

# SEPARATING TREE-CHROMATIC NUMBER FROM PATH-CHROMATIC NUMBER

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ABSTRACT. We apply Ramsey theoretic tools to show that there is a family of graphs which have tree-chromatic number at most 2 while the path-chromatic number is unbounded. This resolves a problem posed by Seymour.

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## 1. INTRODUCTION

Let  $G$  be a graph. A *tree-decomposition* of  $G$  is a pair  $(T, \mathcal{B})$  where  $T$  is a tree and  $\mathcal{B} = (B_t \mid t \in V(T))$  is a family of subsets of  $V(G)$ , satisfying:

- (T1) for each  $v \in V(G)$  there exists  $t \in V(T)$  with  $v \in B_t$ ; and for every edge  $uv \in E(G)$  there exists  $t \in V(T)$  with  $u, v \in B_t$ ;
- (T2) for each  $v \in V(G)$ , if  $v \in B_t \cap B_{t'}$  for some  $t, t' \in V(T)$ , and  $t'$  lies on the path in  $T$  between  $t$  and  $t''$ , then  $v \in B_{t''}$ .

Many researchers refer to the subset  $B_t$  as a *bag* and they consider  $B_t$  as an induced subgraph of  $G$ . With this convention,  $|B_t|$  is just the number of vertices of  $G$  in the bag  $B_t$ , while  $\chi(B_t)$  is the chromatic number of the induced subgraph of  $G$  determined by the vertices in  $B_t$ .

The quality of a tree-decomposition  $(T, (B_t \mid t \in V(T)))$  is usually measured by its *width*, i.e. the maximum of  $|B_t| - 1$  over all  $t \in V(T)$ . Then the *tree-width* of  $G$  is the minimum width of a tree-decomposition of  $G$ . In this paper we study the tree-chromatic number of a graph, a concept introduced by Seymour in [6]. The *chromatic number* of a tree-decomposition  $(T, (B_t \mid t \in V(T)))$  is the maximum of  $\chi(B_t)$  over all  $t \in V(T)$ . The *tree-chromatic number* of  $G$ , denoted by  $\text{tree-}\chi(G)$ , is the minimum chromatic number of a tree-decomposition of  $G$ . A tree-decomposition  $(T, (B_t \mid t \in V(T)))$  is a *path-decomposition* when  $T$  is a path. The *path-chromatic number* of  $G$ , denoted by  $\text{path-}\chi(G)$ , is the minimum chromatic number of a path-decomposition of  $G$ . Clearly, for every graph  $G$  we have

$$\omega(G) \leq \text{tree-}\chi(G) \leq \text{path-}\chi(G) \leq \chi(G).$$

Furthermore, if  $G = K_n$  is the complete graph on  $n$  vertices, then  $\omega(G) = \chi(G) = n$ , so all these inequalities can be tight. Accordingly, it is of interest to ask whether for consecutive parameters in this series of inequalities, there is a sequence of graphs for which one parameter is bounded while the next parameter is unbounded.

In [6], Seymour proved that the classic Erdős construction [1] for graphs with large girth and large chromatic number yields a sequence  $\{G_n : n \geq 1\}$  with  $\omega(G_n) = 2$  and  $\text{tree-}\chi(G_n)$  unbounded.

For an integer  $n \geq 2$ , the *shift graph*  $S_n$  is a graph whose vertex set consists of all closed intervals of the form  $[a, b]$  where  $a, b$  are integers with  $1 \leq a < b \leq n$ . Vertices  $[a, b]$ ,  $[c, d]$  are adjacent in  $S_n$  when  $b = c$  or  $d = a$ . As is well known (and first shown in [2]),  $\chi(S_n) = \lceil \lg n \rceil$ , for every  $n \geq 2$ . On the other hand, the natural path decomposition of  $S_n$  shows that  $\text{path-}\chi(S_n) \leq 2$ , for every  $n \geq 2$ , so as noted in [6], the family of shift graphs has bounded path-chromatic number and unbounded chromatic number.

Accordingly, it remains only to determine whether there is an infinite sequence of graphs with bounded tree-chromatic number and unbounded path-chromatic number. However, these two parameters appear to be more subtle in nature. As a first step, Huynh and Kim [4] showed that there is an infinite sequence  $\{G_n : n \geq 1\}$  of graphs with  $\text{tree-}\chi(G_n) \rightarrow \infty$  and  $\text{tree-}\chi(G_n) < \text{path-}\chi(G_n)$  for all  $n \geq 1$ .

In [6], Seymour proposed the following construction. Let  $T_n$  be the complete (rooted) binary tree with  $2^n$  leaves. When  $y$  and  $z$  are distinct vertices in  $T_n$ , the path from  $y$  to  $z$  is called a “ $V$ ” when the unique point on the path which is closest to the root of  $T_n$  is an intermediate point  $x$  on the path which is *strictly* between  $y$  and  $z$ . We refer to  $x$  as the *low point* of the  $V$  formed by  $y$  and  $z$ .

For a fixed value of  $n$ , we can then form a graph  $G_n$  whose vertices are the  $V$ 's in  $T_n$ . We take  $V$  adjacent to  $V'$  in  $G_n$  when an end point of one of the two paths is the low point of the other. Clearly,  $\omega(G_n) \leq 2$ . Furthermore, it is easy to see that  $\chi(G_n) \rightarrow \infty$  with  $n$  (we will say more about this observation later in the paper), and Seymour [6] suggested that the family  $\{G_n : n \geq 1\}$  has unbounded path-chromatic number.

However, we will show that graphs in the family  $\{G_n : n \geq 1\}$  have bounded path-chromatic number. In fact, we will use Ramsey theoretic tools developed by Milliken [5] to show that if we fix  $r \geq 2$ , and assume we have a path-decomposition of  $G_n$  witnessing that  $\text{path-}\chi(G_n) \leq r$ , then this decomposition is (essentially) uniquely determined. Furthermore, this decomposition actually witnesses that  $\text{path-}\chi(G_n) \leq 3$ .

