Intersection Graphs of Pseudosegments: Chordal Graphs[*]

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Abstract

We investigate which chordal graphs have a representation as intersection graphs of pseudosegments. For positive we have a construction which shows that all chordal graphs that can be represented as intersection graphs of subpaths on a tree are pseudosegment intersection graphs. We then study the limits of representability. We identify certain intersection graphs of substars of a star which are not representable as intersection graphs of pseudosegments. The degree of the substars in these examples, however, has to be large. A more intricate analysis involving a Ramsey argument shows that even in the class of intersection graphs of substars of degree three of a star there are graphs that are not representable as intersection graphs of pseudosegments.

Motivated by representability questions for chordal graphs we consider how many combinatorially different k-segments, i.e., curves crossing k distinct lines, an arrangement of n pseudolines can host. We show that for fixed k this number is in $O(n^2)$.

1   Introduction

A family of pseudosegments is understood to be a set of Jordan arcs in the Euclidean plane that are pairwise either disjoint or intersect at a single crossing point. A family of pseudosegments represents a graph G, the vertices of G are the Jordan arcs and two vertices are adjacent if and only if the corresponding arcs intersect. A graph represented by a family of pseudosegments is a *pseudosegment intersection graph*, for short a PSI-graph.

PSI-graphs are sandwiched between the larger class of string-graphs (intersection graphs of Jordan arcs without condition on their intersection behavior) and of segment-graphs (intersection graphs of straight line segments). In one of the first papers on the subject Ehrlich et al. [5] proved that all planar graphs are string-graphs. In fact this follows from Koebe's coin graph theorem. Scheinerman in his thesis [19] conjectured that planar-graphs are segment graphs. Some special cases have been resolved, most notably de Castro et al. [4] proved that triangle free planar graphs can be represented by segments in three directions. Recently

Chalopin, Gonçalves and Ochem [2] managed to prove that all planar graphs are PSI-graphs. Based on this result Chalopin and Gonçalves manageed to show that all planar graphs are segment intersection graphs [1]. Thus Scheinerman's conjecture is finally verified .

Interesting research has been conducted regarding the membership complexity of these classes. The problem for string-graphs was stated by Graham in '76. It was not known to be decidable for almost thirty years. In two independent papers it was shown that an exponential number of intersections is sufficient to represent string-graphs [16], [18]. Later in [17] it was shown that the recognition problem for string-graphs is in NP. Proofs for NP-hardness have been obtained by Kratochvíl: In [10] he shows that recognizing string-graphs is NP-hard. Recognition of PSI-graphs is shown to be NP-complete in [11]. Recognition of segment-graphs is NP-hard [12]. Interestingly it is open whether it is NP-complete. It is known [13], however, that a representation via segments with integer endpoints may require endpoints of size $2^{2^{\sqrt{n}}}$.

There are some classes of graphs where segment representation, hence, as well PSI-representations, are trivial (e.g. permutation graphs and circle graphs) or very easy to find (e.g. interval graphs). A large superclass of interval graphs is the class of chordal graphs. This paper originated from investigating chordal graphs in view of their representability as intersection graphs of pseudosegments. From the subtree representation of chordal graphs it is immediate that they are string-graphs.

1.1 Basic definitions and results

A graph is *chordal* if it has no induced cycles of length greater than three. Gavril [7] characterized chordal graphs as the intersection graphs of subtrees of a tree, that is: a graph $G = (V_G, E_G)$ is chordal iff there exists a tree $T = (V_T, E_T)$ and a set $\mathcal{T}$ of subtrees of T such that there is a mapping $v \to T_v \in \mathcal{T}$ with the property that $vw \in E_G$ whenever $T_v \cap T_w \neq \emptyset$. The pair $(T, \mathcal{T})$ is a *tree representation* of G.

A subclass of chordal graphs is the class of vertex intersection graphs of paths on a tree, *VPT-graphs* for short. The precise definition is as follows: A graph $G = (V_G, E_G)$ is a VPT-graph if there exists a tree $T = (V_T, E_T)$ and a set $\mathcal{P}$ of paths in T such that there is a mapping $v \to P_v \in \mathcal{P}$ with the property that $vw \in E_G$ iff $P_v \cap P_w \neq \emptyset$. Such a pair $(T, \mathcal{P})$ is said to be a *VPT-representation* of G .

VPT-graphs have been introduced by Gavril in [8], he called them path-graph. Gavril provided a characterization and a recognition algorithm. Since then VPT-graphs have been studied continuously, Monma and Wei [15] give some applications and many references. We show:

Theorem 1 *Every VPT-graph has a PSI-representation.*

The proof of the theorem is given in Section 2. At this point we content ourselves with an indication that the result is not as trivial as it may seem at first glance. Let a VPT-representation $(T, \mathcal{P})$ of a graph G be given. If we fix a plane embedding of the tree we obtain an embedding of each of the paths P_v corresponding to $v \in V_G$. The first idea for converting a VPT-representation into a PSI-representation would be to slightly perturb P_v into a pseudosegment s_v and make sure that paths with common vertices intersect exactly once and are disjoint otherwise. Figure 1 gives an example of a set of three paths which can't be perturbed locally as to give a PSI-representation of the corresponding subgraph.
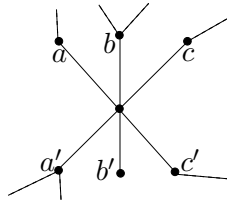
Figure 1: It is impossible to perturb the paths $a \leftrightarrow a'$, $b \leftrightarrow b'$ and $c \leftrightarrow c'$ locally to yield a PSI-representation of the graph, i.e., to make them pairwise intersecting in exactly one point.

It is natural to ask whether all chordal graphs are PSI-graphs. This is not the case, in Theorems 3 and 6 we give examples of chordal graphs which are not PSI-representable. Our examples are quite big, all of them have more than 5000 vertices. This is in contrast to the size of obstructions against PSI-representability within the class of all graphs. There we know of quite small examples, e.g., the complete subdivisions of K_5 and $K_{3,3}$ with 15 vertices.

In Theorem 3 we show that certain chordal graphs K_n^3 are not in PSI for all $n \geq 33$. The vertex set of K_n^3 is partitioned as $V = V_C \cup V_I$ such that $V_C = [n]$ induces a clique and $V_I = \binom{[n]}{3}$ is an independent set. The edges between V_C and V_I represent membership, i.e, $\{i, j, k\} \in V_I$ is connected to the vertices i, j and k from V_C.

The graph K_n^3 can be represented as intersection graph of subtrees of a star S with $\binom{n}{3}$ leaves. The leaves correspond to the elements of V_I. Each $v \in V_I$ is represented by a trivial tree with only one node. A vertex $i \in V_C$ is represented by the star connecting to all leaves of triples containing i. This representation shows that K_n^3 is chordal. The central node of the star S has high degree. If we take a path of $\binom{n}{3}$ nodes and attach a leaf-node to each node of the path we obtain a tree T of maximum degree three such that the graph K_n^3 can be represented as intersection graph of subtrees of T. Actually the tree T and its subtrees are caterpillars of maximum degree three.

These remarks show that the positive result of Theorem 1 and the negative of Theorem 3 only leave a small gap for questions: If the subtrees, in a tree representation of a chordal graph, are paths we have a PSI-representation. If we allow the subtrees to be stars (of large degree), or (large) caterpillars of maximum degree three, then there need not exist a PSI-representation.

What if we restrict the host tree to be a star and the subtrees to be substars of degree three? Let S_n be the chordal graph whose vertices are represented by all substars with three leaves and all leaves on a star with n leaves. In Theorem 6 we show that S_n is not PSI-representable if n is large enough. This resolves our major conjecture from [3]. The proof makes use of a Ramsey argument, hence, we need a really large n for the result. Note that as in the case of K_n^3 the vertex set of S_n again partitions into a clique V_C and an independent set V_I.

