Straight Line Triangle Representations

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Abstract

A straight line triangle representation (SLTR) of a planar graph is a straight line drawing such that all the faces including the outer face have triangular shape. Such a drawing can be viewed as a tiling of a triangle using triangles with the input graph as skeletal structure. In this paper we present a characterization of graphs that have an SLTR. The charcterization is based on flat angle assignments, i.e., selections of angles of the graph that have size π in the representation. We also provide a second characterization in terms of contact systems of pseudosegments. With the aid of discrete harmonic functions we show that contact systems of pseudosegments that respect certain conditions are stretchable. The stretching procedure is then used to get straight line triangle representations. Since the discrete harmonic function approach is quite flexible it allows further applications, we mention some of them.

The drawback of the characterization of SLTRs is that we are not able to effectively check whether a given graph admits a flat angle assignment that fulfills the conditions. Hence it is still open to decide whether the recognition of graphs that admit straight line triangle representation is polynomially tractable.

1 Introduction

In this paper we study a representation of planar graphs in the classical setting, i.e., vertices are represented by points in the Euclidean plane and edges by non-crossing continuous curves connecting the points. We aim at classifying the class of planar graphs that admit a straight line representation in which all faces are triangles. Haas et al. present a necessary and sufficient condition for a graph to be a pseudo-triangulation [12], however this condition is not sufficient for a graph to have a straight line triangle representation (e.g. see Fig. 2 and [1]). There have been investigations of the problem in the dual setting, i.e., in the setting of side contact representations of planar graphs with triangles. Gansner, Hu and Kobourov show that outerplanar graphs, grid graphs and hexagonal grid graphs are Touching Triangle Graphs (TTGs). They give a linear time algorithm to find the TTG [10]. Alam, Fowler and Kobourov [2] consider proper TTGs, i.e., the union of all triangles of the TTG is a triangle and there are no holes. They give a necessary and a stronger sufficient condition for biconnected outerplanar graphs to be TTG, a characterization, however, is missing. Fowler has given a necessary and sufficient condition for a special type of outerplanar graphs to be TTG [9]. Kobourov, Mondal and Nishat present construction algorithms for proper TTGs of 3-connected cubic graphs and some grid graphs. They also present a decision algorithm for testing whether a 3-connected planar graph is proper TTG [14]. Gonçalves, Lévêque and Pinlou consider a primal-dual contact representation by triangles, i.e., both the faces as well as the vertices are represented by triangles. They show that all 3-connected planar graphs admit such a representation [11].

Here is the formal introduction of the main character for this paper.

Definition 1.1 (Straight Line Triangle Representation)**.** A plane drawing of a graph such that

- all the edges are straight line segments and

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- all the faces, including the outer face, bound a non-degenerate triangle

is called a *Straight Line Triangle Representation* (SLTR).

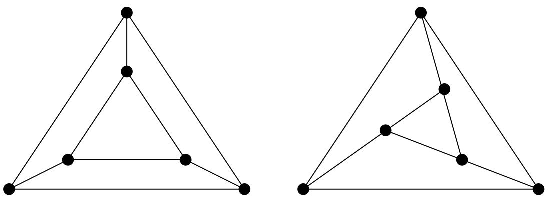

Figure 1: A graph and one of its SLTRs

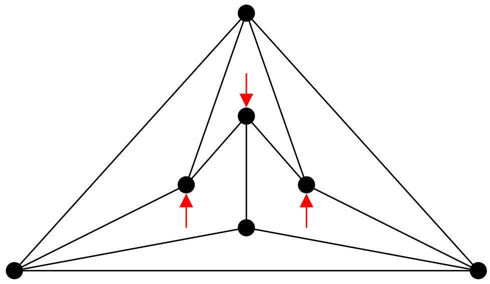

Figure 2: A Flat Angle Assignment arrows) that has no corresponding SLTR.

Clearly every straight line drawing of a triangulation is an SLTR. So the class of planar graphs admitting an SLTR is rich. On the other hand, graphs admitting an SLTR cannot have a cut vertex. Indeed, as shown below (Prop. 1.2), graphs admitting an SLTR are well connected. Being well connected, however, is not sufficient as shown e.g. by the cube graph.

To simplify the discussion we assume that the input graph is given with a plane embedding and a selection of three vertices of the outer face that are designated as corner vertices for the outer face. These three vertices are called *suspension vertices*. If needed, an algorithm may try all triples of vertices as suspensions.

Every degree two vertex that is not a suspension is flat in every SLTR, i.e., it has angles of size π in both incident faces. Such a vertex and its two incident edges can be replaced by a single edge connecting the two neighbors of the vertex. Such an operation is called a *vertex reduction*. We use vertex reductions to eliminate all the degree two vertices that are not suspensions.

A plane graph G with suspensions s_1, s_2, s_3 is said to be *internally 3-connected* when the addition of a new vertex v_∞ in the outer face, that is made adjacent to the three suspension vertices, yields a 3-connected graph.

Proposition 1.2. If a graph G admits an SLTR with s_1, s_2, s_3 as corners of the outer triangle and no vertex reduction is possible, then G is internally 3-connected.

Proof. Consider an SLTR of G. Suppose there is a separating set U of size 2. It is enough to show that each component of $G \setminus U$ contains a suspension vertex, so that $G + v_\infty$ is not disconnected by U. Since G admits no vertex reduction every degree two vertex is a suspension. Hence, if C is a component and $C \cup U$ induces a path, then there is a suspension in C. Otherwise consider the convex hull of $C \cup U$ in the SLTR. The convex corners of this hull are vertices that expose an angle of size at least π. Two of these large angles may be at vertices of U but there is at least one additional large angle. This large angle must be the outer angle at a vertex that is an outer corner of the SLTR, i.e., a suspension. □

From Prop. 1.2 it follows that any graph that is not internally 3-connected but does admit an SLTR, is a subdivision of an internally 3-connected graph. Therefore we may assume that the graphs we consider are internally 3-connected.

In Section 2 we present necessary conditions for the existence of an SLTR in terms of what we call a flat angle assignment. A flat angle assignment that fulfills the conditions is shown to induce a partition of the set of edges into a set of pseudosegments. Finally, with the aid of discrete harmonic functions we show that in our case the set of pseudosegments is stretchable. Hence, the necessary conditions are also sufficient. The drawback of the characterization is that we are not aware of an effective way of checking whether a given graph admits a flat angle assignment that fulfills the conditions.

In Section 3 we consider further applications of the stretching approach. First we look at flat angle assignments that yield faces with more than three corners. Then we proceed to prove a more general result about stretchable systems of pseudosegments with our technique. The result is not new, de Fraysseix and Ossona de Mendez have investigated stretchability conditions for systems of pseudosegments. The counterpart to Theorem 3.2 can be found in [3, Theorem 38]. The proof there is based on a long and complicated inductive construction.

2 Necessary and Sufficient Conditions

Consider a plane, internally 3-connected graph $G = (V, E)$ with suspensions given. Suppose that G admits an SLTR. This representation induces a set of *flat angles*, i.e., incident pairs (v, f) such that vertex v has an angle of size π in the face f.