Moreover, in analyzing this decomposition, we discovered the following minor modification. In the binary tree  $T_n$ , a subtree is called a “ $Y$ ” when it has 3 leaves and the closest vertex in the subtree to the root of  $T_n$  is one of the three leaves. We then let  $H_n$  be the graph whose vertex set consists of the  $V$ 's and  $Y$ 's in  $T_n$ . Furthermore,  $Y$  is adjacent to  $Y'$  in  $H_n$  if and only if one of the two upper leaves of one of them is the lowest leaf in the other. Also, a  $Y$  is adjacent to a  $V$  if and only if one of the upper leaves in the  $Y$  is the low point of the  $V$ .

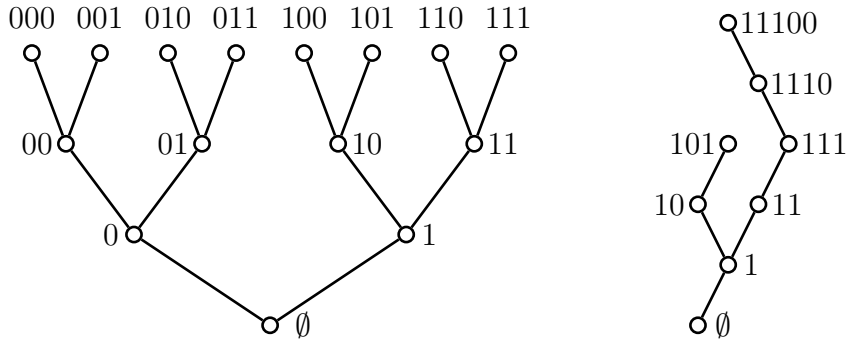
It is clear from the natural tree-decomposition of  $H_n$  that  $\text{tree-}\chi(H_n) \leq 2$ . Using Ramsey theoretic tools, we will then be able to show that  $\text{path-}\chi(H_n) \rightarrow \infty$  with  $n$ , so that Seymour's question has been successfully resolved.

## 2. RAMSEY THEORY ON BINARY TREES

The Ramsey theoretic concepts discussed here are treated in a more comprehensive manner by Milliken [5]<sup>1</sup>, but we will find it convenient to use somewhat different notation and terminology.

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<sup>1</sup>The particular result we need is Theorem 2.1 on page 220. Note that Milliken credits the result to Halpern, Läuchi, Laver and Pincus and comments on the history of the result.

FIGURE 1. Binary Trees: Down sets in  $T_n$ 

For a positive integer  $n$ , we view the *complete binary tree*  $T_n$  as the poset<sup>2</sup> consisting of all binary strings of length at most  $n$ , with  $x \leq y$  in  $T_n$  when  $x$  is an initial segment in  $y$ . The empty string, denoted  $\emptyset$ , is then the zero (least element) of  $T_n$ . For all  $n \geq 1$ ,  $T_n$  has  $2^{n+1} - 1$  elements and height  $n + 1$ . In particular,  $T_0$  is the one-point poset whose only element is the empty string.

When  $n \geq 1$  and  $x$  is a binary string of length  $n$ , we will denote coordinate  $i$  of  $x$  as  $x(i)$  and when a string is of modest length, we may write it explicitly, e.g.,  $x = 01001101$ . When  $n \geq m > p \geq 0$ ,  $x$  is a string of length  $p$ ,  $y$  is a string of length  $m$  and  $x < y$  in  $T_n$ , we say  $y$  is in the *left tree above*  $x$  when  $y(p+1) = 0$  and we say  $y$  is in the *right tree above*  $x$  when  $y(p+1) = 1$ .

Recall that in a poset  $P$ , a subposet  $Q$  of  $P$  is called a *down set* if  $x \in Q$  whenever  $y \in Q$  and  $x \leq y$  in  $P$ . We will refer to down sets of the complete binary tree  $T_n$  as *binary trees*. In Figure 1, we show on the left the complete binary tree  $T_3$ . On the right, we show a binary tree  $Q$  which will be a down set in any complete binary tree  $T_n$  with  $n \geq 5$ .

Let  $n \geq 0$ , let  $Q$  be a binary tree in  $T_n$ , and let  $R$  be a subposet of  $T_n$ . Following Milliken [5], we will say  $R$  is a *strong copy* of  $Q$  when there is a function  $f : Q \rightarrow R$  satisfying the following two requirements:

- (i)  $f$  is a poset isomorphism, i.e.,  $f$  is a bijection and for all  $x, y \in Q$ ,  $x \leq y$  in  $Q$  if and only if  $f(x) \leq f(y)$  in  $R$ .
- (ii) For all  $x, y \in Q$  with  $x < y$  in  $Q$ ,  $y$  is in the left tree above  $x$  in  $Q$  if and only if  $f(y)$  is in the left tree above  $f(x)$  in  $T_n$ .

Since we are concerned with binary trees, we note that when  $f$  satisfies the preceding two conditions, then it automatically implies that  $y$  is in the right tree above  $x$  if and only if  $f(y)$  is in the right tree above  $f(x)$ .

<sup>2</sup>The complete (rooted) binary tree we discussed in an informal manner in the opening section of this paper is just the cover graph of the poset  $T_n$  defined here.

For the remainder of this paper, when  $r \geq 1$ , we let  $[r]$  denote the set  $\{1, 2, \dots, r\}$ . Also, an  $r$ -coloring of a set  $X$  is just a map  $\phi : X \rightarrow [r]$ . In some situations, we will consider a coloring  $\phi$  using a set of  $r$  colors, but the set will not simply be the set  $[r]$ .

