points left and right of the point representing v in Γ and to slightly detour the edges e_i such
 314 that no crossings are introduced and one of the two points is incident to e_1, e_2 and the other
 315 to e_3, e_4 . Addition of an edge connecting the two points completes the split of v into the
 316 edge $uw \in M$. Figure 8 shows a few instances of this *local split*.



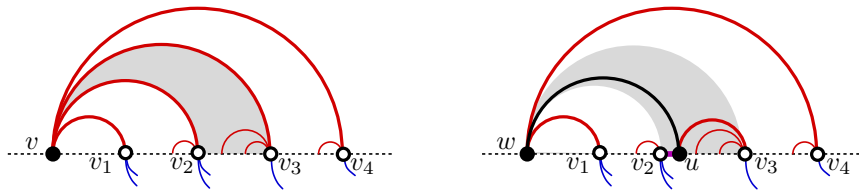
317 ■ **Figure 8** Four cases for the local split of a vertex v .

318 The above condition about the two angles is not fulfilled if all four edges of v emanate into
 319 the same halfspace and the clockwise numbering starting at the x -axis is either e_4, e_1, e_2, e_3
 320 or e_2, e_3, e_4, e_1 . The two cases are the same up to exchanging the names of u and w , therefore,
 321 we can assume the first one. Let v_1, v_2, v_3, v_4 be the neighbors of v in left to right order. A
 322 more important distinction is whether all v_i are left of v or to the right of v . In the first case
 323 we leave w at the former position of v and u slightly left of v_3 while in the second case u is
 324 slightly right of v_2 . Figure 9 shows the first case and Figure 10 the second. To see that in
 325 the first case edges uv_2 and uw are free of crossings we can route them close to the path
 326 $v_3v_2v_1$ and the edge v_3v in the original drawing. This kind of split is a *far split*.



327 ■ **Figure 9** Far split within the gray region of vertex v with edges to the left in the upper half-plane.

330 A special case occurs when a vertex v of degree four has a double-edge. This either leads
 331 to a local split (a few more cases for the complete listing of all possible local splits) or the
 332 double-edge and two simple edges are in the same halfplane such that v_1 and v_4 are the
 333 vertices incident to the simple edges, whence, $v_2 = v_3$. We stick to the rules of the far split.
 334 If v has edges to the left we place u to the left of v_3 and connect to v_2 and v_3 this makes
 335 a short double-edge on the spine. The case where edges from v go to the right is again
 336 symmetric.



328 **Figure 10** Far split within the gray region of vertex v with edges to the right in the upper
 329 half-plane.

337 By now we have shown that starting from the drawing Γ of B we can split a single vertex
 338 v of degree four into an edge $uw \in M$. To split all the degree four vertices we proceed as
 339 follows. First we split all vertices which are subject to a far split. Let U be the set of vertices
 340 of edges $uw \in M$ which are far, i.e., w stays at the position of the contraction vertex v and
 341 u is the vertex close to either v_2 or v_3 .

342 We claim that every vertex has at most one neighbor in U to its left and one to its right.
 343 Indeed if a vertex v' has a neighbor of U to its right (see Figure 10), then v' was the v_2
 344 of some far split and v was the leftmost neighbor of v' . The claim now follows from the
 345 observation that v' only has one leftmost neighbor.

346 It can happen that a vertex v' which was classified as being subject of a local split loses
 347 this property and has to undergo a far split. For example vertex v_2 in Figure 10 has the
 348 neighbor v on the left but may have three neighbors to the right, the split of v generates
 349 a fourth neighbor on the right of v_2 . When no further far split is possible we do all the
 350 local splits. These local splits can not conflict with each other. This completes the proof of
 351 Theorem 4.1.

352 C 2-Trees

353 From the positive results in sections 3 and 4 it could be expected that essentially every
 354 “sufficiently sparse” planar graph is POSH. In this section we show that this is not true.

355 A *2-tree* is a graph which can be obtained, starting from a K_2 , by repeatedly selecting an
 356 edge of the current graph and adding a new vertex which is made adjacent to the endpoints
 357 of that edge. We refer to this operation as *stacking* a vertex over an edge.