Since G is internally 3-connected every vertex has at most one flat angle. Therefore, the flat angles can be viewed as a partial mapping of vertices to faces. Since the outer angle of suspension vertices exceeds π, suspensions have no flat angle. Since each face f (including the outer face) is a triangle, each face has precisely three angles that are not flat. In other words every face f has $|f| - 3$ incident vertices that are assigned to f. This motivates the definition:

Definition 2.1 (FA Assignment)**.** A *flat angle assignment* (FAA) is a mapping from a subset U of the non-suspension vertices to faces such that

(C_v) Every vertex of U is assigned to at most one face,

(C_f) For every face f, precisely $|f| - 3$ vertices are assigned to f.

Not every FAA induces an SLTR. An example is given in Fig. 2. Hence, we have to identify another condition. To state this we need a definition. Let H be a connected subgraph of the plane graph G. The *outline cycle* $\gamma(H)$ *of* H is the closed walk corresponding to the outer face of H. An *outline cycle* of G is a closed walk that can be obtained as outer cycle of some connected subgraph of G. Outline cycles may have repeated edges and vertices, see Fig. 3. The interior $\mathsf{int}(\gamma)$ of an outline cycle $\gamma = \gamma(H)$ consists of H together with all vertices, edges and faces of G that are contained in the area enclosed by γ.

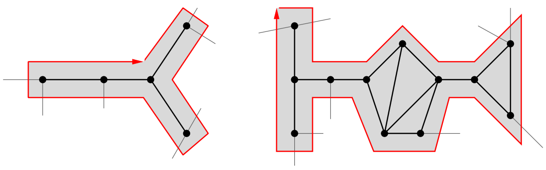

Figure 3: Examples of outline cycles

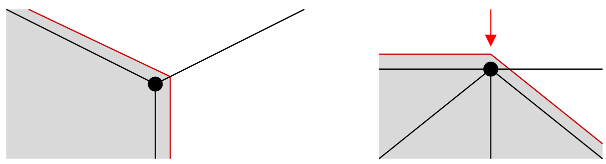

Figure 4: Combinatorially Convex Corners

Proposition 2.2. An SLTR obeys the following condition C_o:

(C_o) Every outline cycle that is not the outline cycle of a path, has at least three geometrically convex corners.

Proof. Consider an SLTR. Suppose that there is a connected subgraph, not a path, such that its outline cycle has less than three geometric convex corners. If the outline cycle has at most two geometric convex corners, then the subgraph is mapped to a line in the plane. The subgraph must either contain a vertex of degree more than three, or a face. If a vertex v together with three its neighbors is mapped onto a line, then the boundary of at least one of the faces incident to v is not a triangle. On the other hand if the subgraph contains a face, then this face is mapped to a line and therefore its boundary is not a triangle. In both cases the properties of an SLTR are violated. This shows that C_o is a necessary condition. □

Condition C_o has the disadvantage that it depends on a given SLTR, hence, it is useless for deciding whether a planar graph G admits an SLTR. The following definition allows to replace C_o by a combinatorial condition on an FAA.

Definition 2.3. Given an FAA ψ. A vertex v of an outline cycle γ is a *combinatorial convex corner* for γ with respect to ψ if

(K1) v is a suspension vertex, or

(K2) v is not assigned and there is an edge e incident to v with $e \notin \mathsf{int}(\gamma)$, or

(K3) v is assigned to a face f, $f \notin \mathsf{int}(\gamma)$ and there exists an edge e incident to v with $e \notin \mathsf{int}(\gamma)$.

In Fig. 4 an unassigned and an assigned combinatorially convex corner are shown. The grey area represents the interior of some outline cycle and the arrow represents the assignment of the vertex to the face in which the arrow is drawn.

Proposition 2.4. Let G admit an SLTR Γ, that induces the FAA ψ and let H be a connected subgraph of G. If v is a geometrically convex corner of the outline cycle $\gamma(H)$ in Γ, then v is a combinatorially convex corner of $\gamma(H)$ with respect to ψ.

Proof. If v is a suspension vertex it is clearly geometrically and combinatorially convex.

Let v be geometrically convex and suppose that v is not a suspension and not assigned by ψ. In this case v is interior and, with respect to γ, the outer angle at v exceeds π. Therefore at least two incident faces of v are in the outside of γ. These faces can be chosen to be adjacent, hence, the edge between them is an edge e with $e \notin \mathsf{int}(\gamma)$. This shows that v is combinatorially convex.

Let v be geometrically convex and suppose that v is assigned to f by ψ. If $f \in \mathsf{int}(\gamma)$, then the inner angle of v with respect to γ is at least π. This contradicts the fact that v is geometrically convex. Hence $f \notin \mathsf{int}(\gamma)$. If there is no edge e incident to v such that $e \notin \mathsf{int}(\gamma)$, then v has an angle of size π with respect to γ. This again contradicts the fact that v is geometrically convex. Therefore, if v is geometrically convex and assigned to f, then $f \notin \mathsf{int}(\gamma)$ and there exists an edge e incident to v such that $e \notin \mathsf{int}(\gamma)$. This shows that v is a combinatorial convex corner for γ. □

The proposition enables us to replace the condition on geometrically convex corners w.r.t. an SLTR by a condition on combinatorially convex corners w.r.t. an FAA.

(C_o^*) Every outline cycle that is not the outline cycle of a path, has at least three combinatorially convex corners.

From Prop. 2.2 and Prop. 2.4 it follows that this condition is necessary for an FAA that induces an SLTR. In Thm. 2.10 we prove that if an FAA obeys C_o^* then it induces an SLTR. The proof is constructive. In anticipation of this result we say that an FAA obeying C_o^* is a *good flat angle assignment* and abbreviate it as a *GFAA*.

Next we show that a GFAA induces a contact family of pseudosegments. This family of pseudosegments is later shown to be stretchable, i.e., it is shown to be homeomorphic to a contact system of straight line segments.

Definition 2.5. A *contact family of pseudosegments* is a family $\{c_i\}_i$ of simple curves $c_i : [0,1] \to \mathbb{R}^2$, with different endpoints, i.e., $c_i(0) \neq c_i(1)$, such that any two curves c_j and c_k ($j \neq k$) have at most one point in common. If so, then this point is an endpoint of (at least) one of them.

A GFAA ψ on a graph G gives rise to a relation ρ on the edges: Two edges, incident to a common vertex v and a common face f are in relation ρ if and only if v is assigned to f. The transitive closure of ρ is an equivalence relation on the edges of G.

Proposition 2.6. The equivalence classes of edges of G defined by ρ form a contact family of pseudosegments.

Proof. Let the equivalence classes of ρ be called arcs.

Condition C_v ensures that every vertex is interior to at most one arc. Hence, the arcs are simple curves and no two arcs cross.

An arc closing to a cycle yields an outline cycle that has no combinatorially convex corner. If an arc touches itself, then by C_v it ends on itself. The outline cycle of this equivalence class has at most one combinatorially convex corner. Both cases contradict C_o^*.

If two arcs share two points, the outline cycle of the union has at most two combinatorially convex corners. This again contradicts C_o^*.