In Section 4 we deal with graphs whose vertex set splits into a clique V_C of size n and a set W. We impose no condition on the subgraph induced by W but require that each $w \in W$ has three neighbors in V_C and no two vertices in W have the same neighbors in V_C. We conjecture that if such a graph has a PSI-representation then $|W| \in o(n^{2-\epsilon})$. Motivated by this problem we consider how many combinatorially different k-segments, i.e., curves crossing k distinct lines, an arrangement of n pseudolines can host. In Theorem 4 we show that for fixed k this number is in $O(n^2)$.

We conclude with some open problems.

2 A PSI-representation for VPT-graphs

2.1 Preliminaries

A path in a tree T with endpoints a and b is denoted by $P(a, b)$. We also allow degenerate paths $P(a, a)$ consisting of just one vertex. If all endpoints of a path P in T are leaves we call P a *leaf-path*.

Lemma 1 *Every VPT-graph has a path-representation $(T, \mathcal{P})$ such that all paths in $\mathcal{P}$ are leaf-paths and no two vertices are represented by the same leaf-path.*

Proof. Let an arbitrary VPT-representation $(T, \mathcal{P})$ of G be given. Now let $\overline{T}$ be the tree obtained from T by taking all paths $P \in \mathcal{P}$ and adding two private new leafes for P to T. If $P = P(a, b)$ we attach a new node a_P to a and a new node b_P to b. Replacing the path P by the path $\overline{P} = P(a_P, b_P)$ in $\overline{T}$ yields a VPT-representation of G using only leaf-paths. And since each new leaf is the end-vertex of a single path the paths in the collection $\overline{\mathcal{P}}$ are all different. □

As a class of intersection graphs the class of VPT-graphs is closed under taking induced subgraphs.This observation together with the previous lemma show that Theorem 1 is implied by the following:

Theorem 2 *Given a tree T we let G be the VPT-graph whose vertices are in bijection to the set of all leaf-paths of T. The graph G has a PSI-representation with pseudosegments $s_{i,j}$ corresponding to the paths $P_{i,j} = P(l_i, l_j)$ in T. In addition there is a collection of pairwise disjoint disks, one disk R_i associated with each leaf l_i of T, such that:*

(a) *The intersection $s_{i,j} \cap R_k \neq \emptyset$ if and only if $k = i$ or $k = j$. Furthermore if $s_{i,j} \cap R_k \neq \emptyset$, then this intersection curve connects two different points on the boundary of R_k.*

(b) *Any two pseudosegments intersecting R_i cross in the interior of this disk, hence the points where they intersect R_i alternate along the boundary of R_i.*

We will prove Theorem 2 by induction on the number of inner nodes of tree T. The construction will have multiple intersections, i.e., there are points where more than two pseudosegments intersect. By perturbing the pseudosegments participating in a multiple intersection locally the representation can easily be transformed into a representation without multiple intersections.

2.2 Theorem 2 for stars

Let T be a star, i.e., a tree with just one inner node v. Let $L = \{l_1, .., l_m\}$ be the set of leaves of T. The subgraph H of G induced by the set $\mathcal{P} = \{P(l_i, l_j) \mid l_i, l_j \in L, l_i \neq l_j\}$ of leaf-paths is a complete graph on $\binom{m}{2}$ vertices, this is because every path in $\mathcal{P}$ contains v.

Take a circle γ and choose m points $c_1, .., c_m$ on γ such that the set of straight lines spanned by pairs of different points from $c_1, .., c_m$ contains no parallel lines. For each i choose a small disk R_i centered at c_i such that these disks are disjoint and put them in one-to-one correspondence with the leaves of T. Let $s_{i,j}$ be the line connecting c_i and c_j. If the disks R_k are small enough we clearly have :

(a) The line $s_{i,j}$ intersects R_i and R_j but no further disk R_k.

(b) Two lines s_I and s_J with $I \cap J = \{i\}$ contain corner c_i, hence s_I and s_J cross in the disk R_i.
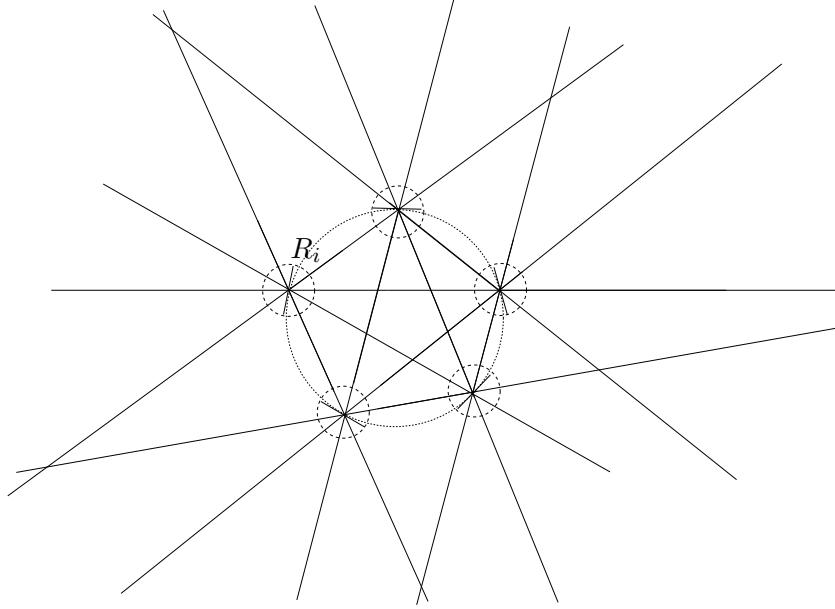

Figure 2: The construction for the star with five leaves.

Prune the lines such that the remaining part of each $s_{i,j}$ still contains its intersection with all the other lines and all segments have their endpoints on a circumscribing circle C. Every pair of segments stays intersecting, hence, we have a segment intersection representation of H.

Add a diameter $s_{i,i}$ to every disk R_i, this segment serves as representation for the leaf-path $P_{i,i}$. Altogether we have constructed a representation of G obeying the required properties (a) and (b), see Figure 2 for an illustration.

2.3 Theorem 2 for trees with more than one inner node

Now let T be a tree with inner nodes $N = \{v_1, .., v_n\}$ and assume the theorem has been proven for trees with at most $n-1$ inner nodes. Let $L = \{l_1, .., l_m\}$ be the set of leaves of T. With $L_i \subset L$ we denote the set of leaves attached to v_i. We have to produce a PSI-representation of the intersection graph G of $\mathcal{P} = \{P_{i,j} \mid l_i, l_j \in L\}$, i.e., of the set of all leaf-paths of T. We suppose that v_n is a leaf node in the tree induced by N:

- The tree T_n is the star with inner node v_n and its leaves $L_n = \{l_k, .., l_m\}$.

- The tree T' contains all nodes of T except the leaves in L_n. The set of inner nodes of T' is $N' = N \backslash \{v_n\}$, the set of leaves is $L' = L \backslash L_n \cup \{v_n\}$. For consistency we rename $l_0 := v_n$ in T', hence $L' = \{l_0, l_1, .., l_{k-1}\}$.