The following result is a straightforward extension of the special case of Theorem 2.1 from [5] for binary trees.

**Theorem 2.1.** *For every triple  $(Q, p, r)$ , where  $Q$  is a binary tree, and  $p$  and  $r$  are positive integers with  $p$  at least as large as the height of  $Q$ , there is a least positive integer  $n_0 = \text{Ram}(Q, p, r)$  so that if  $n \geq n_0$  and  $\phi$  is an  $r$ -coloring of the strong copies of  $Q$  in  $T_n$ , then there is a color  $\alpha \in [r]$  and a subposet  $R$  of  $T_n$  such that  $R$  is a strong copy of  $T_p$  and  $\phi$  assigns color  $\alpha$  to every strong copy of  $Q$  contained in  $R$ .*

### 3. SEPARATING TREE-CHROMATIC NUMBER AND PATH-CHROMATIC NUMBER

For the remainder of the paper, for a positive integer  $n$ , we let  $G_n$  be the graph of the  $V$ 's in the complete binary tree  $T_n$ . Strictly speaking, a vertex  $V$  in  $G_n$  is a path which is determined by its two endpoints, but we find it convenient to specify  $V$  as a triple  $(x, y, z)$ , where  $y$  and  $z$  are the endpoints of the path and  $x$  is the low point on the path. We view  $V$  as a triple and not a 3-element set so we can follow the convention that  $y$  is in the left tree above  $x$  and  $z$  is in the right tree above  $x$ . When  $V_1 = (x_1, y_1, z_1)$  and  $V_2 = (x_2, y_2, z_2)$  are vertices in  $G_n$ , we note that  $V_1$  and  $V_2$  are adjacent if and only if one of the following four statements holds:  $z_1 = x_2$ ,  $y_1 = x_2$ ,  $y_2 = x_1$  or  $z_2 = x_1$ .

Also, for each  $n \geq 1$ , we let  $H_n$  be the graph of  $V$ 's and  $Y$ 's in  $T_n$ . Of course,  $G_n$  is an induced subgraph of  $H_n$ . Furthermore, the natural tree-decomposition of  $H_n$  shows that  $\text{tree-}\chi(H_n) \leq 2$  for all  $n \geq 1$ .

Our goals for this section are to prove the following two theorems.

**Theorem 3.1.** *For all  $n \geq 1$ , the path-chromatic number of the graph  $G_n$  of  $V$ 's in the complete binary tree  $T_n$  is at most 2.*

**Theorem 3.2.** *For every positive integer  $r$ , there is a least positive integer  $n_0$  so that if  $n \geq n_0$ , then the path-chromatic number of the graph  $H_n$  of  $V$ 's and  $Y$ 's in the complete binary tree  $T_n$  has chromatic number larger than  $r$ .*

We elect to follow the line of our research and prove the second of these two theorems first. In accomplishing this goal, we will discover a path-decomposition of  $G_n$  witnessing that  $\text{path-}\chi(G_n) \leq 2$  for all  $n \geq 1$ .

Our argument for Theorem 3.2 will proceed by contradiction, i.e. we will assume that there is some positive integer  $r$  such that  $\text{path-}\chi(H_n) \leq r$  for all  $n \geq 1$ . The contradiction will come when  $n$  is sufficiently large in comparison to  $r$ .

For the moment, we take  $n$  as a large but unspecified integer. Later, it will be clear how large  $n$  needs to be. We then take a path-decomposition of  $H_n$  witnessing that

path- $\chi(H_n) \leq r$ . We may assume that the host path in this decomposition is the set  $\mathbb{N}$  of positive integers with  $i$  adjacent to  $i + 1$  in  $\mathbb{N}$  for all  $i \geq 1$ . For each vertex  $v$  in  $H_n$ , the set of all integers  $i$  for which  $v \in B_i$  is a set of consecutive integers, and we denote the least integer in this set as  $a_v$  and the greatest integer as  $b_v$ . Abusing notation slightly, we will denote this set as  $[a_v, b_v]$ , i.e., this interval notation identifies the integers  $i \in \mathbb{N}$  with  $a_v \leq i \leq b_v$ . Alternatively,  $[a_v, b_v]$  is just the set of integers  $i$  for which  $v$  is in the bag  $B_i$ . We point out the requirement that  $[a_v, b_v] \cap [a_u, b_u] \neq \emptyset$  when  $v$  and  $u$  are adjacent vertices in  $H_n$ .

We may assume that  $a_v < b_v$  for every vertex  $v \in H_n$ . Furthermore, we may assume that for each integer  $i$ , there is at most one vertex  $v \in H_n$  with  $i \in \{a_v, b_v\}$ .

For each  $i \in \mathbb{N}$ , we let  $G_n(i)$  denote the induced subgraph of  $G_n$  determined by those vertices  $v \in G_n$  with  $i \in [a_v, b_v]$ . Alternatively,  $G_n(i)$  is the subgraph of  $G_n$  induced by the vertices in bag  $B_i$ . The graph  $H_n(i)$  is defined analogously.

We pause here to point out an essential detail for the remainder of the proof. Since  $\chi(G_n(i)) \leq \chi(H_n(i)) \leq r$  for all integers  $i$ , then for all  $q > 2^r$ , there is no positive integer  $i$  for which either  $G_n(i)$  or  $H_n(i)$  contains the shift graph  $S_q$  as a subgraph.

To begin to make the connection with Ramsey theory, we observe that there is a natural 1–1 correspondence between  $V$ 's in  $G_n$  and strong copies of  $T_1$  in  $T_n$ . So in the discussion to follow, we will interchangeably view a vertex  $V = (x, y, z)$  of  $G_n$  as a path in  $T_n$  and as a 3-element subposet of  $T_n$  forming a strong copy of  $T_1$ . Of course, we are abusing notation slightly by referring to  $T_n$  as a graph and as a poset, but by now the notion that as a graph, we are referring to the cover graph of the poset should be clear.

