News about Semiantichains and Unichain Coverings

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Abstract. We study a min-max relation conjectured by Saks and West: For any two posets P and Q the size of a maximum semiantichain and the size of a minimum unichain covering in the product $P \times Q$ are equal. For positive we state conditions on P and Q that imply the min-max relation. However, we also have an example showing that in general the min-max relation is false. This disproves the Saks-West conjecture.

1 Introduction

Partial order theory plays an important role in many disciplines of computer science and engineering. It has applications in distributed computing, concurrency theory, programming language semantics and data mining. Posets and particularly products of posets are used for modeling dynamic behavior of complex systems that can be captured by causal relations. Min-max relations in posets, such as Dilworth's Theorem [1], relate to flow problems, perfect matchings and integer programming. Often the proofs are constructive using methods of combinatorial optimization.

This paper is about min-max relations with respect to chains and antichains in posets. In a poset, *chains* and *antichains* are sets of pairwise comparable and pairwise incomparable elements, respectively. By the *height* $h(P)$ and the *width* $w(P)$ of poset P we mean the size of a largest chain and a largest antichain, respectively.

Dilworth [1] proved that any poset P can be covered with $w(P)$ many chains. Greene and Kleitman [4] generalized Dilworth's Theorem. A k-*antichain family* in P is a subset of P which may be decomposed into k disjoint antichains. We denote the size of a maximal k-antichain family of P by $d_k(P)$ or simply

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d_k if the poset is unambiguous from the context. The theorem of Greene and Kleitman says that for every k there is a chain-partition $\mathcal{C}$ of P such that $d_k(P) = \sum_{C \in \mathcal{C}} \min(k, |C|)$. In [9] (also see [13]) Saks proves the theorem of Greene and Kleitman by showing the following equivalent statement.

Theorem 1. *In a product $C \times Q$ where C is a chain, the size of a minimum chain covering with chains of the form $\{c\} \times C'$ and $C \times \{q\}$ equals the size of a maximum subset $S \subseteq C \times Q$ containing no two-element chain of the form $\{c\} \times C'$ or $C \times \{q\}$. In particular this number is $d_{\min(|C|, h(Q))}(Q)$.*

The *Saks-West Conjecture* states a generalization of Theorem 1. In a product $P \times Q$ we call a chain a *unichain* if it is of the form $\{p\} \times C'$ or $C \times \{q\}$. A *semiantichain* is a set $S \subseteq P \times Q$ such that no two distinct elements of S form a unichain. The conjecture states that the size of a largest semiantichain equals the size of a smallest unichain covering. Several partial results and special cases for posets satisfying the conjecture were obtained in the [8, 10, 11, 16, 17].

This paper is structured as follows. In Section 2 we provide a sufficient criterion for pairs of posets to satisfy the Saks-West Conjecture. This allows to reproduce several known results and to contribute new classes satisfying the conjecture. Moreover, we present a new class of posets, such that all P from this class satisfy the conjecture with any Q. For negative, in Section 3 we provide a counterexample to the Saks-West Conjecture. The example can be modified to produce an arbitrary large gap between the size of a largest semiantichain and the size of a smallest unichain covering. In Section 4 we comment on some natural dual versions of the Saks-West Conjecture raised by Trotter and West [12] and ask a few open questions related to algorithmic complexity

2 Constructions

In this section we obtain positive results for posets admitting certain chain and antichain partitions. Dually to the concept of k-antichain family we call a subset of P a k-chain family if it is the union of k disjoint chains. Similarly to $d_k(P)$ we denote the size of a maximal k-chain family of P by $c_k(P)$ or simply c_k. We will make use of the following theorem of Greene [3]:

Theorem 2. *For any poset P there exists a partition $\lambda^P = \{\lambda_1^P \geq \ldots \geq \lambda_w^P\}$ of $|P|$ such that $c_k(P) = \lambda_1^P + \ldots + \lambda_k^P$ and $d_k(P) = \mu_1^P + \ldots + \mu_k^P$ for each k, where μ^P denotes the partition conjugate to λ^P, i.e., $\mu_i^P = \max\{j \mid \lambda_j^P \geq i\}$ for $i = 1, \ldots, h(P)$.*

Following Viennot [14] we call the Ferrers diagram of λ^P the *Greene diagram* of P, denoted by $G(P)$. We say that P is *d-decomposable* if it has an antichain partition $A_1, A_2, \ldots, A_h$ with $|\bigcup_{i=1}^{k} A_i| = d_k$ for each k. This is, $|A_k| = \mu_k^P$ for all k.