Let G_n and G' be the VPT-graphs induced by all leaf-paths in T_n and T'. Both these trees have fewer inner nodes than T. Therefore, by induction we can assume that we have PSI-representations PS' of G' and PS_n of G_n as claimed in Theorem 2. We will construct a PSI-representation of G using PS' and PS_n. The idea is as follows:
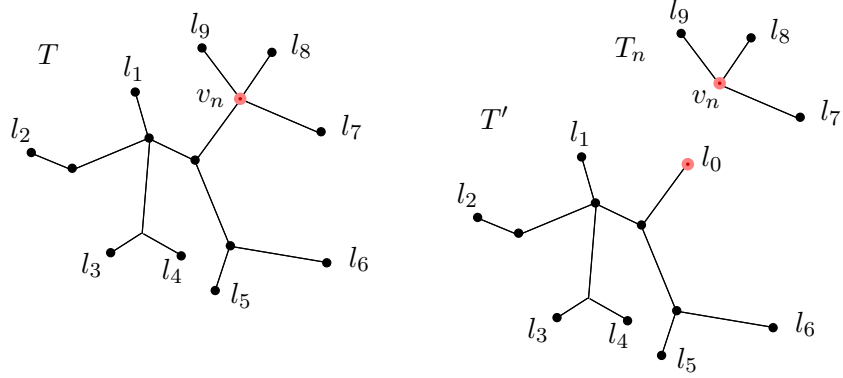
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Figure 3: A tree T and the two induced subtrees T' and T_n.

1. Replace every pseudosegment of PS' representing a leaf-path ending in l_0 by a bundle of pseudosegments. This bundle stays within a narrow tube around the original pseudosegment. (The inner structure of the bundle will be determined later.)

2. Remove all pieces of pseudosegments from the interior of the disk R_0 and patch an appropriately transformed copy of PS_n into R_0.

3. The crucial step is to connect the pseudosegments of the bundles through the interior of R_0 such that the induction invariants for the transformed disks of R_r with $k \le r \le m$ are satisfied.

The set $\mathcal{P}$ of leaf-paths of T can be partitioned into three parts. The subsets $\mathcal{P}'$ and $\mathcal{P}_n$ are represented by leaf-paths of T' or T_n. Let $\mathcal{P}^* = \mathcal{P} - \mathcal{P}' - \mathcal{P}_n$ be the remaining subset. The paths in $\mathcal{P}^*$ connect leaves l_i and l_r with $1 \le i < k \le r \le m$, in other words they connect a leaf l_i from T' through v_n with a leaf in T_n. We subdivide these paths into classes $\mathcal{P}_1^*, .., \mathcal{P}_{k-1}^*$ such that $\mathcal{P}_i^*$ consists of those paths from $\mathcal{P}^*$ which start in the leaf l_i of T'. Each $\mathcal{P}_i^*$ consists of $|L_n|$ paths. In T' we have the pseudosegment $s_{i,0}$ which leads from l_i to l_0. Replace each such pseudosegment $s_{i,0}$ by a bundle of $|L_n|$ parallel pseudosegments routed in a narrow tube around $s_{i,0}$.

We come to the second step of the construction. Remove all pieces of pseudosegments from the interior of R_0. Recall that the representation PS_n of G_n from 2.2 has the property that all long pseudosegments have their endpoints on a circle C. Choose two arcs A_b and A_t on C such that every segment connecting a point in A_b and a point in A_t intersects each pseudosegment $s_{i,j}$ with $i \ne j$, this is possible by the choice of C. This partitions the circle into four arcs which will be called A_b, A_l, A_t, A_r in clockwise order. The choice of A_b and A_t implies that each pseudosegment touching C has one endpoint in A_l and the other in A_r.

Map the interior of C with an homeomorphism h into a wide rectangular box Γ such that A_t and A_b are mapped to the top and bottom sides of the box, A_l is the left side and A_r the right side. This makes the images of all long pseudosegments traverse the box from left to right. We may also require that the homomorphism maps the disks R_r to disks and arranges them in a nice left to right order in the box, Figure 4 shows an example. The figure was generated by sweeping the representation from Figure 2 and converting the sweep into a wiring diagram (the diametrical segments $s_{r,r}$ have been re-attached horizontally).

In the box we have a left to right order of the disks R_r, $l_r \in L_n$. By possibly relabeling the leaves of L_n we can assume that the disks are ordered from left to right as $R_k, .., R_m$.
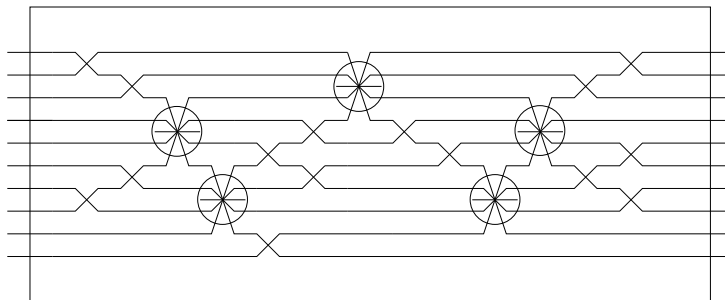
Figure 4: A box containing a deformed copy of the representation from Figure 2.

This step of the construction is completed by placing the box Γ appropriately resized in the disk R_0 such that each of the segments $s_{i,0}$ from the representation of G' traverses the box from bottom to top and the sides A_l and A_r are mapped to the boundary of R_0. The boundary of R_0 is thus partitioned into four arcs which are called A_l, A'_t, A_r, A'_b in clockwise order. We assume that the segments $s_{i,0}$ touch the arc A'_b in PS' in counterclockwise order as $s_{1,0}, .., s_{k-1,0}$, this can be achieved by renaming the leaves appropriately.

We now move to the third step of the construction where we have to connect the pseudosegments of the bundles through the interior of R_0. The result of such a construction is shown in Figure 5.

By removing everything from the interior of R_0 we have disconnected all the pseudosegments which had been inserted in bundles replacing the original pseudosegments $s_{i,0}$. Let B_i^{in} be the half of the bundle of $s_{i,0}$ which touches A'_b and let B_i^{out} be the half which touches A'_t. By the above assumption the bundles $B_1^{in}, .., B_{k-1}^{in}$ touch A'_b in counterclockwise order, with (b) from the statement of the theorem and induction it follows that $B_1^{out}, .., B_{k-1}^{out}$ touch A'_t in counterclockwise order. Within a bundle B_i^{in} we label the segments as $s_{i,k}^{in}, .., s_{i,m}^{in}$, again counterclockwise. The segment in B_i^{out} which was connected to $s_{i,r}^{in}$ is labeled $s_{i,r}^{out}$ The pieces $s_{i,r}^{in}$ and $s_{i,r}^{out}$ will be part of the pseudosegment representing the path $P_{i,r}$.

To have property (b) for the pseudosegments of a bundle we twist whichever of the bundles B_i^{in} or B_i^{out} traverses R_i within this disk R_i thus creating a multiple intersection point. Note that they all cross $s_{i,i}$ as did $s_{i,0}$.

Also due to (b) the pseudosegments of paths $P_{i,r}$ for fixed $r \in \{k, .., m\}$ have to intersect in the disks R_r inside of the box Γ. To prepare for this we take a narrow bundle of $k-1$ parallel vertical segments reaching from top to bottom of the box Γ and intersecting the disk R_r. This bundle is twisted in the interior of R_r. Let $\breve{a}_1^r, .., \breve{a}_{k-1}^r$ be the bottom endpoints of this bundle from left to right and let $\hat{a}_1^r, .., \hat{a}_{k-1}^r$ be the top endpoints from right to left, due to the twist the endpoints $\breve{a}_j^r$ and $\hat{a}_j^r$ belong to the same pseudosegment.

We are ready now to construct the pseudosegment $s_{i,r}$ that will represent the path $P_{i,r}$ in T for $1 \leq i < k \leq r \leq m$. The first part of $s_{i,r}$ is $s_{i,r}^{in}$, this pseudosegment is part of the bundle B_i^{in} and has an endpoint on A'_b. Connect this endpoint with a straight segment to $\breve{a}_i^r$, from this point there is the connection up to $\hat{a}_i^r$. This point is again connected by a straight segment to the endpoint of $s_{i,r}^{out}$ on the arc A'_t. The last part of $s_{i,r}$ is the pseudosegment $s_{i,r}^{out}$ in the bundle B_i^{out}. The construction is illustrated in Figure 5.