We conclude that the family of arcs satisfies the properties of a contact family of pseudosegments. □

Definition 2.7. Let Σ be a family of pseudosegments and let S be a subset of Σ. A point p of a pseudosegment from S is a *free point* for S if

(F1) p is an endpoint of a pseudosegment in S, and

(F2) p is not interior to a pseudosegment in S, and

(F3) p is incident to the unbounded region of S, and

(F4) p is incident to the unbounded region of Σ or
p is incident to a pseudosegment that is not in S.

With Lem. 2.8 we prove that the family of pseudosegments Σ that arises from a GFAA has the following property

(C_P) Every subset S of Σ with $|S| \geq 2$ has at least three free points.

Lemma 2.8. Let ψ a GFAA on a plane, internally 3-connected graph G. For every subset S of the family of pseudosegments associated with ψ, it holds that, if $|S| \geq 2$ then S has at least 3 free points.

Proof. Let S be a subset of the contact family of pseudosegments defined by the GFAA (Prop. 2.6).

Each pseudosegment of S corresponds to a path in G. Let H be the subgraph of G obtained as union of the paths of pseudosegments in S. We assume that H is connected and leave the discussion of the cases where it is not to the reader. If H itself is not a path, then by C_o^* the outline cycle $\gamma(H)$ must have at least three combinatorially convex corners. Every combinatorially convex corner of $\gamma(H)$ is a free point of S.

If S induces a path, then the two endpoints of this path are free points for S. Moreover, there exists at least one vertex v in this path which is an endpoint for two pseudosegments and not an interior point for any. Now there must be an edge e incident to v, such that $e \notin S$, therefore v is a free point for S. □

Given an internally 3-connected, plane graph G with a GFAA. To find a corresponding SLTR we aim at representing each of the pseudosegments induced by the FAA as a straight line segment. If this can be done, every assigned vertex will be between its two neighbors that are part of the same pseudosegment. This property can be modeled by requiring that the coordinates $p_v = (x_v, y_v)$ of the vertices of G satisfy a harmonic equation at each assigned vertex.

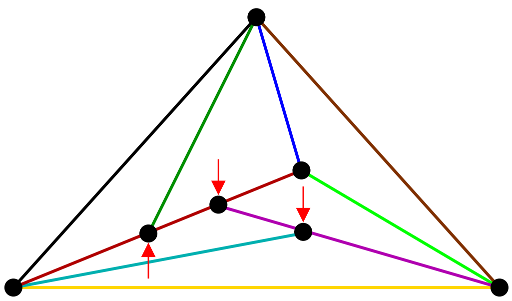

Figure 5: A stretched representation of a contact family of pseudosegments that arises from a GFAA.

Indeed if uv and vw are edges belonging to a pseudosegment s, then the coordinates satisfy

$$x_v = \lambda_v x_u + (1 - \lambda_v) x_w \qquad \text{and} \qquad y_v = \lambda_v y_u + (1 - \lambda_v) y_w \tag{1}$$

For some λ_v. In our model we can choose λ_v as a parameter from $(0, 1)$. With fixed λ_v the equations of (1) are the harmonic equations for v.

In the SLTR every unassigned vertex v is placed in the convex hull of its neighbors. In terms of coordinates this means that there are $\lambda_{vu} > 0$ with $\sum_{u \in N(v)} \lambda_{vu} = 1$ such that

$$x_v = \sum_{u \in N(v)} \lambda_{vu} x_u, \qquad y_v = \sum_{u \in N(v)} \lambda_{vu} y_u \,. \tag{2}$$

Again for the model we can choose the $\lambda_{vu} > 0$ arbitrarily subject to $\sum_{u \in N(v)} \lambda_{vu} = 1$. With fixed parameters the equations (2) enforce that v is located in the a weighted barycenter of its neighbors. These are the harmonic equations for an unassigned vertex v.

Vertices whose coordinates are not restricted by harmonic equations are called *poles*. In our case the suspension vertices are the three poles of the harmonic functions for the x and y-coordinates. The coordinates for the suspension vertices are fixed as the corners of some non-degenerate triangle, this adds six equations to the linear system.

The theory of harmonic functions and applications to (plane) graphs are nicely explained by Lovász [15]. The following proposition is taken from Chapter 3 of [15].

Proposition 2.9. For every choice of the parameters λ_v and λ_{vu} complying with the conditions, the system has a unique solution.

Proof. The proof has three steps, first we show that every non-constant harmonic function has at least two poles. Then we show that for every map $\psi_0 : P \to \mathbb{R}^2$ from the set of poles to the plane, there is a unique extension $\psi : V \to \mathbb{R}^2$ that is harmonic in all the vertices that are not poles. Last we show that a solution of the system is an extension of a mapping of the poles.

Let f be a non-constant harmonic function on the vertices of a connected graph G and let P be the set of poles of f. Let $Q = \{v \in V : f(v) \text{ maximum}\}$ and $Q' = \{v \in Q : v \text{ has a neighbor not in } Q\}$. The set Q is not empty as f is not constant. Since G is connected also Q' is not empty and every element in Q' must be a pole. Similarly we can consider the vertices where the minimum is attained, hence there are at least two poles.

Consider $\psi_0 : P \to \mathbb{R}^2$, a map from the set of poles to the plane and suppose there are two extensions $\psi, \psi^* : V \to \mathbb{R}^2$ that satisfy the harmonic equations of all non-poles. Then the function $\omega = \psi - \psi^*$ is also harmonic in all vertices not in P. If ω is the zero function, then $\psi = \psi^*$. Otherwise, ω is a non-constant harmonic function hence has a minimum or maximum different from 0 and there is a pole. This pole can not be an element of P as ψ and ψ^* are extensions of ψ_0 hence for all poles ω is zero. But then there exists a vertex, not in P, in which ω is not harmonic, contradiction. Hence the extension is unique.

For every choice of the parameters λ_v and λ_{vu} complying with the conditions, the solution of the system extends the mapping of the poles (suspension vertices). As the functions are harmonic in each non-suspension vertex, the solution is unique for every choice of the parameters. □

Now we state our main result, it shows that the necessary conditions are also sufficient.

Theorem 2.10. Let G be an internally 3-connected, plane graph and Σ a family of pseudosegments associated to an FAA, such that each subset $S \subseteq \Sigma$ has three free points or cardinality at most one. The unique solution of the system of equations that arises from Σ is an SLTR.

Proof. The proof consists of 7 arguments, which together yield that the drawing induced from the GFAA is a non-degenerate, plane drawing. The proof has been inspired by proof for convex straight line drawings of plane graphs via spring embeddings shown to us independently by Günter Rote and Éric Fusy, both attribute key ideas to Éric Colin de Verdière.

1. *Pseudosegments become Segments.* Let $(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)$ be the set of edges of a pseudosegment defined by ψ. The harmonic conditions for the coordinates force that v_i is placed between v_{i-1} and v_{i+1} for $i = 2, .., k-1$. Hence all the vertices of the pseudosegment are placed on the segment with endpoints v_1 and v_k.

2. *Convex Outer Face.* The outer face is bounded by three pseudosegments and the suspensions are the endpoints for these three pseudosegments. The coordinates of the suspensions (the poles of the harmonic functions) have been chosen as corners of a non-degenerate triangle and the pseudosegments are straight line segments, therefore the outer face is a triangle and in particular convex.