For posets P and Q with families of disjoint antichains $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_\ell\}$, respectively, the set $A_1 \times B_1 \cup \ldots \cup A_{\min(k,\ell)} \times B_{\min(k,\ell)}$ is a semiantichain of $P \times Q$. A semiantichain that can be obtained this way is called *decomposable semiantichain*, see [10]. By our definitions we have the following

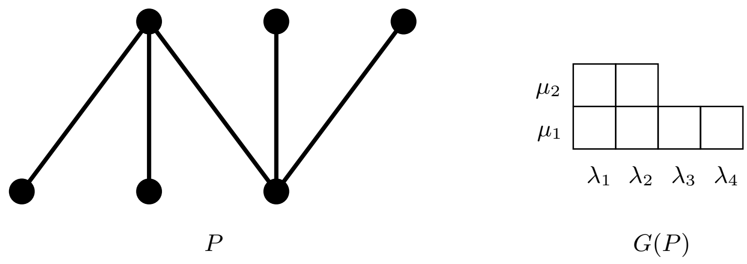

Fig. 1. A poset P with its Greene diagram $G(P)$. Note that P is not d-decomposable but c-decomposable.

Observation 3 *If P and Q are d-decomposable, then $P \times Q$ has a decomposable semiantichain of size*

$$\sum_{i=1}^{\min(h_P,h_Q)} \mu_i^P \mu_i^Q.$$

In order to construct unichain coverings for $P \times Q$ one can apply Theorem 1 repeatedly. The resulting coverings are called *quasi-decomposable* in [10]. More precisely

Proposition 1. *In a product $P \times Q$ where $\mathcal{C}$ is a chain covering of P there is a unichain covering of size*

$$\sum_{C \in \mathcal{C}} d_{\min(|C|,h(Q))}(Q).$$

Proof. Use Theorem 1 on every $C \times Q$ for $C \in \mathcal{C}$. The union of the resulting unichain coverings is a unichain covering of $P \times Q$ and is of the claimed size. $\square$

Dually to d-decomposable we call P *c-decomposable* if it has a chain partition $C_1, C_2, \ldots, C_w$ with $|\bigcup_{i=1}^k C_i| = c_k$, i.e., $|C_k| = \lambda_k^P$ for all k. Chain partitions with the latter property have been referred to as *completely saturated*, see [10, 7]. The following theorem has already been noted implicitly by Tovey and West in [10].

Theorem 4. *If P is c- and d-decomposable and Q is d-decomposable, then the size of a maximum semiantichain and the size of a minimum unichain covering in the product $P \times Q$ are equal. The size of these is obtained by the two above constructions, i.e.,*

$$\sum_{i=1}^{\min(h_P,h_Q)} \mu_i^P \mu_i^Q = \sum_{j=1}^{w(P)} d_{\min(\lambda_j^P,h(Q))}(Q).$$

Proof. Since P and Q are d-decomposable, there is a semiantichain of size $\sum_{i=1}^{\min(h_P,h_Q)} \mu_i^P \mu_i^Q$ by Observation 3. On the other hand if we take a chain covering $\mathcal{C}$ of P witnessing that P is c-decomposable we obtain a unichain covering of size $\sum_{j=1}^{w(P)} d_{\min(\lambda_j^P, h(Q))}(Q)$ by Proposition 1. We have to prove that these values coincide. Therefore consider the Greene diagrams $G(P)$ and $G(Q)$. Their *merge* $G(P,Q)$ is the set of unit-boxes at coordinates (i,j,k) with $j \leq w(P)$, $i \leq \min(\lambda_j^P, h(Q))$, and $k \leq \mu_i^Q$, see Figure 2. Note that the merge is not symmetric in P and Q. Now counting the boxes in $G(P,Q)$ by j-slices is just the right hand side of our formula. On the other hand if we count the boxes by i-slices we obtain the left hand side of the formula. This concludes the proof. $\square$