The following list of claims collects some crucial properties of the construction.

Claim 1. There is exactly one pseudosegment $s_{i,j}$ for every pair l_i, l_j of leaves of T.
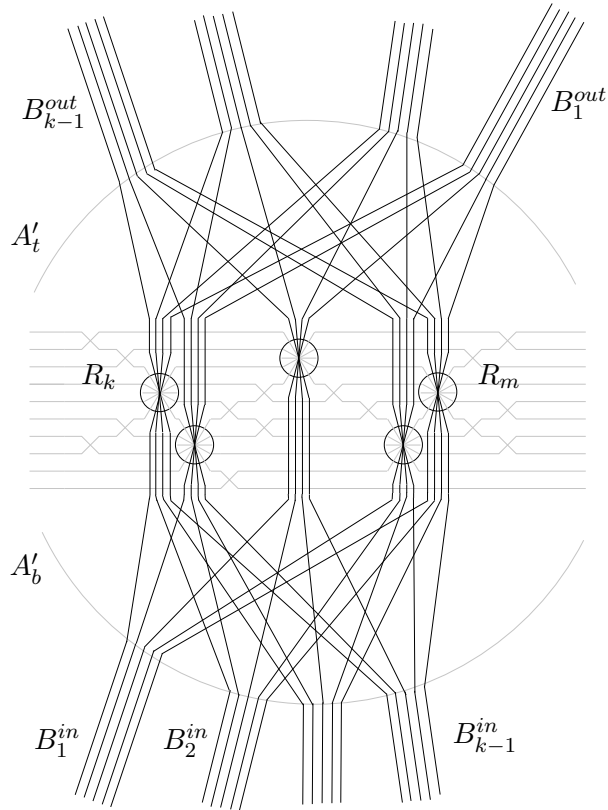
Figure 5: The routing of pseudosegments in the disk R_0, an example.

Claim 2. The pseudosegment $s_{i,j}$ traverses R_i and R_j but stays disjoint from every other of the disks.

Claim 3. Any two pseudosegments intersecting the disk R_i cross in R_i.

Claim 4. Two pseudosegments $s_{i,j}$ and $s_{i',j'}$ intersect exactly if the corresponding paths $P_{i,j}$ and $P_{i',j'}$ intersect in T.

Claim 5. Two pseudosegments $s_{i,j}$ and $s_{i',j'}$ intersect at most once, i.e., to call them pseudosegments is justified.

These claims follow by induction. Hence the construction indeed yields a representation of G as intersection graph of pseudosegments and this representation has properties (a) and (b).

3 Chordal graphs that are not PSI

Recall the definition of the graphs K_n^3: The graph K_n^3 has two groups of vertices. The set $V_C = [n]$ induces a clique of K_n^3 and in addition every triple $\{i, j, k\} \subset [n]$ is a vertex adjacent only to the three vertices i, j and k, hence, the triples form an independent set V_I. In the introduction we have already described tree-representations of K_n^3, in particular K_n^3 is chordal.

Theorem 3 *For $n \geq 33$ the graph K_n^3 admits no PSI-representation.*

Assume that there is a representation of K_n^3 as intersection graph of pseudosegments. Let PS_C and PS_I be the sets of pseudosegments representing vertices from V_C and V_I.

The pseudosegments in PS_C form a set of pairwise crossing pseudosegments, we refer to the configuration of these pseudosegments as the arrangement A_n. The set $S = PS_I$ of *small* pseudosegments has the following properties:

(i) Any two pseudosegments $t \neq t'$ from S are disjoint,

(ii) Every pseudosegment $t \in S$ has nonempty intersection with exactly three pseudosegments from the arrangement A_n and no two pseudosegments $t \neq t'$ intersect the same three pseudosegments from A_n.

The idea for the proof is to show that a set of pseudosegments with properties (i) and (ii) only has $O(n^2)$ elements. The theorem follows, since $|S| = \binom{n}{3} = \Omega(n^3)$.

3.1 K_n^3 and planar graphs

Every pseudosegment $p \in A_n$ is cut into n pieces by the $n-1$ other pseudosegments of A_n. Let W be the set of all the pieces obtained from pseudosegments from A_n, note that $|W| = n^2$. Pseudosegments in S intersect exactly three pieces from three different pseudosegments of A_n. Elements of S are called *3-segments*. Every 3-segment has a unique middle and two outer intersections. Let $S(w)$ be the set of 3-segments with middle intersection on the piece $w \in W$. This yields the partition $S = \bigcup_{w \in W} S(w)$ of the set of 3-segments.

Define $G_p = (W, E_p)$ as the simple graph where two pieces w, w' are adjacent if and only if there exists a 3-segment $t \in S$ such that t has its middle intersection on w and an outer intersection on w'.

Lemma 2 $G_p = (W, E_p)$ *is planar.*

Proof. A planar embedding of G_p is induced by A_n and S. Contract all pieces from A_n, the contracted pieces represent the vertices of G_p. The 3-segments are pairwise non-crossing, this property is maintained during contraction of pieces, see Figure 6. If a 3-segment $t \in S$ has middle piece w and outer pieces w' and w'', then t contributes the two edges (w, w') and (w, w'') to G_p. Hence, the multi-graph obtained through these contractions is planar and its underlying simple graph is indeed G_p. $\square$
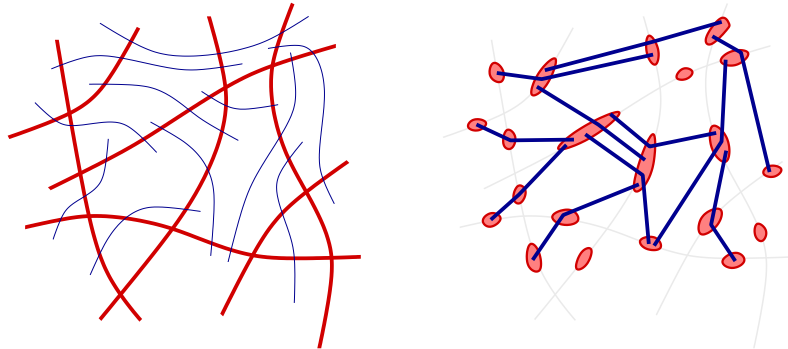

Figure 6: A part of A_n with some 3-segments and the edges of G_p induced by them.

Let $N(w)$ be the set of neighbors of $w \in W$ in G_p and let $d_{G_p}(w) = |N(w)|$.

Lemma 3 *The size of a set $S(w)$ of 3-segments with middle piece w is bounded by $3d_{G_p}(w)-6$ for every $w \in W$.*

Proof. The idea is to ignore all pieces that do not belong to elements in $N(w)$ and to contract the pieces corresponding to elements of $N(w)$ to points. The 3-segments in $S(w)$ together with the vertices obtained by contraction form a planar graph denoted by NG_w, see Figure 7. Note that this graph is not a multi-graph: every 3-segment leading to an edge also intersects the piece w. The number of vertices of this graph is $d_{G_p}(w)$, hence, there are at most $3d_{G_p}(w)-6$ edges. □

In [3] we have shown the stronger statement, that the graph NG_w is cycle free, hence a forest. This implies that the number of edges is at most $d_{G_p}(w) - 1$. We omit the more involved proof. Still we will use the stronger bound in the following calculations. Following the computation given below the weaker bound of Lemma 3 implies that K_n^3 is not PSI-representable for $n \geq 75$.
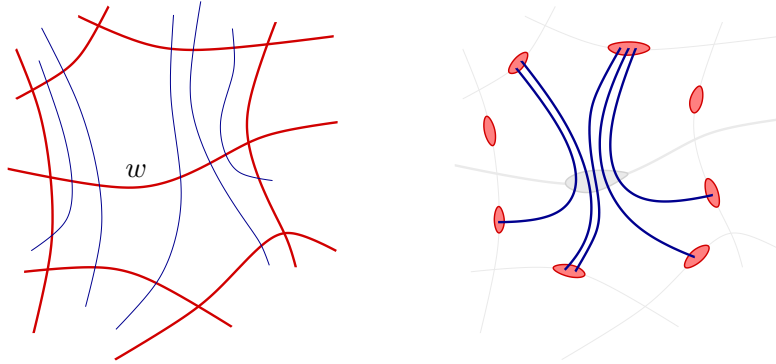

Figure 7: The planar graph NG_w induced by the 3-segments with middle intersection.