In the discussion to follow, when we discuss a family  $\{V_j : j \in [t]\}$  of  $V$ 's in  $G_n$ , we will let  $V_j = (x_j, y_j, z_j)$ , and we will let  $[a_j, b_j]$  be the interval in the path-decomposition corresponding to  $V_j$ , for each  $j \in [t]$ .

Let  $(V_1, V_2)$  be an ordered pair of vertices in  $G_n$ . Referring to the binary trees in Figure 2, we consider the 7 different ways this pair can appear in  $T_n$  so that the two paths  $V_1$  and  $V_2$  have at most one vertex from  $T_n$  in common:

- (i)  $V_1$  and  $V_2$  are adjacent with  $z_1 = x_2$ . In this case, we associate the pair  $(V_1, V_2)$  with a strong copy of the poset  $Q_1$ .
- (ii)  $V_1$  and  $V_2$  are adjacent with  $y_1 = x_2$ . In this case, we associate the pair  $(V_1, V_2)$  with a strong copy of the poset  $Q_2$ .
- (iii)  $V_1$  and  $V_2$  are non-adjacent with  $x_2$  in the right tree above  $z_1$ . In this case, we associate the pair  $(V_1, V_2)$  with a strong copy of the poset  $Q_3$ .
- (iv)  $V_1$  and  $V_2$  are non-adjacent with  $x_2$  in the left tree above  $y_1$ . In this case, we associate the pair  $(V_1, V_2)$  with a strong copy of the poset  $Q_4$ .
- (v)  $V_1$  and  $V_2$  are non-adjacent with  $x_2$  in the left tree above  $z_1$ . In this case, we associate the pair  $(V_1, V_2)$  with a strong copy of the poset  $Q_5$ .
- (vi)  $V_1$  and  $V_2$  are non-adjacent with  $x_2$  in the right tree above  $y_1$ . In this case, we associate the pair  $(V_1, V_2)$  with a strong copy of the poset  $Q_6$ .

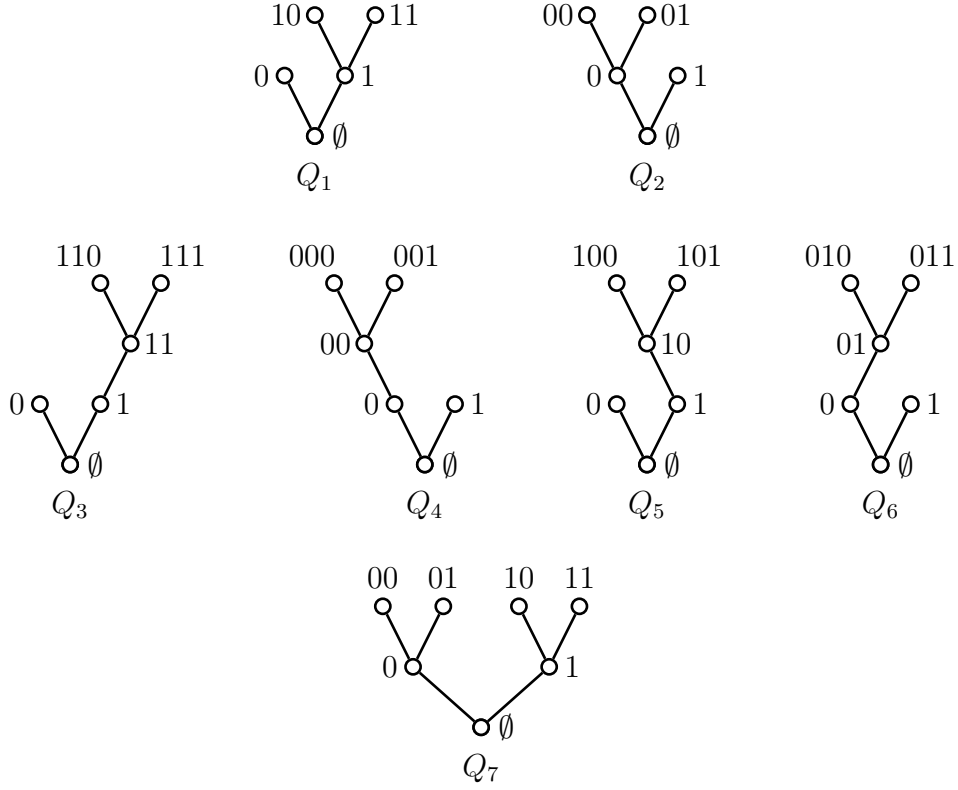


FIGURE 2. Applying Ramsey with Seven Binary Trees

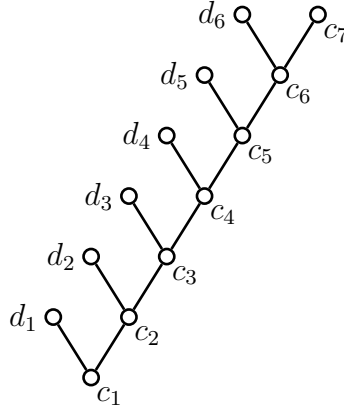
(vii)  $V_1$  and  $V_2$  are non-adjacent and there is a vertex  $w$  in  $T_n$  so that  $x_1$  is in the left tree above  $w$  while  $x_2$  is in the right tree above  $w$ . In this case, we associate the pair  $(V_1, V_2)$  with a strong copy of the poset  $Q_7$ .