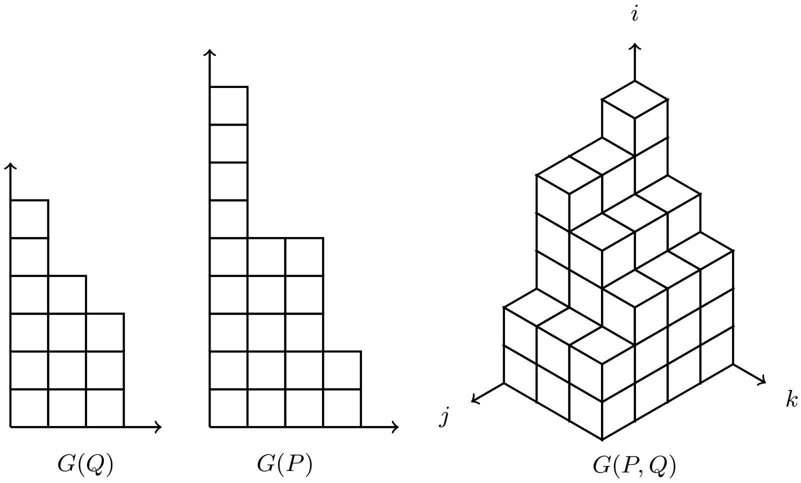

Fig. 2. The merge of two Greene diagrams.

Theorem 4 includes some interesting cases for the min-max relation that have been known but also adds a few new cases. These instances follow from proofs that certain classes of posets are d-decomposable, respectively c-decomposable.

A graded poset P whose ranks yield an antichain partition witnessing that P is d-decomposable is called *strongly Sperner*, see [6]. For emphasis we repeat

- Strongly Sperner posets are d-decomposable.

For a chain C in P denote by $r(C)$ the set of ranks used by C. A chain-partition $\mathcal{C}$ of P is called *nested* if for each $C, C' \in \mathcal{C}$ we have $r(C) \subseteq r(C')$ if $|C| \leq |C'|$. The most famous class of nested chain partitions are the symmetric chain partitions. In [6] it is proved that nested chain-partitions are completely saturated and that posets admitting a nested chain partition are strongly Sperner. Hence we have the following

- Posets that have a nested chain partition are c- and d-decomposable.

The fact that products of posets with nested chain partitions satisfy the Saks-West Conjecture was proved earlier in [16]. A special class of strongly Sperner posets are *LYM posets*. A conjecture of Griggs [5] that remains open [2, 15] and seems interesting in our context is that LYM posets are c-decomposable.

- Orders of width at most 3 are d-decomposable.

Proof. Since P has width at most 3 there are only three possible values of μ_i in the Greene diagram of P. Let a, b, c be the numbers of 3s, 2s, and 1s in $\mu_1, \ldots, \mu_k$, respectively. We have to find an antichain partition of P such that a antichains will be of size 3, b antichains will be of size 2 and c antichains will have size 1. Let $A \subseteq P$ be a maximum $(a+b)$-antichain. We have the following $|A| = 3a+2b$, $h(A) = a+b$ and $|P - A| = c$. The last gives us c antichains of size 1. To find the other antichains consider a partition $A = \bigcup_{i=1}^{h(A)} B_i$ such that B_i is the set of minimal points in $B_i \cup \ldots \cup B_{h(A)}$. Since $|B_i| \leqslant 3$ we may consider a', b', c' as the numbers of 3s, 2s, and 1s in all $|B_i|$ (for $i = 1, \ldots, h(A)$). With these numbers we have $|A| = 3a' + 2b' + c'$ and $h(A) = a' + b' + c'$. Obviously, $a' \leqslant a$ because otherwise there would be an $(a+1)$-antichain in A bigger than $3a+2$. Also, note that $|A| - 2h(A) = a = a' - c'$. Thus $c' = 0$, $a' = a$ and $b' = b$. This concludes the proof. □

- Series-parallel orders are c- and d-decomposable.

Proof. This can be proved along a composition sequence. Therefore let the antichain-partitions $\{A_1, \ldots, A_{h(P)}\}$ and $\{A'_1, \ldots, A'_{h(P')}\}$ be witnesses for d-decomposability of P and P', respectively. In a parallel composition $P \parallel P'$ any k-antichain family decomposes into a k-antichain family of P and one of P' and vice-versa. Hence $d_k(P \parallel P') = d_k(P) + d_k(P')$. Say $h(P) \leq h(P')$, then $\{A_1 \cup A'_1, \ldots, A_{h(P)} \cup A'_{h(P)}, A'_{h(P)+1}, \ldots A'_{h(P')}\}$ is an antichain partition of $P \parallel P'$ proving it to be d-decomposable. In a series composition $P; P'$ any k-antichain family decomposes into a $(k - \ell)$-antichain family of P and an ℓ-antichain family of P' and vice-versa. Ordering $A_1, \ldots, A_{h(P)}, A'_1, \ldots, A'_{h(P')}$ by decreasing size yields a witness for d-decomposability of $P; P'$. The proof for c-decomposability goes along the same lines. □

Since weak orders form a subclass of series-parallel orders we immediately get:

- Weak orders are c- and d-decomposable.