3. *No Concave Angles.* Every vertex, not a pole, is forced either to be on the line segment between two of its neighbors (if assigned) or in a weighted barycenter of all its neighbors (otherwise). Therefore every non-pole vertex is in the convex hull of its neighbors. This implies that there are no concave angles at non-poles.

4. *No Degenerate Vertex.* A vertex is degenerate if it is placed on a line, together with at least three of its neighbors. Suppose there exists a vertex v, such that v and at least three of its neighbors are placed on a line ℓ. Let S be the connected component of pseudosegments that are aligned with ℓ, such that S contains v. The set S contains at least two pseudosegments. Therefore S must have at least three free points, v_1, v_2, v_3.

By property 4 in the definition of free points, each of the free points is incident to a segment that is not aligned with ℓ. Suppose the free points are not suspension vertices. If v_i is interior to $s_i \in S$, then s_i has an endpoint on each side of ℓ. If v_i is not assigned by the GFAA it is in the strict convex hull of its neighbors, hence, v_i is an endpoint of a segment reaching into each of the two half-planes defined by ℓ.

Now suppose v_1 and v_2 are suspension vertices. Since not all three suspension vertices lie on one line, at least one of the three free points is not a suspension. Let v_3 be such a free point. If v_3 is interior to a pseudosegment not on ℓ, then one endpoint of this pseudosegment lies outside the convex hull of the three suspensions, which is a contradiction. Hence it is not interior to any pseudosegment and at least one of its neighbors does not lie on ℓ, but then v_3 should be in a weighted barycenter of its neighbors, hence again we would find a vertex outside the convex hull of the suspension vertices. Therefore at most one of the free points is a suspension and ℓ is incident to at most one of the suspension vertices.

In any of the above cases each of v_1, v_2, v_3 has a neighbor on either side of ℓ.

Let n^+ and $n^- = -n^+$ be two normals for line ℓ and let p^+ and p^- be the two poles, that maximize the inner product with n^+ resp. n^-. Starting from the neighbors of the v_i in the positive halfplane of ℓ we intend to move to a neighbor with larger inner product with n^+ until we reach p^+. If n^+ is perpendicular to another segment this may not be possible. In this case, however, we can use a slightly perturbed vector n^+_ε to break ties and make the intended progress towards p^+ possible.

Hence v_1, v_2, v_3 have paths to p^+ in the upper halfplane of ℓ and paths to p^- in the lower halfplane. Since v_1, v_2, v_3 also have a path to v we can contract all vertices of the upper and lower halfplane of ℓ to p^+ resp. p^- and all inner vertices of these paths to v to produce a $K_{3,3}$ minor of G. This is in contradiction to the planarity of G. Therefore, there is no degenerate vertex.

5. *Preservation of Rotation System.* Let $\theta(v) = \sum_f \theta(v, f)$ denote the sum of the angles around an inner vertex. Here f is a face incident to v and $\theta(v, f)$ is the (smaller!) angle between the two edges incident to v and f in the drawing obtained by solving the harmonic system. If the incident faces are oriented consistently around v, then the angles sum up to 2π. In general there may be some folding, see Fig. 6 but we can argue that this increases the angle sum. Indeed v has three neighbors x, y, z such that every closed halfspace containing v also containes one of these three. The angular sum to get from x via y to z is at least the larger of the two angles between x and z, i.e., some $\rho \geq \pi$. The angular sum to get back from z to x is at least $2\pi - \rho$ or if it again included a visit at y at least ρ. In either case the angular sum exeeds 2π, i.e., $\theta(v) \geq \pi$ for all inner vertices v.

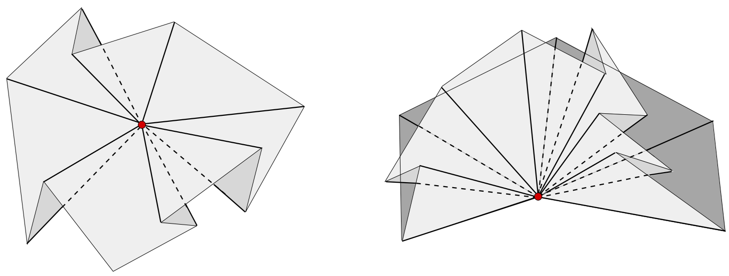

Figure 6: If the incident faces are not oriented consistently around v, then the angles sum up to more than 2π.

We do not include the outer face in the sums so that the b vertices incident to the outer face contribute a total angle of at least $(b-2)\pi$ to the inner faces.

Now consider the sum $\theta(f) = \sum_v \theta(v, f)$ of the angles of a face f. A triangulation of the face f in the planar drawing consists of $|f| - 2$ triangles. The angle sum of these triangles in the straight line is $(|f| - 2)\pi$. The angles of the triangles incident to v cover at least the smaller of the two angles formed by the two edges incident to v and f. Hence, $(|f|-2)\pi \geq \theta(f)$.

The sum over all vertices $\sum_v \theta(v)$ and the sum over all faces $\sum_f \theta(f)$ must be equal since they count the same angles in two different ways.

$$(|V|-b)2\pi + (b-2)\pi \leq \sum_v \theta(v) = \sum_f \theta(f) \leq ((2|E|-b) - 2(|F|-1))\,\pi \tag{3}$$

This yields $|V|-|E|+|F| \leq 2$. Since G is planar Euler's formula implies equality. Therefore $\theta(v) = 2\pi$ for every interior vertex v and the faces must be oriented consistently around every vertex, i.e. the rotation system is preserved. Note that the rotation system could have been flipped, between clockwise and counterclockwise but then it is flipped at every vertex.

6. *No Crossings.* Suppose two edges cross. On either side of both of the edges there is a face, therefore there must be a point p in the plane which is covered by at least two faces. Outside of the drawing there is only the unbounded face. Move along a ray, that does not pass through a vertex of the graph, from p to infinity. A change of the cover number, i.e. the number of faces by which the point is covered, can only occur when crossing an edge. But if the cover number changes then the rotation system at a vertex of that edge must be wrong. This would contradict the previous item. Therefore a crossing cannot exist.

7. *No Degeneracy.* Suppose there is an edge of length zero. Since every vertex has a path to each of the three suspensions there has to be a vertex a that is incident to an edge of length zero and an edge ab of non-zero length. Following the direction of forces we can even find such a vertex-edge pair with b contributing to the harmonic equation for the coordinates of a. We now distinguish two cases.

If a is assigned, it is on the segment between b and some b', together with the neighbor of the zero length edge this makes three neighbors of a on a line. Hence, a is a degenerate vertex. A contradiction.

If a is unassigned it is in the convex hull of its neighbors. However, starting from a and using only zero-length edges we eventually reach some vertex a' that is incident to an edge $a'b'$ of non-zero length, such that b' is contributing to the harmonic equation for the coordinates of a'. Vertex a' has the same position as a and is also in the convex hull of its neighbors. This makes a crossing of edges unavoidable. A contradiction. Hence, there are no edges of length zero.