358 From the recursive construction it follows that a 2-tree on n vertices is a planar graph
 359 with $2n - 3$ edges. We also mention that 2-trees are series-parallel planar graphs. Another
 360 well studied class which contains 2-trees as a subclass is the class of (planar) Laman graphs.

361 Fulek and Tóth have shown that planar 3-trees admit n -universal point sets of size
 362 $O(n^{3/2} \log n)$. Since every 2-tree is an induced subgraph of a planar 3-tree the bound carries
 363 over to this class.

364 **► Proposition 3.1.** There is a 2-tree G on 499 vertices which is not POSH.

365 Throughout the following discussion we assume that a 2-tree G is given together with a
 366 left to right placement v_1, \dots, v_n of the vertices on the x -axis such that adding the spine
 367 edges and the reverse edge $v_n v_1$ to G we obtain a plane graph with a one-sided Hamiltonian
 368 cycle.

369 For an edge e of G we let $X(e)$ be the set of vertices which are stacked over e and $S(e)$ the
 370 set of edges which have been created by stacking over e , i.e., each edge in $S(e)$ has one vertex
 371 on e and one vertex in $X(e)$. We partition the set $X(e)$ of an edge $e = v_i v_j$ with $i < j$ into a

372 left part $XL(e) = \{v_k \in X(e) : k < i\}$, a middle part $XM(e) = \{v_k \in X(e) : i < k < j\}$, and
 373 a right part: $XR(e) = \{v_k \in X(e) : j < k\}$.

374 ► **Claim 1.** For every edge $|XR(e)| \leq 2$.

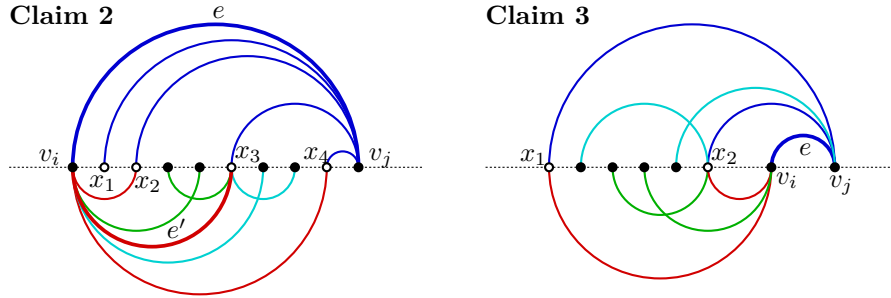
375 Suppose that $|XR(e)| \geq 3$. Each vertex in this set has all its back-edges on the same side.
 376 Two of them use the same side for the back edges to the vertices of e . This implies a crossing
 377 pair of edges, a contradiction.

378 ► **Claim 2.** If for all $e' \in S(e)$ we have $|X(e')| \geq 3$, then $|XM(e)| \leq 3$.

379 Suppose that $e = v_i v_j$ with $i < j$ is in the upper half-plane and there are four vertices
 380 x_1, x_2, x_3, x_4 in $XM(e)$. One-sidedness implies that the four edges $x_k v_j$ are in the upper
 381 half-plane. Now if x_1, x_2, x_3, x_4 is the left to right order, then the edges $v_i x_2, v_i x_3$, and $v_i x_4$
 382 have to be in the lower half-plane. Now let $e' = v_i x_3$ and consider the three vertices in $X(e')$.
 383 Two of them, say y_1, y_2 , are on the same side of x_3 .

384 First suppose $y_1, y_2 \in X(e')$ are left of x_3 . Note that the edges of $v_i x_2$ and $x_2 v_j$ enforce
 385 that y_1, y_2 are between x_2 and x_3 . Due to the edge $x_2 v_j$ the edges $v_i y_1$ and $v_i y_2$ are in
 386 the lower half-plane. One-sidedness at x_3 requires that $y_1 x_3$ and $y_2 x_3$ are also in the lower
 387 half-plane. This makes a crossing unavoidable.