Recall that from simple counting and from the planarity of G_p we have:

- $|W| = n^2$,
- $\sum_{w \in W} d_{G_p}(w) = 2|E_p| < 6|V_p| = 6|W|$.

Using $|S(w)| \leq d_{G_p}(w) - 1$ we thus obtain:

- $|S| = \sum_{w \in W} |S(w)| \leq \sum_{w \in W}(d_{G_p}(w) - 1) < 6|W| - |W| = 5n^2$.

Since $\binom{n}{3} > 5n^2$ for all $n \geq 33$ we conclude that K_n^3 does not belong to PSI for $n \geq 33$. This completes the proof of Theorem 3. □

4 k-Segments in Arrangements of Pseudolines

In the previous section we have shown that the chordal graph K_n^3, $n \geq 33$, has no PSI-representation. For this purpose we partitioned the graph K_n^3 into two induced subgraphs, the maximal clique and the maximal independent set. In any PSI-representation of K_n^3, the clique has to correspond to an arrangement, that is a set of n pairwise intersecting pseudosegments. Let $\mathcal{A}_n$ denote a representation of the clique. To obtain a PSI-representation of the whole graph K_n^3, it is necessary to assign the $\binom{n}{3}$ vertices of the independent set to a set of disjoint

pseudosegments (*triple-segments*) in the base arrangement $\mathcal{A}_n$. What if we omit the condition of disjointness for the triple-segments?

To be more precise: Let $\mathcal{A}_n$ be an arrangement of n pseudosegments and $T \subset \binom{[n]}{3}$ be a set of triples. We say that T *is hosted on* $\mathcal{A}_n$ if for every $t \in T \subset \binom{[n]}{3}$ there is an S_t such that $\mathcal{A}_n \cup \{S_t : t \in T\}$ is a family of pseudosegments and the segment S_t with $t = \{i, j, k\}$ is intersecting the lines labeled i, j, k in $\mathcal{A}_n$ and no others.

Problem 1 *What is the growth of the largest function $f_3(n)$ such that there is a family $T \subset \binom{[n]}{3}$ hosted on some $\mathcal{A}_n$ with $f_3(n) = |T|$?*

Although we conjecture that $f_3(n)$ is $o(n^{2+\epsilon})$ for all $\epsilon > 0$ we have not even been able to show that $f_3(n)$ is $o(n^3)$.

The results of this section were motivated by Problem 1. We have simplified the situation by dealing with arrangements of pseudolines instead of arrangements of pseudosegments. The advantage of this model lies in the fact that the elements of the arrangement cannot be bypassed at their ends. We think that this more geometric model is of independent interest.

Definition 1 *A* pseudoline *in the Euclidean plane is a simple curve that approaches a point at infinity in either direction. An* arrangement of pseudolines *is a family of pseudolines with the property that each pair of pseudolines has a unique point of intersection, where the two pseudolines cross.*

We now ask: How many triple-segments can an arrangement $\mathcal{A}_n$ of n pseudolines host?

A pseudoline in an arrangement of pseudolines is split into a sequence of edges and vertices by the crossing lines.

Definition 2 *A k-segment* in an arrangement $\mathcal{A}$ of pseudolines is a sequence $e_1, e_2, \ldots, e_k$ of edges from k different pseudolines of $\mathcal{A}_n$ with the property that there exists a curve crossing these edges in the given order that has no further intersections with $\mathcal{A}$.
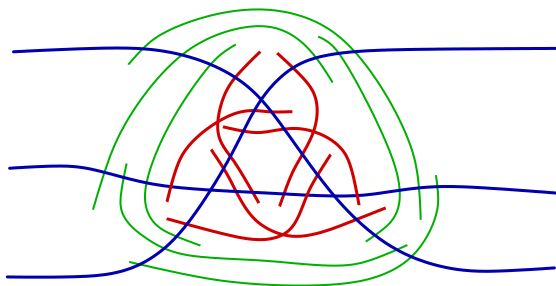

Figure 8: An arrangement of 3 pseudolines and representative curves for all its 3-segments.

The main theorem of this section is:

Theorem 4 *For k fixed the number of different k-segments in an arrangement of n pseudolines is at most $c^k n^2 \in O(n^2)$ for some c.*

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Compared to the setting of Problem 1 we now consider k-segments instead of triple-segments, moreover, we can count the same set of k pseudolines with many k-segments and we have dropped the condition that the k-segments must be compatible, their representing curves may cross any number of times.

If we drop the compatibility restriction for the k-segments in the case where the clique is represented by a collection of pseudosegments (instead of pseudolines), then all k subsets become representable, in particular for $k = 3$ the number of representable triples becomes $\Theta(n^3)$. For a construction place the segments of the clique in the halfplane $x \leq 0$ with an endpoint on the line $x = 0$ and route the k-segments in the halfplane $x \geq -\varepsilon$.

The proof of Theorem 4 It is based the notion of k-zones in arrangements.

Definition 3 *Let $\mathcal{A}$ be an arrangement of pseudolines, $p \in \mathcal{A}$ and $k \geq 2$. The k-zone of p is defined as the collection of vertices, edges and faces of $\mathcal{A}$ that can be connected to p by a curve that has at most k intersections with lines of $\mathcal{A} \setminus p$.*

We are interested in the edges of the k-zone. They can also be defined as the set of edges that are contained in a $(k + 1)$-segment whose initial edge is on p. With $\mathrm{Zone}_k(p)$ we denote the set of edges of the k-zone of p.

Theorem 5 *The number of edges in $\mathrm{Zone}_k(p)$ is at most $36(k + 1)n$, i.e., it is in $O(nk)$.*

The proof is an easy consequence of the theory developed in Chapter 6 of Matoušek's book [14]. The proof is like the proof of Theorem 6.3.1 in that book. In addition we need the classical 2-D Zone Theorem: $\mathrm{Zone}_0(p) \leq 6n$, and the observation that in a simple arrangement $\mathrm{Zone}_k(p)$ is at most twice the number of vertices of the k-zone.

We now come to the proof of Theorem 4. To bound the number of k-segments we proceed as follows: Consider p and some $e \in \mathrm{Zone}_{k-1}(p)$. There are n choices for p. Theorem 5 bounds the number of choices for e. From Lemma 4 below we get a bound on the number of k-segments that have e as extremal edge and the other extremal edge on p. This estimate counts every k-segment twice, hence

$$\#k\text{-segments} < n \cdot \left(36(k+1)n\right) \cdot k \left(\frac{5}{2}\right)^{k-2} \cdot \frac{1}{2} < 3(k+1)\left(\frac{5}{2}\right)^k n^2.$$

Lemma 4 *If e is an edge of $\mathcal{A}$ not on p then then there are at most $k \left(\frac{5}{2}\right)^{k-2}$ k-segments that have e as extremal edge and the other extremal edge on p.*

Proof. Any two curves that represent k-segments with e and e' as extremal edges cross the same $k - 2$ lines. This can be shown by considering the region enclosed by the two curves. It follows that we can view the k-segments with e and e' as extremal edges as *cut-paths* in an arrangement of $k - 2$ pseudolines. In [9] Knuth proves the upper bound 3^n for the maximum number of cut-paths in arrangements of n pseudolines. A simpler proof and the improved upper bound of $(5/2)^n$ can be found in [6].