Suppose there is an angle of size zero. Since every vertex is in the convex hull of its neighbors there are no angles of size larger than π. Moreover there are no crossings, hence the face with the angle of size zero is stretching along a line segment with two angles of size zero. Since there are no edges of length zero and all vertices are in the convex hull of their neighbors, all but two vertices of the face must be assigned to this face. Therefore, there are two pseudosegments bounding this face, which have at least two points in common, this contradicts that Σ is a family of pseudosegments. We conclude that there is no degeneracy.

From 1–7 we conclude that the drawing is plane and thus an SLTR. □

For later use we will show that it is sufficient to verify condition C_o^* for outline cycles that are simple cycles, i.e., outline cycles without cut vertices.

Lemma 2.11. Given a planar 3-connected graph G and an FAA such that every simple outline cycle has at least three combinatorially convex corners. Then every outline cycle, not the outline cycle of a path, has at least three combinatorially convex corners.

Proof. Let $\bar{\gamma}$ be an arbitrary outline cycle (not the outline cycle of a path) and let γ be the largest simple outline cycle contained in $\bar{\gamma}$.

1. γ *consists of a single vertex.* In this case $\bar{\gamma}$ is the outline cycle of a tree with ≥ 3 leaves. Each leaf of $\bar{\gamma}$ is a combinatorially convex corner for $\bar{\gamma}$.

2. γ *is a cycle of length at least three.* As $\bar{\gamma}$ is not a simple cycle, $\bar{\gamma}-\gamma$ has at least one component. Such a component can connect to at most one vertex of $\bar{\gamma}$ as otherwise γ is not the largest simple cycle in $\bar{\gamma}$. Each component in $\bar{\gamma}-\gamma$ contributes at least one combinatorially convex corner, even if the component is just a path. As γ has at least three combinatorially convex corners, it follows that $\bar{\gamma}$ has at least three combinatorially convex corners. □

3 Further Applications of the Proof Technique

We have shown that a graph G has an SLTR exactly if it admits an FAA satisfying C_v, C_f and C_o^*. Conditions C_v and C_o^* are necessary for the proof that the system of pseudosegments corresponding to the FAA is stretchable. Condition C_f, however, is only needed to make all the faces triangles. Modifying condition C_f allows for further applications of the stretching technique.

We still need that least three poles (suspensions) in convex position. Also we have to make sure that no vertex of the outer face is assigned to an inner face. And of course we still need at least three corners for every face. Together this makes the modified face condition:

(C_f^*) For every face f, at most $|f| - 3$ vertices are assigned to f and no vertex of the outer face f^o are assigned to an inner face.

If we use the empty flat angle assignment, i.e., if the harmonic equations of all non-suspensions are of type (2), then we obtain a drawing such that all non-suspension vertices are in the barycenter of their neighbors. If all vertices from the outer face are suspensions this is the Tutte drawing with asymmetric elastic forces given by the parameters λ_{uv}, see [19] and [15]. Note that in this case the existence of at least three combinatorially convex corners at an outline cycle (condition C_o^*) follows from the internally 3-connectedness of the graph.

The construction of Section 2 also applies when

- the assignment has $|f| - i$ vertices assigned to every inner face f, for $i = 4, 5$ (drawing with only convex 4-gon or only convex 5-gon faces.)
- the assignment has some number c_f of corners at inner face f (drawing with convex faces of prescribed complexity).

The drawback is that again in these cases we do not know how to find an FAA that fulfills C_o^*.

In [13] Kenyon and Sheffield study T-graphs in the context of dimer configurations (weighted perfect matchings). In our terminology T-graphs correspond to straight line representations such that each non-suspension is assigned. In [13] the straight line representations of T-graphs are obtained by analyzing random walks. Cf. [15] for further connections between discrete harmonic functions and Markov chains.

Stretchability of Systems of Pseudosegments. A contact system of pseudosegments is *stretchable* if it is homeomorphic to a contact system of straight line segments. De Fraysseix and Ossona de Mendez characterized stretchable systems of pseudosegments [3]. They use the notion of an extremal point.

Definition 3.1. Let Σ be a family of pseudosegments and let S be a subset of Σ. A point p is an *extremal point* for S if

(E1) p is an endpoint of a pseudosegment in S, and

(E2) p is not interior to a pseudosegment in S, and

(E3) p is incident to the unbounded region of S.

Theorem 3.2 (De Fraysseix & Ossona de Mendez hbox[3, Theorem 38])**.** A contact family Σ of pseudosegments is stretchable if and only if each subset $S \subseteq \Sigma$ of pseudosegments with $|S| \geq 2$, has at least 3 extremal points.

Our notion of a free point (Def. 2.7) contains the three properties of an extremal point but adds a fourth condition. In the following we show that there is no big difference. First in Prop. 3.3 we show that in the case of families of pseudosegments that live on a plane graph via an FAA, the two notions coincide. Then we continue by reproving Thm. 3.2 as a corollary of Thm. 2.10. The proof of Thm 3.2 in [3] is based on a long and complicated inductive construction.

Proposition 3.3. Let G be an internally 3-connected, plane graph and Σ a family of pseudosegments associated to an FAA, such that each subset $S \subseteq \Sigma$ has three extremal points or cardinality at most one. The unique solution of the system of equations corresponding to Σ, is an SLTR.

Proof. Note that in the proof of Thm. 2.10 the notion of free points is only used to show that there is no degenerate vertex. We show how to modify this part of the argument for the case of extremal points:

Consider again the set S of pseudosegments aligned with ℓ. We will show that all extremal points are also free points. Let p an extremal point of S. Assuming that p is not free we can negate condition 4. from Def. 2.7, i.e., all the pseudosegments for which p is an endpoint are in S. Since p is not interior to a pseudosegment in S it follows from 3-connectivity that p is incident to at least three pseudosegments, all of which lie on the line ℓ. Since all regions are bounded by three pseudosegments and p is not interior to a segment of S, all the regions incident to p must lie on ℓ. But then p is not incident to the unbounded region of S, hence p is not an extremal point. Therefore all extremal points of S are also free points of S. Prop. 3.3 now follows from Thm. 2.10. □

Proof (of Thm. 3.2). Let Σ a contact family of pseudosegments which is stretchable. Consider a set $S \subseteq \Sigma$ of cardinality at least two in the stretching, i.e., in the segment representation. Endpoints (of segments) on the boundary of the convex hull of S are extremal points. There are at least three of them unless S lies on a line ℓ. In the collinear case, there is a point q on ℓ that is the endpoint of two segments for S. This is a third extremal point.

Conversely, assume that each subset $S \subseteq \Sigma$ of pseudosegments, with $|S| \geq 2$, has at least 3 extremal points. We aim at applying Prop 3.3. To this end we construct an extended system Σ^+ of pseudosegments in which every region is bounded by precisely three pseudosegments.

First we take a set Δ of three pseudosegments that intersect like the three sides of a triangle so that Σ is in the interior. The corners of Δ are chosen as suspensions and the sides of Δ are deformed such that they contain all extremal points of the family Σ. Let the new family be Σ'.