Also, given a pair  $(V_1, V_2)$  of distinct  $V$ 's in  $G_n$ , there are 6 ways the intervals  $[a_1, b_1]$  and  $[a_2, b_2]$  can appear in the path-decomposition:

$a_1 < a_2 < b_1 < b_2$	Overlapping, moving right
$a_2 < a_1 < b_2 < b_1$	Overlapping, moving left
$a_1 < b_1 < a_2 < b_2$	Disjoint, moving right
$a_2 < b_2 < a_1 < b_1$	Disjoint, moving left
$a_1 < a_2 < b_2 < b_1$	Inclusion, second in first
$a_2 < a_1 < b_1 < b_2$	Inclusion, first in second

In the arguments to follow, we will abbreviate these 6 options as OMR, OML, DMR, DML, ISF and IFS, respectively.

We then define for each  $i \in [7]$  a 6-coloring  $\phi_i$  of the strong copies of  $Q_i$  in  $T_n$ . The colors will be the six labels  $\{\text{OMR}, \text{OML}, \dots, \text{IFS}\}$  listed above. When  $i \in [7]$  and  $Q$

FIGURE 3. A Shift Graph in  $G_n$ 

is a strong copy of  $Q_i$ , then  $Q$  is associated with a pair  $(V_1, V_2)$  of vertices from  $G_n$ . It is then natural to set  $\phi_i(Q)$  as the label describing how the pair  $([a_1, b_1], [a_2, b_2])$  of intervals are positioned in the path decomposition.

Now let  $p = 4 \cdot 2^r$ . By iterating on Theorem 2.1, we may assume that  $n$  is sufficiently large to guarantee that there is a subposet  $R$  of  $T_n$  and a vector  $(\alpha_1, \alpha_2, \dots, \alpha_7)$  of colors such that  $R$  is a strong copy of  $T_p$  and for each  $i \in [7]$ ,  $\phi_i$  assigns color  $\alpha_i$  to all strong copies of  $Q_i$  in  $R$ . In the remainder of the argument, we will abuse notation slightly and simply consider that  $R = T_p$ .

**Claim 1.**  $\alpha_1$  is either OMR or OML.

*Proof.* A pair  $(V_1, V_2)$  of vertices in  $G_n$  associated with a strong copy of  $Q_1$  in  $T_p$  is adjacent in  $G_n$  so that  $[a_1, b_1]$  and  $[a_2, b_2]$  intersect. So  $\alpha_1$  cannot be DMR or DML. We assume that  $\alpha_1$  is ISF and argue to a contradiction. The argument when  $\alpha_1$  is IFS is symmetric. Consider the subposet of  $T_p$  consisting of all non-empty strings for which each bit, except possibly the last, is a 1. We suggest how this subposet appears (at least for a modest value of  $p = 7$ ) in Figure 3.

Using the labelling given in Figure 3, for each interval  $[i, j]$  with  $1 \leq i < j \leq p$ , we consider the vertex  $V[i, j] = (c_i, d_i, c_j)$ . Clearly,  $V[i, j]$  is adjacent to  $V[j, k]$  when  $1 \leq i < j < k \leq p$ , i.e., these vertices form the shift graph  $S_p$ .

Let  $[a, b] = [a_{V[p-1, p]}, b_{V[p-1, p]})$  be the interval for the vertex  $V[p-1, p]$ . We claim that for each  $[i, j]$  with  $1 \leq i < j \leq p-1$ , the interval for  $V[i, j]$  in the path-decomposition for  $H_n$  contains  $[a, b]$ . This is immediate if  $j = p-1$ , since  $(V[i, p-1], V[p-1, p])$  is assigned color ISF. Now suppose  $j < p-1$ . Then  $(V[i, j], V[j, p-1])$  is also ISF, so that in the path-decomposition, the interval for  $V[j, p-1]$  is included in the interval for  $V[i, j]$ . By transitivity, we conclude that  $[a, b]$  is included in the interval for  $V[i, j]$ .



So the  $V$ 's in  $\{V[i, j] : 1 \leq i < j \leq p - 1\}$  form a copy of the shift graph  $S_{p-1}$ , and all of them are in the bag  $G_n(a)$ . Since  $p = 4 \cdot 2^r$ , this is a contradiction.  $\square$

Without loss of generality, we take  $\alpha_1$  to be OMR, since if  $\alpha_1$  is OML, we may simply reverse the entire path-decomposition. To help keep track of the configuration information as it is discovered, we list this statement as a property.

**Property 1.**  $\alpha_1 = \text{OMR}$ , i.e.,  $\phi_1$  assigns color OMR to a pair  $(V_1, V_2)$  of adjacent vertices in  $G_n$  when  $z_1 = x_2$ .

Although it may not be a surprise, once the color  $\alpha_1$  is set, colors  $\alpha_2, \alpha_3, \dots, \alpha_7$  are determined.

**Property 2.**  $\alpha_3 = \text{DMR}$ , i.e.,  $\phi_3$  assigns color DMR to a pair  $(V_1, V_2)$  of non-adjacent vertices in  $G_n$  when  $x_2$  is in the right tree above  $z_1$ .

*Proof.* Let  $(V_1, V_2)$  be a pair of non-adjacent vertices in  $G_n$  with  $x_2$  in the right tree above  $z_1$ . Then let  $w_3$  be the string formed by attaching a 0 at the end of  $z_1$ , and set  $V_3 = (z_1, w_3, x_2)$ . Then  $V_3$  is adjacent to both  $V_1$  and  $V_2$ . Furthermore,  $\phi_1(V_1, V_3) = \text{OMR}$  and  $\phi_1(V_3, V_2) = \text{OMR}$ . Accordingly,  $\alpha_3$  is either OMR or DMR. We assume that  $\alpha_3 = \text{OMR}$  and argue to a contradiction.