We call a poset P *rectangular* if P contains a poset L consisting of the disjoint union of w chains of length h and P is contained in a weak order U of height h with levels of size w each. Here containment is meant as an inclusion among binary relations. Clearly, rectangular posets are graded with nested chain-decomposition. Thus they are c-and d-decomposable, but more can be said.

According to our knowledge the next result is the strongest generalization of Theorem 1 that has been obtained so far.

Theorem 5. *In a product $P \times Q$ where P is rectangular of width w and height h the size of a largest semiantichain equals the size of a smallest unichain covering. Moreover, this number is $w \cdot d_{\min(h,h(Q))}(Q)$.*

Proof. P contains a poset L consisting of the disjoint union of w chains of length h and P is contained in a weak order U of height h with levels of size w. By Proposition 1 we have a unichain covering of $L \times Q$ of size $\sum_{i=1}^{w} d_{\min(h,h(Q))}(Q)$. Moreover this is an upper bound on the size of a minimum unichain covering of $P \times Q$. On the other hand in $U \times Q$ we can find a decomposable semiantichain as a product of the ranks of U with the antichain decomposition $B_1, \ldots, B_{\min(h,h(Q))}$ of a maximal $\min(h, h(Q))$-antichain family in Q. The size of this semiantichain is then $\sum_{i=1}^{\min(h,h(Q))} w|B_i| = w \cdot d_{\min(h,h(Q))}(Q)$ in $U \times Q$. This is a lower bound on the size of the largest semiantichain in $P \times Q$. This concludes the proof. □

3 A bad example

To analyze the upcoming example we need the following property of weak orders.

Proposition 2. *If P is a weak order and Q is an arbitrary poset, then the maximal size of a semiantichain in $P \times Q$ can be expressed as $\sum_{i=1}^{k} \mu_i^P \cdot |B_i|$ where $B_1, B_2, \ldots, B_k$ is a family of disjoint antichains in Q.*

Proof. Let S be a semiantichain in $P \times Q$. For any $X \subseteq P$ denote by $S(X) := \{q \in Q \mid p \in X, (p, q) \in S\}$. Recall that for any $p \in P$ the set $S(\{p\})$ (or shortly $S(p)$) is an antichain in Q. Now take a level $A_i = \{p_1, \ldots, p_k\}$ of P and let B_i be a maximum antichain among $S(p_1), \ldots, S(p_k)$. Replacing $\{p_1\} \times S(p_1), \ldots, \{p_k\} \times S(p_k)$ in S by $A_i \times B_i$ we obtain S' with $|S'| \geq |S|$. Moreover, since P is a weak order the $S(A_i)$ are mutually disjoint. This remains true in S'. Thus S' is a semiantichain. Applying this operation level by level we construct a decomposable semiantichain of the desired size. □

Let P and Q be the posets shown on in Figure 3. Since Q is a weak-order we can use Proposition 2 to determine the size of a maximum semiantichain in $P \times Q$ as $15 = 5 \cdot 2 + 5 \cdot 1 = 6 \cdot 2 + 3 \cdot 1$. We focus on two of the maximum semiantichains:

$$S_1 = \{1, 2, 3, 4, 9, 10\} \times \{b, c\} \cup \{6, 7, 8\} \times \{a\}$$

and

$$S_2 = \{1, 2, 7, 8, 9, 10\} \times \{b, c\} \cup \{3, 4, 5\} \times \{a\}$$

If there is an optimal unichain covering of size 15 then every unichain has to contain one element of each of the two semiantichains S_1 and S_2. We say that the element of a maximum semiantichain is *paying* for the unichain that contains it. Now look at the three points $(11, a)$, $(12, a)$ and $(13, a)$, they have to be covered