It remains to show that there are at most k choices for an edge e_p on p such that e and e_p are the extremal edges of a k-segment.

Let γ and γ' be representatives of k-segments whose extremal edges are e and some edges on p. If γ and γ' are disjoint there is a clear notion of which is left and which is right. If γ and γ' intersect then there exist two representatives of k-segments γ_l and γ_r such that γ_l is

left of both and γ_r is right of both. This implies that there is a leftmost S_l and a rightmost S_r among these k-segments. Consider the interval I on p between the edge $p \cap S_l$ and the edge $e \cap S_r$. Every line intersecting p in I must have an edge on S_l or S_r. Since p is out of question and edge e belongs to both there are at most $2(k-2)+1$ vertices on I, hence, I contains at most $2k-2$ edges of p. Figure 9 shows an example.

Finally, observe that only every other edge on p can be extreme for a k-segment connecting e to p. The only exception is with two edges incident to the vertex where the supporting line of e intersects p. This shows that from the $2k-2$ candidate edges on I only at most k can indeed contribute to one of the k-segments. □
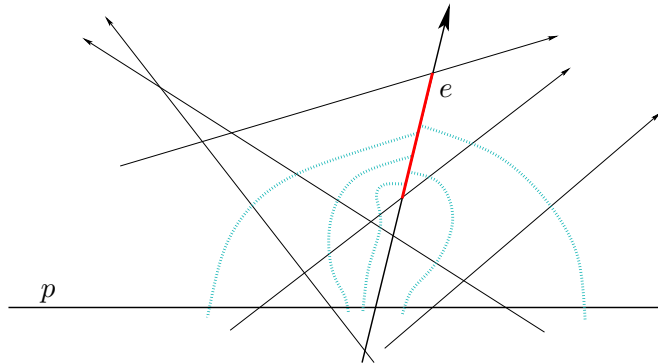

Figure 9: A line p, an edge e and curves representing all 4-segments with extreme edges on e and on p.

Improved bounds for the number of cut-paths in an arrangement imply improved bounds for the number of arrangements of n pseudolines [6].

From the point of view of our investigations the following also seems to be very interesting: Call a set of cut-paths *compatible* if they can be represented by a set of curves such that every pair of curves from the set has at most one intersection.

Problem 2 *How large can a family of compatible cut-paths of an arrangement of n pseudolines be?*

5 Chordal Graphs that are not PSI Revisited: An Application of Ramsey Theory

In Theorem 3 we have seen that even in the class of chordal graphs which can be represented by substars of a star there are graphs which are not PSI. If the degree of the subtrees is bounded by 2, then the subtrees are just paths and the graphs are PSI by Theorem 1.

What if we restrict the host tree to be a star and the substars to be of degree three? Again we have graphs in the class that are not PSI. This is the topic of this section.

Theorem 6 *Let S_n be the chordal graph whose vertices are represented by all substars with three leaves and all leaves on a star with n leaves. For n large enough S_n is not PSI-representable*

Suppose that there is a PSI representation of S_n. Let $\mathcal{D}$ be the set of pseudosegments representing the leaves of the star, these segments are pairwise disjoint. To simplify the picture we use an homeomorphism of the plane that aligns the pseudosegments of $\mathcal{D}$ as vertical segments of unit length which touch the X-axis with their lower endpoints at positions $1, 2, \ldots, n$. For ease of reference we will call these vertical segments *sticks* and number them such that p_i is the stick containing the point $(i, 0)$.

With every ordered triple (i, j, k), $1 \leq i < j < k \leq n$, there is a 3-segment γ_{ijk} intersecting the sticks p_i, p_j and p_k from $\mathcal{D}$. Let ϕ_{ijk} denote the middle of the three sticks intersected by γ_{ijk}. We partition the ordered triples (i, j, k) into three classes depending of the position of ϕ_{ijk} in the list (p_i, p_j, p_k). If $\phi_{ijk} = p_i$, i.e., the middle intersection of γ_{ijk} is left of the other two, we assign (i, j, k) to class $[L]$. The class of (i, j, k) is $[M]$ if $\phi_{ijk} = p_j$, i.e., the middle intersection of γ_{ijk} is between the other two. The class of (i, j, k) is $[R]$ if $\phi_{ijk} = p_k$, i.e., the middle intersection of γ_{ijk} is to the right of the other two. We use this notation rather flexible and also write $\gamma_{ijk} \in [X]$ or say that γ_{ijk} is of class $[X]$ when the triple (i, j, k) is of class $[X]$, for $X \in L, M, R$.
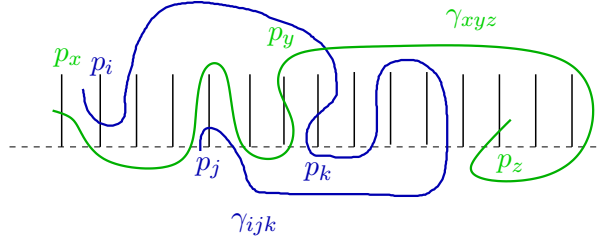

Figure 10: Two 3-segments γ_{ijk} and γ_{xyz}. Note that $\phi_{ijk} = p_k$ and $\phi_{xyz} = p_y$, hence, $\gamma_{ijk} \in [R]$ and $\gamma_{xyz} \in [M]$.

Cutting γ_{ijk} at the intersection points with the three sticks yields two arcs γ_{ijk}^1 and γ_{ijk}^2 each connecting two sticks and up to two ends. The ends are of no further interest. For the arcs we adopt the convention that γ_{ijk}^1 connects ϕ_{ijk} to the stick further left and and γ_{ijk}^2 connects ϕ_{ijk} to the stick further right. In the example of Figure 10 $\phi_{ijk} = p_k$ so that γ_{ijk}^1 is the arc connecting p_i and p_k while arc γ_{ijk}^2 connects p_j and p_k.

For a contradiction we will show that if n is large enough there have to be two 3-segments γ_{ijk} and γ_{xyz} such that γ_{ijk}^1 and γ_{xyz}^1 intersect and γ_{ijk}^2 and γ_{xyz}^2 intersect, hence, γ_{ijk} and γ_{xyz} intersect at least twice which is not allowed in a PSI-representation.

To get to that contradiction we need some control over the behavior of 3-segments between the sticks. Let $\vec{r}_x$ be a vertical ray downwards starting at $(x, 0)$, i.e., the ray pointing down from the lower end of stick p_x. For $s = 1, 2$ let $I_x^s(ijk)$ be the number of intersections of ray $\vec{r}_x$ with γ_{ijk}^s and let $J_x^s(ijk)$ be the parity of $I_x^s(ijk)$, i.e., $J_x^s(ijk) = I_x^s(ijk) \pmod 2$.

Given an ordered 7-tuple (a, i, b, j, c, k, d), $1 \leq a < i < b < j < c < k < d \leq n$, we call γ_{ijk} the induced 3-segment and define $T_x^s = J_x^s(ijk)$. The *pattern* of the tuple is the binary 8-tuple

$$(T_a^1, T_b^1, T_c^1, T_d^1, T_a^2, T_b^2, T_c^2, T_d^2).$$

The *color* of a 7-tuple (a, i, b, j, c, k, d) is the pair consisting of the class of the induced 3-segment and the pattern. The 7-tuples are thus colored with the 768 colors from the set
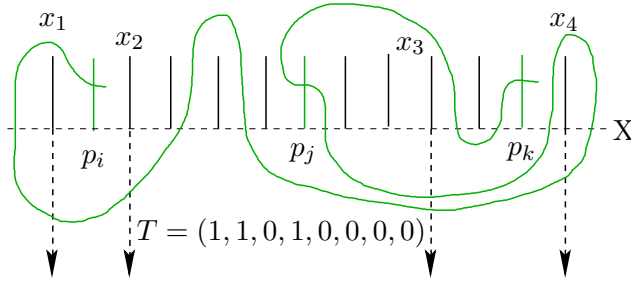
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Figure 11: A pattern associated to a 3-segment γ_{ijk}.