Consider the shift graph used in the proof of Claim 1. Let  $a = a_{V[p-1, p]}$  be the left endpoint of the interval for  $V[p - 1, p]$  in the path-decomposition. We claim that  $a$  is in the interval for  $V[i, j]$  in the path-decomposition whenever  $1 \leq i < j \leq p - 1$ . Again, this holds when  $j = p - 1$  since  $\phi_1(V[i, p - 1], V[p - 1, p]) = \text{OMR}$ . Also, when  $j < p - 1$ , the color assigned by  $\phi_3$  to the pair  $(V[i, j], V[p - 1, p])$  is also OMR, so that the interval for  $V[i, j]$  in the path-decomposition also contains  $a$ . This now implies that  $G_n(a)$  contains the shift graph  $S_{p-1}$ . The contradiction completes the proof.  $\square$

**Property 3.**  $\alpha_2 = \text{OML}$ , i.e.,  $\phi_2$  assigns color OML to a pair  $(V_1, V_2)$  of adjacent vertices in  $G_n$  when  $y_1 = x_2$ . Also,  $\alpha_4 = \text{DML}$ , i.e.,  $\phi_4$  assigns color DML to a pair  $(V_1, V_2)$  of non-adjacent vertices in  $G_n$  when  $x_2$  is in the left tree above  $y_1$ .

*Proof.* We can repeat the arguments given previously to conclude that one of two cases must hold: Either (1)  $\alpha_2 = \text{OMR}$  and  $\alpha_4 = \text{DMR}$ , or (2)  $\alpha_2 = \text{OML}$  and  $\alpha_4 = \text{DML}$ . We assume that  $\alpha_2 = \text{OMR}$  and  $\alpha_4 = \text{DMR}$  and argue to a contradiction. Consider the binary tree contained in  $T_p$  as shown on the left side of Figure 4. Let  $V_1 = (f, g, h)$ ,  $V_2 = (i, j, k)$ ,  $V_3 = (c, f, e)$  and  $V_4 = (c, d, i)$ .

Since  $\phi_4(V_4, V_1) = \text{DMR}$ , we know  $b_4 < a_1$ . Since  $\phi_1(V_4, V_2) = \text{OMR}$ , we know  $a_2 < b_4$ , so  $a_2 < a_1$ . Since  $\phi_3(V_3, V_2) = \text{DMR}$ , we know  $b_3 < a_2$  so  $b_3 < a_1$ . But  $\phi_2(V_3, V_1) = \text{OMR}$ , which requires  $a_1 < b_3$ . The contradiction completes the proof of Property 3.  $\square$

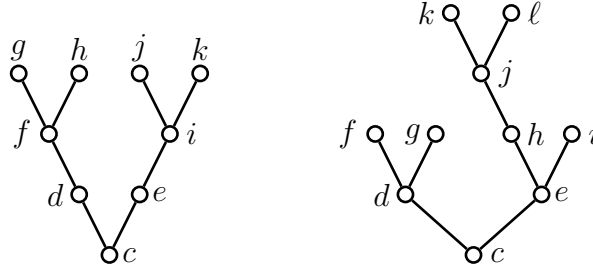


FIGURE 4. Two Useful Small Examples

**Property 4.**  $\alpha_7 = \text{DMR}$ , i.e.,  $\phi_7$  assigns color DMR to a pair  $(V_1, V_2)$  of non-adjacent vertices in  $G_n$  when there is a vertex  $w$  in  $T_n$  such that  $x_1$  is in the left tree above  $w$  while  $x_2$  is in the right tree above  $w$ .

*Proof.* We again consider the binary tree shown on the left side of Figure 4. Again, we take  $V_1 = (f, g, h)$  and  $V_2 = (i, j, k)$ . Noting that  $f$  is in the left tree above  $c$  and  $i$  is in the right tree above  $a$ ,  $\phi_7(V_1, V_2) = \alpha_7$ .

Now let  $V_5 = (c, d, e)$ . Then  $\phi_4(V_5, V_1) = \text{DML}$  and  $\phi_3(V_5, V_2) = \text{DMR}$ . These statements imply  $\alpha_7 = \text{DMR}$ .  $\square$

**Property 5.**  $\alpha_5 = \alpha_6 = \text{ISF}$ , i.e.,  $\phi_5$  assigns color ISF to a pair  $(V_1, V_2)$  of non-adjacent vertices in  $G_n$  when  $x_2$  is in the left tree above  $z_1$  and  $\phi_6$  assigns this pair color IFS when  $x_2$  is in the right tree above  $y_1$ .

*Proof.* We prove that  $\alpha_5 = \text{ISF}$ . The argument to show that  $\alpha_6 = \text{ISF}$  is symmetric. Consider the binary tree shown on the right side of Figure 4. Let  $V_1 = (c, d, e)$  and  $V_2 = (j, k, l)$ . Then  $j$  is in the left tree above  $e$ , so  $\phi_5(V_1, V_2) = \alpha_5$ .

Now set  $V_3 = (d, f, g)$  and  $V_4 = (e, h, i)$ . We observe that  $\phi_2(V_1, V_3) = \text{OML}$ ,  $\phi_7(V_3, V_2) = \text{DMR}$ ,  $\phi_1(V_1, V_4) = \text{OMR}$  and  $\phi_4(V_4, V_2) = \text{DML}$ . Together, these statements imply  $\alpha_5 = \text{ISF}$ .  $\square$

Up to this point in the proof, our entire focus has been on the  $V$ 's in  $G_n$ . We now turn our attention to properties that the  $Y$ 's in  $H_n$  must satisfy.