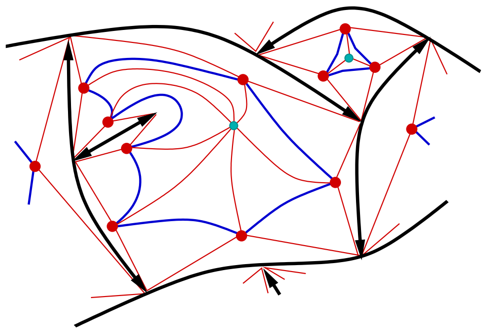

Figure 7: Protection points in red and the triangulation point in cyan for two faces of some Σ'.

Next we add *protection points*, these additional points ensure that the pseudosegments of Σ' will be mapped to straight lines. For each inner region R in Σ', for each pseudosegment s in R, we add a protection point for each visible side of s. The protection point is connected to the endpoints of s, with respect to R from the visible side of s.

Now the inner part of R is bounded by an alternating sequence of endpoints of Σ' and protection points. We connect two protection points if they share a neighbor in this sequence. Last we add a *triangulation point* in R and connect it to all protection points of R.

This construction yields a family Σ^+ of pseudosegments such that every region is bounded by precisely three pseudosegments and every subset $S \subseteq \Sigma^+$ has at least 3 extremal points, unless it has cardinality one.

Let V be the set of points of Σ^+ and E the set of edges induced by Σ^+. It follows from the construction that $G = (V, E)$ is internally 3-connected.

By Prop. 3.3 the graph $G = (V, E)$ together with Σ^+ is stretchable to an SLTR. Removing the protection points, triangulation points and their incident edges yields a contact system of straight line segments homeomorphic to Σ. □

3.1 Schnyder Woods and Primal-Dual Contact Representations

Schnyder woods were introduced in the context of order dimension [17]. In a second publication Schnyder used them for compact straight line drawings of planar graphs [18]. Schnyder woods have since found many additional applications to various graph drawing models as well as to the enumeration and encoding of planar maps. The notion of Schnyder woods was generalized to 3-connected planar graphs [5]. Gonçalves, Lévêque and Pinlou [11] used Schnyder woods of 3-connected planar graphs for the construction of primal-dual contact representations with triangles. They proof that each Schnyder wood induces a stretchable contact family of pseudosegments which represents the primal-dual contact graph. In this sections we give a simpler proof of this result using geodesic embeddings on orthogonal surfaces. The theory was again developed in the context of order dimension [16, 6, 8].

Definition 3.4 (Schnyder Wood)**.** Let G a 3-connected plane graph with three suspensions s_1, s_2, s_3 in clockwise order on the boundary of the outer face. A Schnyder wood is an orientation and labeling of the edges of G with the labels 1,2 and 3 such that the following four conditions are satisfied[1].

(S1) Each edge is either unidirected or bidirected. In the latter case the two directions have distinct labels.

(S2) At each suspension s_i there is an additional half edge with label i pointing into the outer face.

(S3) Each vertex v has outdegree one in each label. Around v in clockwise order there is an outgoing edge of label 1, zero or more incoming edges of label 3, an outgoing edge of label 2, zero or more incoming edges of label 1, an outgoing edge of label 3 and zero or more incoming edges of label 2.

(S4) There is no directed cycle in one color.

Primal-Dual Triangle Contact representation. In a triangle contact representation a graph the vertices are represented by collection of interiourly disjoint triangles and edges correspond to point-to-side contacts between the triangles. De Fraysseix, Ossona de Mendez and Rosenstiehl proved that every planar graph has a triangle contact representation [4].

A primal-dual contact representation of a plane graph by triangles is a dissection of a triangle into triangles with a correspondence between the triangles of the dissection and the union of vertices and dual vertices (faces) of the graph such that point contacts between triangles correspond to edges of the graph and its dual while side contacts correspond to incidences between vertices and faces. The enclosing triangle of the primal-dual contact representation corresponds to the outer face. Note that a triangle contact representations of a triangulation inmediately yields a primal-dual contact representation, the only detail that needs to be adjusted is that the outer face has to get triangular shape.

Gonçalves, Lévêque and Pinlou have shown that every 3-connected planar graph G has a primal-dual contact representation by triangles [11]. They use a Schnyder wood of the primal graph to define a family of pseudosegments and then use the results of [3] to show that this system is stretchable. Moreover they have shown that primal-dual contact representations are in one-to-one correspondence with Schnyder woods of planar 3-connected graphs.

We give a simpler proof of the first part. The proof is based on outline cycles and a geodesic embedding of the graph. To begin with we need some definitions.

With a point $p \in \mathbb{R}^d$ associate its *cone* $C(p) = \{q \in \mathbb{R}^d : p \leq q\}$. The *filter* $\langle \mathcal{V} \rangle$ generated by a finite set $\mathcal{V} \subset \mathbb{R}^d$ is the union of all cones $C(v)$ for $v \in \mathcal{V}$. The *orthogonal surface* $S_{\mathcal{V}}$ generated by $\mathcal{V}$ is the boundary of $\langle \mathcal{V} \rangle$. A point $p \in \mathbb{R}^d$ belongs to $S_{\mathcal{V}}$ if and only if p shares a coordinate with all $v \leq p$, $v \in \mathcal{V}$. The generating set $\mathcal{V}$ is an antichain if and only if all elements of $\mathcal{V}$ appear as minima on $S_{\mathcal{V}}$. Figure 8 shows an example of an orthogonal surface with an embedded graph. The vertices of the graph are the elements of $\mathcal{V}$. Each vertex is incident to three ridges, we call them *orthogonal arcs*. The set of all orthogonal arcs of the surface yields the partition into plane patches, we call them *flats*. An *elbow geodesic* is a connection between two two vertices u and v, it connects the two vertices with line segments

[1]The labels are considered in a cyclic structure, such that $(i-1)$ and $(i+1)$ are always well defined.

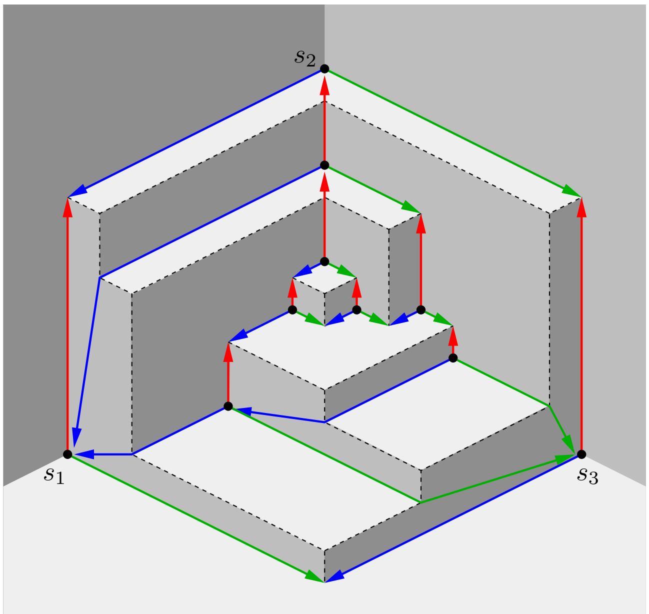

Figure 8: A geodesic embedding. The vertices of the graph are the local minima of the orthogonal surface. The edges carry the coloring and orientation of a Schnyder wood.

on the surface to a saddle-point s of $S_{\mathcal{V}}$. One or both of the line segments forming an elbow geodesic are orthogonal arcs.