$[3] \times \mathbf{2}^8$. Ordered 7-tuples and 7-element subsets of $[n]$ are essentially the same. Therefore we can apply the hypergraph Ramsey theorem with parameters $768, 7, 13$.

Theorem 7 (Hypergraph Ramsey) *For all numbers r, p, k there exists a number N such that whenever X is an n-element set with $n \geq N$ and c is a coloring of the system of all p-element subsets of X with r colors, i.e. $c : \binom{X}{p} \to \{1, 2, .., r\}$, then there is an k-element subset $Y \subseteq X$ such that all the p-subsets in $\binom{Y}{p}$ have the same color.*

Given two curves γ and γ', closed or not, we let $X(\gamma, \gamma')$ be the number of crossing points of the two curves. The following is essential for the argument:

Fact 1 *If γ and γ' are closed curves, then $X(\gamma, \gamma') \equiv 0 \pmod{2}$.*

With an arc γ_{ij} connecting sticks p_i and p_j we associate a closed curve $\breve{\gamma}_{ij}$ as follows: At the intersection of γ_{ij} with either of the sticks we append long vertical segments and connect the lower endpoints of these two segments horizontally. The union of the three connecting segments will be called the *bow* β_{ij} of the curve $\breve{\gamma}_{ij}$. If this construction is applied to several arcs we assume that the vertical segments of the bows are long enough as to avoid any intersection between the arcs and the horizontal part of the bows.

Given $\breve{\gamma}_{ij}$ and $\breve{\gamma}_{xy}$ we can count their crossings in parts:

$$X(\breve{\gamma}_{ij}, \breve{\gamma}_{xy}) = X(\gamma_{ij}, \gamma_{xy}) + X(\gamma_{ij}, \beta_{xy}) + X(\beta_{ij}, \gamma_{xy}) + X(\beta_{ij}, \beta_{xy})$$

With Fact 1 we obtain

Fact 2 $X(\gamma_{ij}, \gamma_{xy}) \equiv X(\gamma_{ij}, \beta_{xy}) + X(\beta_{ij}, \gamma_{xy}) + X(\beta_{ij}, \beta_{xy}) \pmod{2}$.

The application of the Ramsey theorem left us with a uniformized configuration. We have kept only a subset Y of sticks such that all 3-segments connecting three of them are of the same class and all 7-tuples on Y have the same pattern $T = (T_1^1, T_2^1, T_3^1, T_4^1, T_1^2, T_2^2, T_3^2, T_4^2)$.

The uniformity allows to apply Fact 2 to detect intersections of arcs of type γ_{ijk}^s. Depending on the entries of the pattern T we choose two appropriate 3-segments γ_{ijk} and γ_{xyz} and show that they intersect twice. Assume that the class of all 3-segments is $[L]$ or $[M]$, hence there is an arc connecting the two sticks with smaller indices. Let $\gamma_{ij} = \gamma_{ijk}^1$, and $\gamma_{xy} = \gamma_{xyz}^1$.

Lemma 5 *If $T_1^1 = T_3^1$ and $i < x < j < y < k$, then there is an intersection between the arcs γ_{ij} and γ_{xy}.*

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Proof. We evaluate the right side of the congruence given in Fact 2.

$X(\gamma_{ij}, \beta_{xy})$ is the number of intersections of arc γ_{ij} with the bow connecting p_x and p_y. These intersections happen on the vertical part, hence on the rays $\vec{r}_x$ and $\vec{r}_y$. The parity of these intersections can be read from the pattern. The position of x between i and j implies $T_x^1 = T_2^1$ and the position of y between j and k implies $T_y^1 = T_3^1$. Hence, $X(\gamma_{ij}, \beta_{xy}) \equiv T_2^1 + T_3^1$ (mod 2).

From the positions of i left of x and of j between x and y we conclude that $X(\beta_{ij}, \gamma_{xy}) \equiv T_1^1 + T_2^1$ (mod 2).

Since the pairs ij and xy interleave the two bows are intersecting, i.e., $X(\beta_{ij}, \beta_{xy}) = 1$.

Together this yields $X(\gamma_{ij}, \gamma_{xy}) \equiv T_2^1 + T_3^1 + T_1^1 + T_2^1 + 1$ (mod 2). With $T_1^1 = T_3^1$ we see that $X(\gamma_{ij}, \gamma_{xy})$ is odd, hence, there is at least one intersection between the arcs. $\square$

Lemma 6 *If $T_1^1 \neq T_3^1$ and $x < i < j < y < k$, then there is an intersection between the arcs γ_{ij} and γ_{xy}.*

Proof. Since x is left of i and y is between j and k we obtain $X(\gamma_{ij}, \beta_{xy}) \equiv T_1^1 + T_3^1$ (mod 2). Both i and j are between x and y, thus $X(\beta_{ij}, \gamma_{xy}) \equiv T_2^1 + T_2^1 \equiv 0$ (mod 2). For the bows we observe that either they don't intersect or they intersect twice, in both cases $X(\beta_{ij}, \beta_{xy}) \equiv 0$ (mod 2).

Put together $X(\gamma_{ij}, \gamma_{xy}) \equiv T_1^1 + T_3^1$ (mod 2). With $T_1^1 \neq T_3^1$ we see that $X(\gamma_{ij}, \gamma_{xy})$ is odd, hence, there is at least one intersection between the arcs. $\square$

Now consider the case where the class of all 3-segments is $[M]$. In addition to the arcs γ_{ij} and γ_{xy} we have the arcs $\gamma_{jk} = \gamma_{ijk}^2$, and $\gamma_{yz} = \gamma_{xyz}^2$. The following two lemmas are counterparts to lemmas 5 and 6. They show that depending on the parity of $T_1^2 + T_3^2$ an alternating or a non-alternating choice of jk and yz force an intersection of the arcs γ_{jk} and γ_{yz}. For the proofs note that reflection at the y-axis keeps class $[M]$ invariant but exchanges the first and the second arc, the relevant effect on the pattern is $T_1^1 \leftrightarrow T_4^2$ and $T_3^1 \leftrightarrow T_2^2$.

Lemma 7 *If $T_2^2 = T_4^2$ and $x < j < y < k < z$, then there is an intersection between the arcs γ_{jk} and γ_{yz}.*

Lemma 8 *If $T_2^2 \neq T_4^2$ and $x < j < y < z < k$, then there is an intersection between the arcs γ_{jk} and γ_{yz}.*

The table below shows that it is possible to select ijk and xyz out of six numbers such that the positions of ij and xy respectively jk and yz are any combination of alternating and non-alternating. Hence, according to the lemmas we have at least two intersections between 3-segments γ_{ijk} and γ_{xyz} chosen appropriately depending on the entries of pattern T. We represent elements of ijk by a box $\square$ and elements of xyz by circles $\bullet$.