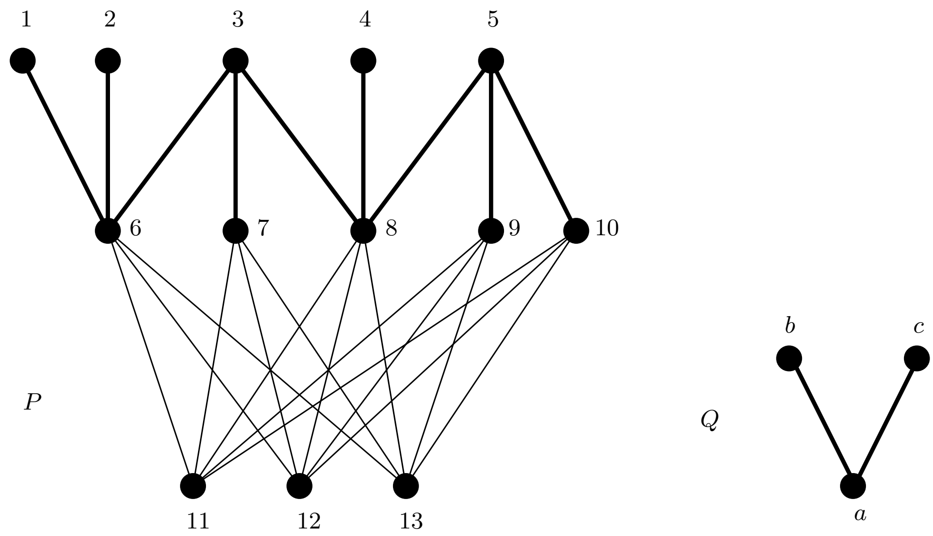

Fig. 3. A pair (P, Q) of posets disproving the conjecture.

with P-chains. To pay for these three P-chains we need the elements $(6, a), (7, a)$ and $(8, a)$ from S_1 and the elements $(3, a), (4, a)$ and $(5, a)$ from S_2. This implies that $(6, a), (7, a)$ and $(8, a)$ and $(3, a), (4, a)$ and $(5, a)$ have to be covered with the same three chains of the unichain covering. Since the poset induced by these 6 points has width 4 we have reached a contradiction. Consequently there is no unichain covering of $P \times Q$ with 15 unichains.

The above construction can be improved to get the gap between a maximum semiantichain and a minimum unichain covering as big as we want. To see that just replace Q by a height 2 weak order Q' with a set A_1 of k minima and a set A_2 of $k + 1$ maxima. Now consider P' arising from P by blowing up the antichains $\{1, 2\}$ and $\{9, 10\}$ to antichains of size $k + 1$. As in the above proof the size of the maximum semiantichains of $P' \times Q'$ can be determined to be $(k + 4)(k + 1) + (k + 4)k = (2k + 4)(k + 1) + 3k$. As above, the chains containing the $3k$ elements $\{11, 12, 13\} \times A_1$ have to cover the subposet induced by $\{3, 4, 5, 6, 7, 8\} \times A_1$. The latter is of width $4k$. Hence the size of a minimum unichain covering is of size $(2k + 4)(k + 1) + 4k$. We have:

Remark 1. The gap between the size of a maximum semiantichain and a minimum unichain covering in $P' \times Q'$ is k.

Recall that there is no gap if one factor of the product is rectangular (see Theorem 5). Here Q' is almost rectangular but the gap is large.

In [8] some partial results concerning the conjecture were obtained for the class of two-dimensional posets. Note that the two factors in our counterexample are two-dimensional.

4 Further comments

4.1 A question of Trotter and West

In [12] Leslie E. Trotter, Jr. and Douglas B. West define a *uniantichain* to be an antichain in $P \times Q$ in which one of the coordinates is fixed, and define a *semichain* to be a collection of elements of $P \times Q$ in which pairs of elements are comparable if they agree in either coordinate. In [12] it is also shown that the size of a minimum semichain covering equals the size of a largest uniantichain. They state the open problem whether the size of a minimum uniantichain covering always equals the size of a largest semichain. We remark that taking the two-dimensional conjugates of P and Q from the previous section yields a negative answer to that question. Furthermore our positive results may be "dualized" to provide classes of posets where the size of a minimum uniantichain covering always equals the size of a largest semichain.

4.2 Open problems

In the present paper the concept of d-decomposability is of some importance. Also, it seems to be a natural concept in the context of Greene-Kleitman Theory. We wonder if there is any "nice" characterization of d-decomposable posets. Let P be the six-element poset on $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ with $x_i \leq y_2$ and $x_2 \leq y_i$ for all $i = 1, 2, 3$, see also Figure 1. Is it true that any poset excluding P as an induced subposet is d-decomposable? Note that this is not a necessary condition: The Boolean lattice $\mathcal{B}_5$ is strongly Sperner. Hence $\mathcal{B}_5$ is d-decomposable but the set

$$\{\{1,2\},\{2,3\},\{1,3\},\{2,3,5\},\{1,2,3\},\{2,3,4\}\}$$

induces P.

Another set of questions arises when considering complexity issues. How hard are the optimization problems of determining the size of a largest semiantichain or smallest a unichain covering for a given $P \times Q$? What is the complexity of deciding whether $P \times Q$ satisfies the Saks-West Conjecture?

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