Figure 8 shows a geodesic embedding, in fact the geodesic embedding is decorated with the orientation and coloring of a Schnyder wood. Miller [16] was the first to observe the connection between Schnyder woods and orthogonal surfaces in $\mathbb{R}^3$.

Definition 3.5 (Geodesic Embedding)**.** Let G a plane 3-connected graph. A drawing of G onto an orthogonal surface $\mathcal{S}_{\mathcal{V}}$ generated by an antichain $\mathcal{V}$ is a geodesic embedding if the following axioms are satisfied.

(G1) There is a bijection between the vertices of G and the points in $\mathcal{V}$.

(G2) Every edge of G is an elbow geodesic in $\mathcal{S}_{\mathcal{V}}$ and every bounded orthogonal arc in $\mathcal{S}_{\mathcal{V}}$ belongs to an edge in G.

(G3) There are no crossing edges in the embedding of G on $\mathcal{S}_{\mathcal{V}}$.

Let G be a 3-connected plane graph with suspensions a_1, a_2, a_3 and let T be a Schnyder wood of G. There is an orthogonal surface S such that G has a geodesic embedding on S that induces T. Taking the maxima of S as vertices we obtain a geodesic embedding of the dual G^* of G without the vertex v^*_∞ representing the outer face (edges of G^* connecting to v^*_∞ are unbounded rays). The geodesic embedding of G^* is naturally decorated with colors and orientations adding one suspension for the unbounded rays of each color yields a Schnyder wood T^* of the dual. The pair (T, T^*) is denoted a primal dual Schnyder wood. For more detailed background see [7] and [8].

Let a 3-connected plane graph G and a primal and dual Schnyder wood for G be given. Following the approach of Gonçalves, Lévêque and Pinlou we first construct an auxiliary graph H. The SLTR of H will be the dissection of a triangle which is the primal-dual contact representation of G. In contrast to [11] we work with a FAA on H and not with a contact family of pseudosegments.

The vertices of H are the edges of G including the half edges at the suspensions. The vertices corresponding to the half edges are the suspensions of H. The edges of H correspond to the angles of G, i.e., if e and e' are both incident to a common vertex v and a common face f, then (e, e') is an edge of H.

The faces of H are in bijection to vertices and faces (dual vertices) of G. In the context of knot theory this graph H is known as the *medial graph* of G.

The graph H inherits a plane drawing from G. The faces of H are in bijection to the vertices and faces of G. In an SLTR of H we need three corners in every face, moreover, every vertex of H (except the three suspensions) has to be the corner for three of its four incident faces. A corner assignment with these two properties is obtained form the orthogonal arcs of the surface, i.e., if s is a vertex and g is a face of H, then s is one of the three designated corners for g iff in g there is an orthogonal arc ending in s. The corner assignment is equivalent to an FAA, an angle of s is to be flat if the two edges of H forming the angle belong to the same flat of the orthoginal surface. An example is shown in Figure 9.

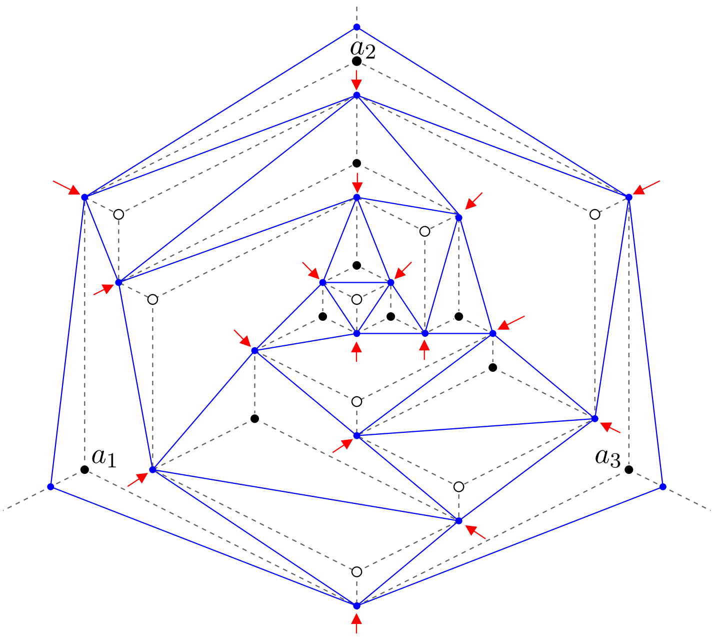

Figure 9: The graph H (in blue) is drawn on top of an orthogonal surface (in dashed grey). The flat angles of an FAA are given by the red arrows.

The family of pseudosegments corresponding to this FAA is precisely the family defined by Gonçalves, Lévêque and Pinlou. This family of pseudosegments also has has a nice description in terms of the flats. In fact there is a bijection between the pseudosegments and bounded flats. A flat F whose boundary consists of $2k$ orthogonal arcs contains k saddle-points of the surface, these are the vertices of H on F. These vertices induce a path P_F in H, every internal vertex of P_F has an angle in F and is, hence, assigned, see Figure 10. If F is a flat which is constant in coordinate i, then within P_F one of the endpoints is maximal in coordinate $i-1$ and the other is maximal in coordinate $i+1$. We call them the *left-end* and the *right-end* of P_F, respectively. In each of the three unbounded flats we have two suspensions of H as end-vertices for the path.

A flat is called *rigid* if P_F is a monotone path with respect to coordinates $i-1$ and $i+1$. The flat F shown in the left part of Figure 10 is not-rigid, the path P_F is not monotone with respect to coordinate $i+1$. An orthogonal surface is rigid if all its bounded flats are rigid. It has been shown in [6] and [8] that every Schnyder wood has a geodesic embedding on some rigid orthogonal surface. From now on we assume that the given orthogonal surface is rigid, this assumption will be critical in the proof of Proposition 3.6.

To prove that the FAA thus defined is a good FAA we use the structure of the flats. First we note that the flats are naturally partitioned into three classes, let $\mathcal{F}_i$ be the set of flats of color i, i.e, of the flats whose boundary consists of orthogonal arcs in directions $i-1$ and $i+1$.

Proposition 3.6. The flat angle assignment in H as defined above is a Good FAA.

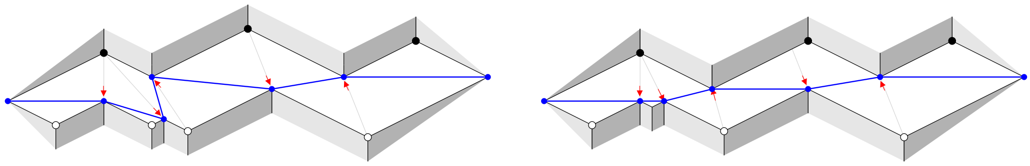

Figure 10: Two combinatorially equivalent sketches of a typical flat. The left one is non-rigid, the right one is rigid. The gray edges belong to the primal dual Schnyder wood, they prescribe the edges of H on the flat. The sequence of H edges is a pseudosegment of the FAA.

Proof. It is enough to show that every simple outline cycle has at least three combinatorially convex corners (Lemma 2.11). Let γ a simple outline cycle in H. We concider γ with its embedding into the rigid orthogonal surface.