Consider the binary tree shown in Figure 5. Of course, we intend that this tree appear inside  $T_p$ . In our figure, the “size” of this construction is  $m = 6$ , but since  $p = 4 \cdot 2^r$ , we know we can make  $m > 2^r$ . For each interval  $[i, j]$  with  $1 \leq i < j \leq m$ , we let  $Y[i, j]$  be the  $Y$  whose three leaves are  $x_i, x_j$  and  $w_j$ . Clearly, the family  $\{Y[i, j] : 1 \leq i < j \leq m\}$  forms a copy of the shift graph  $S_m$ . To reach a final contradiction, it remains only to show that there is some integer  $k \in \mathbb{N}$  for which all vertices in  $\{Y[i, j] : 1 \leq i < j \leq m\}$  belong to  $H_n(k)$ .

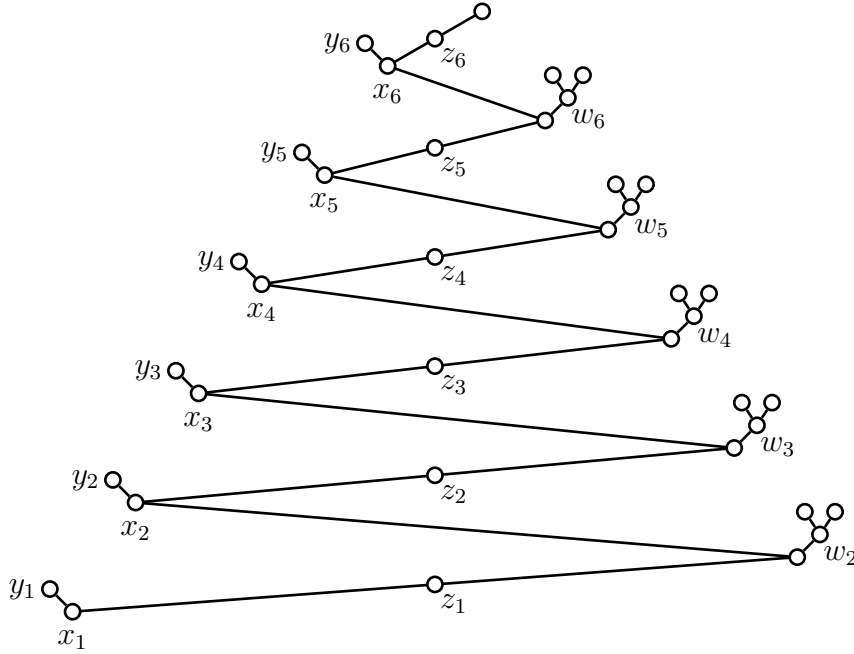


FIGURE 5. The Final Counter-Example

For each  $j \in [m]$ , we let  $V_j = (x_j, y_j, z_j)$ , and as usual, we let  $[a_j, b_j]$  be the corresponding interval for  $V_j$  in the path decomposition. By Property 2, we have  $\alpha_3 = \text{DMR}$ , so that:

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_{m-1} < b_{m-1} < a_m < b_m.$$

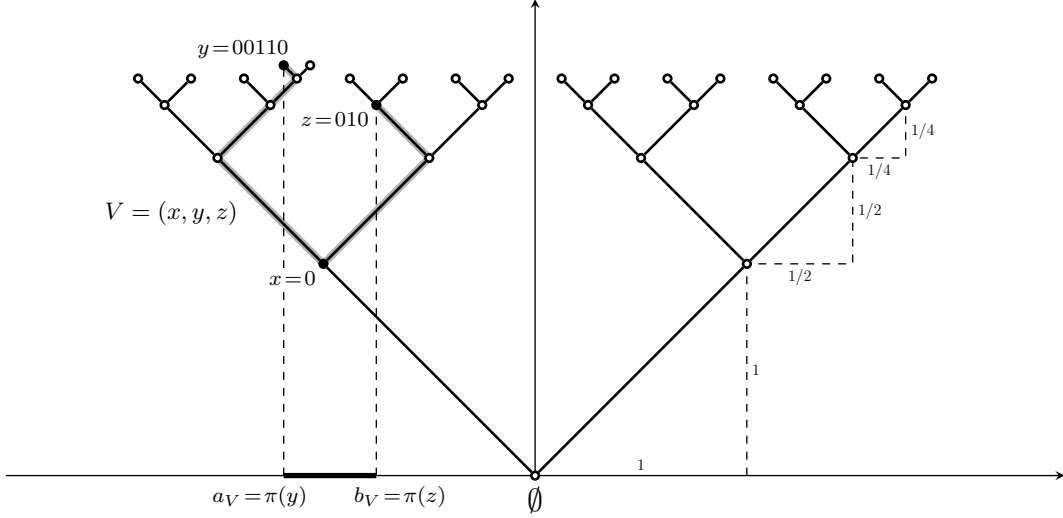
For each  $j = 2, 3, \dots, m$ , let  $V'_j = (w_j, w_{j0}, w_{j1})$ , and we let  $[a'_j, b'_j]$  be the corresponding interval in the path-decomposition. By Property 4,  $\alpha_7 = \text{DMR}$  so that:

$$a'_m < b'_m < a'_{m-1} < b'_{m-1} < \cdots < a'_3 < b'_3 < a'_2 < b'_2.$$

Again, since  $\alpha_7 = \text{DMR}$ , we know that  $a_m < b_m < a'_m < b'_m$ .

Now consider a pair  $i, j$  with  $1 \leq i < j \leq m$ . The vertex  $Y[i, j]$  is adjacent in  $H_n$  to both  $V_j$  and  $V'_j$ . This implies that the interval for  $Y[i, j]$  must overlap both  $[a_j, b_j]$  and  $[a'_j, b'_j]$ . However, this forces the interval for  $Y[i, j]$  to contain  $[b_m, a'_m]$ . Therefore,  $G_n(b_m)$  contains the shift graph  $S_m$ . With this observation, the proof of Theorem 3.2 is complete.