388 Now suppose that $y_1, y_2 \in X(e')$ are right of x_3 . The edges $v_i x_4$ and $x_4 v_j$ enforce that
 389 y_1, y_2 are between x_3 and x_4 . Due to the edge $x_3 v_j$ the edges $v_i y_1$ and $v_i y_2$ are in the lower
 390 half-plane. Now let y_1 be left of y_2 . One-sidedness at y_2 requires that $x_3 y_2$ is also in the
 391 lower half-plane, whence, there is a crossing between $v_i y_1$ and $x_3 y_2$. This completes the proof
 392 of the claim.



393 ■ **Figure 11** Illustrating the proofs of the claims.

394 ► **Claim 3.** If $XL(e) \geq 2$ and x is the rightmost element of $XL(e)$, then $XL(e') \leq 2$ for some
 395 $e' \in S(e)$ incident with x .

396 Suppose that $e = v_i v_j$ with $i < j$ is in the upper half-plane and there are two vertices
 397 x_1, x_2 in $XL(e)$. We assume that x_2 is the rightmost element of $XL(e)$. From one-sidedness
 398 at v_j we know that $x_1 v_j$ and $x_2 v_j$ are in the upper half-plane. Now $x_1 v_i$ and hence also
 399 $x_2 v_i$ are in the lower half-plane. All the vertices of $X(x_2 v_i)$ and $X(x_2 v_j)$ are in the region
 400 bounded by $x_2 v_j, v_j v_i, v_i x_2$. Suppose that we have $y_1, y_2 \in XL(x_2 v_i)$ and $z_1, z_2 \in XL(x_2 v_j)$.
 401 By one-sidedness the edges from x_2 to the four vertices y_1, y_2, z_1, z_2 are in the same half-plane.
 402 If they are in the lower half-plane and y_1 is left of y_2 there is a crossing between $y_1 x_2$ and
 403 $y_2 v_i$. If they are in the upper half-plane and z_1 is left of z_2 there is a crossing between $z_1 x_2$
 404 and $z_2 v_j$. The contradiction shows that $XL(x_2 v_i) \leq 2$ or $XL(x_2 v_j) \leq 2$, since $x = x_2$ this
 405 completes the proof of the claim.

406 We are ready to define the graph G and then use the claims to verify that G is not a
 407 POSH. The graph G contains a *base edge* e , and there are seven vertices stacked on e , i.e.,
 408 $|X(e)| = 7$. Edges in $S(e)$ are the *primary* edges of G . For each primary edge e' there are
 409 five vertices stacked on e' . Edges introduced by stacking over a primary edge are *secondary*
 410 edges. Finally, for each secondary edge e'' there are three vertices stacked on e'' . Note that
 411 there are $7 \cdot 2 = 14$ primary edges, $14 \cdot 5 \cdot 2 = 140$ secondary edges and $140 \cdot 3 \cdot 2 = 840$ edges
 412 introduced by stacking on a secondary edge. In total the number of edges is $995 = 2n - 3$,
 413 hence, the graph has 499 vertices.

414 Now suppose that G is POSH and let v_1, \dots, v_n be the order of vertices on the spine of
 415 a certifying 2-book embedding. Let $v_i v_j$ with $i < j$ be the vertices of the base edge e . By
 416 symmetry we may assume that e is in the upper half-plane. From Claim 1 we get $|XR(e)| \leq 2$
 417 and from Claim 2 we get $|XM(e)| \leq 3$, it follows that $|XL(e)| \geq 2$. Let x_1 and x_2 be elements
 418 of $XL(e)$ such that x_2 is the rightmost element of $XL(e)$. Let $e' = x_2 v_i$ and $e'' = x_2 v_j$ and
 419 observe that $XR(e') = \emptyset = XR(e'')$ (the argument can be found in the proof of Claim 3).
 420 From Claim 2 applied to e' and e'' we deduce that $|XM(e')| \leq 3$ and $|XM(e'')| \leq 3$. Hence
 421 $|XL(e')| \geq 2$ and $|XL(e'')| \geq 2$. This is in contradiction with Claim 3. We conclude that
 422 there is no spine ordering for G which leads to a one-sided crossing free 2-book embedding.