On γ we specify some special combinatorially convex vertices, they will be called *candidates*. The candidates are not necessarily distinct but we can show that at least three of them are pairwise distinct. This is sufficient to prove the proposition.

The candidates come with a color. We now describe how to identify the candidates of color i. If γ contains the suspension of color i then by (K1) this is a combinatorially convex vertex for γ and we take it as the candidate. Otherwise, consider the flat F that has the maximal i coordinate among all flats in $\mathcal{F}_i$ that contain a vertex from γ. Let I be a path in $\gamma \cap F$. As candidates of color i we take the the endpoints of I. Of course if I consists of just one vertex we only have one candidate.

Claim. The candidates are combinatorially convex.

An primal-saddle of F is a corner between to vertices of G and a dual-saddle is a corner between two dual vertices. The vertices of H in F come in the four types left-end, right-end, primal-saddle and dual-saddle.

A primal-saddle of F has two edges in H that reach to a flat in $\mathcal{F}_i$ with i coordinate larger than F. From the choice of F_i we know that these two edges do not belong to γ. Therefore, with a primal saddle in I both neighbors in P_F also belong to γ and hence to I.

If an end z of I is a dual-saddle, then it has an edge e of P_F that does not belong to $\mathsf{int}(\gamma)$. The edge e is part of the angle at z that belongs to the face to which z is assigned, i.e., z is assigned to a face outside of γ. This shows that z is combinatorially convex by (K3).

If z is an end of P_F. Consider the flat F' that contains two H-edges incident to z. The rigidity of F' implies that $P_{F'}$ contains an edge e incident to z that reaches to a flat in $\mathcal{F}_i$ with i coordinate larger than F. Hence, edge e does not belong to γ and not to $\mathsf{int}(\gamma)$. The edge e ins part of the angle at z that belongs to the face to which z is assigned. Again z is combinatorially convex by (K3).

This concludes the proof of the claim. △

I can happen that a candidate z_i of color i and a candidate z_j of color j coincide. We have to show that in total we obtain at least three different candidates.

Let candidate z_i be a dual-saddle at a flat F_i of color i. Let G_{i-1} and G_{i+1} be the other two flats incident to z_i. The two edges of H in G_{i-1} and G_{i+1} belong to $\inf(\gamma)$ and show that G_{i-1} and G_{i+1} are not maximal in their respective colors. Hence, z_i is a candidate only in color i.

It remains to look at the left-ends and right-ends of paths P_F. Let z be the endpoint of paths P_{F_i} and P_{F_j}. We claim that the two edges in F_i and F_j incident to z belong to γ. Otherwise, consider an edge e of γ on the third flat G incident to z. This edge eather reaches a flat of color i higher than F_i in coordinate i or a flat of color j higher than F_j in coordinate j, This contradicts the maximality of either F_i or F_j. Since z is incident to edges in F_i and F_j we know that it is not the only candidate of color i and not the only candidate of color j.

This is enough to show that there are at least three pairwise different candidates. □

As every 3-connected plane graph G has a Schnyder wood, we can define the auxiliary graph H and an

FAA of H can be obtained as described. Proposition 3.6 shows that this FAA is good. We have thus reproved the theorem:

Theorem 3.7. Every 3-connected plane graph admits a primal-dual triangle contact representation.

To prove the theorem we have worked with the skeleton graph H of the primal dual triangle representation. We continue by asking which graphs H can serve as skeleton graphs for a primal-dual representation of some graph.

If a dissection of a triangle is a primal-dual triangle contact representation of some graph, then there is a 2 coloring of the triangles. Hence, the skeleton graph H is eulerian, i.e., all the vertex degrees are even. It is also evident that only degrees 4 and 2 are possible.

Definition 3.8 (Almost 4-regular)**.** A plane graph is called almost 4-regular[2] if:

- There are three vertices of degree 2 on the outer face,
- All the other vertices have degree 4.

With the following theorem we show that deciding whether an almost 4-regular plane graph has an SLTR is equivalent to deciding whether the underlying graph is 3-connected.

Theorem 3.9. An almost 4-regular plane graph H has an SLTR if and only if it is the medial graph of an interiorly 3-connected graph, or $H = C_3$.

Proof. Let $H \neq C_3$ be an almost 4-regular plane graph and let R be a SLTR of H. The three suspensions in R are the three degree two vertices. Since H is even, the dual is a bipartite graph. We abuse notation and denote the bounded faces in R that contain the suspension vertices, with *suspension of the dual.* Since they are all adjacent to the outer face of R the suspensions are in the same color class of the bipartition, say in the white class.

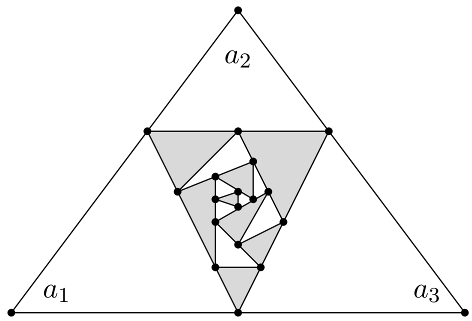

Figure 11: An SLTR.

Let $G^{\triangle}$ be the graph whose vertices correspond to the white triangles of R together with an extra vertex v_∞. The edges of $G^{\triangle}$ are the contacts between white triangles together with an edge between each of the suspensions and v_∞. The degree of v_∞ is three and each corner of a white triangle is reponsible for a contact, hence, every vertex of $G^{\triangle}$ has degree at least three.

Claim. $G^{\triangle}$ is 3-connected.

Suppose there is a separating set U of size at most 2. Let C be be a component of $G^{\triangle} \setminus U$ such that $v_\infty \notin C$. The convex hull H_C of the corners of triangles in C has at least 3 corners. Covering all the corners of H_C with only two triangles results in a corner p of H_C that has a contact to a triangle $T \in U$ such that p has an angle larger than π in the skeleton of $C + T$. Since p is a vertex of H and angles larger than π do not occur at vertices of degree 4 of an SLTR this is a contradiction. $\triangle$.

By construction H is just the medial graph of $G^{\triangle}$. $\square$

[2]Almost 4-regular graphs are Laman graphs. The number of edges is twice the number of vertices minus three and this is an upper bound for each subset of the vertices.

4 Conclusion and Open Problems

We have given necessary and sufficient conditions for a 3-connected planar graph to have an SLT Representation. Given an FAA and a set of rational parameters $\{\lambda_i\}_i$, the solution of the harmonic system can be computed in polynomial time. Checking whether a solution is degenerate can also be done in polynomial time. Hence, we can decide in polynomial time whether a given FAA corresponds to an SLTR. In other words, checking whether a given FAA is a GFAA can be done in polynomial time. However, most graphs admit different FAAs of which only some are good. We are not aware of an effective way of finding a Good FAA. Therefore we have to leave this problem open: Is the recognition of graphs that have an SLTR (GFAA) in P?

Given a 3-connected planar graph and a GFAA, interesting optimization problems arise, e.g. find the set of parameters $\{\lambda_i\}_i$ such that the smallest angle in the graph is maximized, or the set of parameters such that the length of the shortest edge is maximized.

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