We now return to the task of proving Theorem 3.1, i.e., the assertion that  $\text{path-}\chi(G_n) \leq 2$  for all  $n \geq 1$ . Our proof for Theorem 3.2 suggests a natural way to define a path-decomposition of the graph  $G_n$  of  $V$ 's in the binary tree  $T_n$ , one that satisfies all five properties we have developed to this point. We simply take a drawing in the plane of  $T_n$  using a geometric series approach. Taking a standard cartesian coordinate system in the plane, we place the zero of  $T_n$  at the origin. If  $m \geq 0$  and we have placed a

FIGURE 6. A Path-Decomposition of  $G_n$ 

string  $x$  of length  $m$  at  $(h, v)$ , we set  $\delta = 2^{-m}$  and place  $x_1$  and  $x_0$  at  $(h + \delta, v + \delta)$  and  $(h - \delta, v + \delta)$ , respectively.

For each  $x$  in  $T_n$ , let  $\pi(x)$  denote the vertical projection of  $x$  down onto the horizontal axis. In turn, for each  $V = (x, y, z)$ , we take  $a_V = \pi(y)$  and  $b_V = \pi(z)$ . To illustrate this construction, we show in Figure 6 the interval  $[a_V, b_V]$  corresponding to the vertex  $V = (0, 00110, 010)$  in  $G_n$ .

Clearly, we may consider the host path  $P$  for the decomposition as consisting of all points on the horizontal axis of the form  $\pi(x)$  where  $x \in T_n$ . Also, in the natural manner,  $\pi(x)$  is adjacent to  $\pi(x')$  in  $P$  when there is no string  $x'' \in T_n$  with  $\pi(x'')$  between  $\pi(x)$  and  $\pi(x')$ .

So let  $x_0 \in T_n$  and consider the bag  $B = B_{\pi(x_0)}$  consisting of all vertices  $V = (x, y, z)$  in  $G_n$  with  $\pi(y) \leq \pi(x_0) \leq \pi(z)$ . We partition  $B$  as  $C_1 \cup C_2 \cup C_3$  where:

- (i) A vertex  $V = (x, y, z)$  of  $B$  belongs to  $C_1$  if  $\pi(x) < \pi(x_0)$ .
- (ii) A vertex  $V = (x, y, z)$  of  $B$  belongs to  $C_2$  if  $\pi(x) > \pi(x_0)$ .
- (iii) A vertex  $V = (x, y, z)$  of  $B$  belongs to  $C_3$  if  $\pi(x) = \pi(x_0)$ . In this case,  $x = x_0$ .

We now explain why  $C_1$ ,  $C_2$  and  $C_3$  are independent sets in  $G_n$ . This is trivial for  $C_3$ . We give the argument for  $C_1$ , noting that the argument for  $C_2$  is symmetric.

Suppose that  $V_1$  and  $V_2$  are adjacent vertices in  $C_1$ . If the pair  $(V_1, V_2)$  determines a strong copy of  $Q_1$ , then  $\pi(z_1) = \pi(x_2) < \pi(x_0)$ , which is a contradiction. On the other hand, if the pair  $(V_1, V_2)$  determines a strong copy of  $Q_2$ , then  $y_1 = x_2$  so that  $\pi(y_1) = \pi(x_2) < \pi(x_1) < \pi(x_0)$ . Now the geometric series nature of the construction implies that  $\pi(z_2) < \pi(x_1) < \pi(x_0)$ , which is again a contradiction.

With these observations, we have now proved that  $\text{path-}\chi(G_n) \leq 3$  for all  $n \geq 1$ . This inequality is tight as evidenced by the following five elements of  $G_n$  which form a 5-cycle:  $V_1 = (\emptyset, 0, 1)$ ,  $V_2 = (1, 10, 11)$ ,  $V_3 = (10, 100, 101)$ ,  $V_4 = (101, 1010, 1011)$  and  $V_5 = (1, 101, 11)$ . Note that  $\pi(101)$  is in  $[a_i, b_i]$  for each  $i \in [5]$ .

Nevertheless, we are able to make a small but important change in the path-decomposition to obtain a decomposition witnessing that  $\text{path-}\chi(G_n) \leq 2$ . For the integer  $n$ , let  $\varepsilon = 2^{-2n}$ . Then for each vertex  $V = (x, y, z)$  of  $G_n$ , we change the interval in the path decomposition for  $V$  from  $[\pi(y), \pi(z)]$  to  $[\pi(y) + \varepsilon, \pi(z) - \varepsilon]$ . Our choice of  $\varepsilon$  guarantees that we still have a path-decomposition of  $G_n$ .

Again, we consider an element  $x_0$  of  $T_n$  and the bag  $B$  consisting of all  $V = (x, y, z)$  with  $\pi(y) + \varepsilon \leq \pi(x_0) \leq \pi(z) - \varepsilon$ . As before,  $C_1$ ,  $C_2$  and  $C_3$  are independent sets, although membership in these three sets has been affected by the revised path-decomposition. We claim that  $C_1 \cup C_3$  is also an independent set, so that the partition  $B = (C_1 \cup C_3) \cup C_2$  witnesses that  $\text{path-}\chi(G_n) \leq 2$ .

Suppose to the contrary that  $V_1 \in C_1$  and  $V_3 \in C_3$  with  $V_1$  adjacent to  $V_3$  in  $G_n$ . Clearly, this requires that  $(V_1, V_3)$  is associated with a strong copy of the binary tree  $Q_1$  as shown in Figure 2. This implies that  $z_1 = x_0 = x_3$  so that  $b_1 = \pi(z_1) - \varepsilon = \pi(x_0) - \varepsilon$ , which contradicts the assumption that  $a_1 < \pi(x_1) < \pi(x_0) \leq b_1$ . The contradiction completes the proof of Theorem 3.1.

#### 4. ACKNOWLEDGEMENTS

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