$\square$	$\bullet$	$\square$	$\bullet$	$\square$	$\bullet$	alt / alt	$[T_1^1 = T_3^1$ and $T_2^2 = T_4^2]$
$\square$	$\bullet$	$\square$	$\bullet$	$\bullet$	$\square$	alt / non-alt	$[T_1^1 = T_3^1$ and $T_2^2 \neq T_4^2]$
$\bullet$	$\square$	$\square$	$\bullet$	$\square$	$\bullet$	non-alt / alt	$[T_1^1 \neq T_3^1$ and $T_2^2 = T_4^2]$
$\bullet$	$\square$	$\square$	$\bullet$	$\bullet$	$\square$	non-alt / non-alt	$[T_1^1 \neq T_3^1$ and $T_2^2 \neq T_4^2]$

Now consider the case where the class of all 3-segments is $[L]$. In addition to the arcs γ_{ij} and γ_{xy} we have the arcs $\gamma_{ik} = \gamma_{ijk}^2$, and $\gamma_{xz} = \gamma_{xyz}^2$. The following two lemmas show that depending on the parity of $T_1^2 + T_2^2 + T_3^2 + T_4^2$ an alternating or a non-alternating choice of ik and xz force an intersection of the arcs γ_{ik} and γ_{xz}.

Lemma 9 *If $T_1^2 + T_2^2 + T_3^2 + T_4^2 \equiv 0 \pmod 2$ and $i < x < \{j,y\} < k < z$, then there is an intersection between the arcs γ_{ik} and γ_{xz}.*

Proof. Since x is between i and j and z is right of k we obtain $X(\gamma_{ik}, \beta_{xz}) \equiv T_2^2 + T_4^2 \pmod 2$. Since i is left of x and k is between y and z we obtain $X(\beta_{ik}, \gamma_{xz}) \equiv T_1^2 + T_3^2 \pmod 2$. Since the pairs ik and xz interleave the two bows are intersecting, i.e., $X(\beta_{ik}, \beta_{xz}) = 1$.

Put together $X(\gamma_{ij}, \gamma_{xy}) \equiv T_1^2 + T_2^2 + T_3^2 + T_4^2 + 1 \pmod 2$. Hence there is at least one intersection between the arcs. □

Lemma 10 *If $T_1^2 + T_2^2 + T_3^2 + T_4^2 \equiv 1 \pmod 2$ and $i < x < \{j,y\} < z < k$, then there is an intersection between the arcs γ_{ik} and γ_{xz}.*

Proof. Since x is between i and j and z is between j and k we have $X(\gamma_{ik}, \beta_{xz}) \equiv T_2^2 + T_3^2 \pmod 2$. Since i is left of x and k is right of z we have $X(\beta_{ik}, \gamma_{xz}) \equiv T_1^2 + T_4^2 \pmod 2$. For the bows we observe that either they don't intersect or they intersect twice, hence $X(\beta_{ik}, \beta_{xz}) \equiv 0 \pmod 2$.

Put together $X(\gamma_{ij}, \gamma_{xy}) \equiv T_1^2 + T_2^2 + T_3^2 + T_4^2 \pmod 2$. Hence there is at least one intersection between the arcs. □

As in the previous case we provide a table showing that it is possible to select ijk and xyz out of six numbers such that the positions of ij and xy respectively ik and xz are any combination of alternating and non-alternating. We represent elements of ijk by a box □ and elements of xyz by circles •.

$$
\begin{array}{lll}
□\ •\ □\ •\ □\ • & \text{alt / alt} & [T_1^1 = T_3^1 \text{ and } T_1^2 + T_2^2 + T_3^2 + T_4^2 \equiv 0] \\
□\ •\ □\ •\ •\ □ & \text{alt / non-alt} & [T_1^1 = T_3^1 \text{ and } T_1^2 + T_2^2 + T_3^2 + T_4^2 \equiv 1] \\
•\ □\ □\ •\ •\ □ & \text{non-alt / alt} & [T_1^1 \neq T_3^1 \text{ and } T_1^2 + T_2^2 + T_3^2 + T_4^2 \equiv 0] \\
•\ □\ □\ •\ □\ • & \text{non-alt / non-alt} & [T_1^1 \neq T_3^1 \text{ and } T_1^2 + T_2^2 + T_3^2 + T_4^2 \equiv 1]
\end{array}
$$

To deal with the case where the class of all 3-segments is $[R]$ we refer to symmetry. Reflecting the picture at the y-axis yields a configuration which is in class $[L]$. Hence it is impossible to have a uniform configuration on six or more sticks. Well, the definition of the pattern alone involves seven sticks. This is why we said that we want to have a uniform family on 13 sticks, if we use the odd numbered sticks for the choice of triples there are enough candidates to complete the selection of seven positions for a pattern. This completes the proof of Theorem 6.

6   Conclusion

Besides the problems that have already been presented in the text we would like to mention some more. In a large part of the paper we have been considering questions of the following general type: How big can a family W of 3-segments living on a base arrangement B of size n be?

- In Theorem 3 we requested that the 3-segments are disjoint.
- In Theorem 4 we requested that B is given by an arrangement of pseudolines.

- In Theorem 6 we requested that the segments in B are disjoint and that the segments in W are compatible.

The theorems give upper bounds. Good lower bound constructions might give some additional insight that can help improve the upper bounds. In fact we think that in the situation of Theorem 6 the true size of W should be in $o(n^{2+\epsilon})$.

References

[1] J. Chalopin and D. Gonçalves, *Every planar graph is the intersection graph of segments in the plane*, Proc. of the 41st ACM-Symp. Theory of Computing, STOC, (2009), pp. 631–638.

[2] J. Chalopin, D. Gonçalves, and P. Ochem, *Planar graphs are in 1-STRING*, Proc. of the 18th ACM-SIAM Symp. on Discr. Alg., SODA, (2007), pp. 609–617.

[3] C. Dangelmayr and S. Felsner, *Chordal graphs as intersection graphs of pseudosegments*, in Proceedings of Graph Drawing 2006, vol. 4372 of Lect. Notes Comput. Sci., Springer Verlag, 2006, pp. 208–219.

[4] N. de Castro, F. J. Cobos, J. C. Dana, A. Márquez, and M. Noy, *Triangle-free planar graphs as segment intersection graphs*, J. Graph Algor. and Appl., 6 (2002), pp. 7–26.

[5] G. Ehrich, S. Even, and R. E. Tarjan, *Intersection graphs of curves in the plane*, J. Combin. Theory Ser. B, 21 (1976), pp. 8–20.

[6] S. Felsner and P. Valtr, *Coding and counting arrangements of pseudolines*. submitted, 2009.

[7] F. Gavril, *The intersection graphs of subtrees in trees are exactly the chordal graphs*, J. Combin. Theory Ser. B, 16 (1974), pp. 47–56.

[8] F. Gavril, *A recognition algorithm for the intersection graphs of paths in trees*, Discr. Math., 23 (1978), pp. 211–227.

[9] D. E. Knuth, *Axioms and Hulls*, Lecture Notes in Computer Science, vol. 606, Springer-Verlag, 1992.

[10] J. Kratochvíl, *String graphs ii: Recognizing string graphs is NP-hard*, J. Combin. Theory Ser. B, 52 (1991), pp. 67–78.

[11] J. Kratochvíl, *A special planar satisfiability problem and a consequence of its NP-completeness*, Discr. Appl. Math., 52 (1994), pp. 233–252.

[12] J. Kratochvíl and J. Matoušek, *NP-hardness results for intersection graphs*, Comment. Math. Univ. Carolin, 30 (1989), pp. 761–773.

[13] J. Kratochvíl and J. Matoušek, *Intersection graphs of segments*, J. Combin. Theory Ser. B, 62 (1994), pp. 289–315.

[14] J. MATOUŠEK, *Lectures on Discrete Geometry*, vol. 212 of Graduate Texts in Mathematics, Springer-Verlag, 2002.

[15] C. MONMA AND V. K. WEI, *Intersection graphs of paths in a tree*, J. Combin. Theory Ser. B, 41 (1986), pp. 141–181.

[16] J. PACH AND G. TÓTH, *Recognizing string graphs is decidable*, Discr. and Comput. Geom., 28 (2002), pp. 593–606.

[17] M. SCHAEFER, E. SEDGWICK, AND D. ŠTEFANKOVIČ, *Recognizing string graphs in NP*, J. Comput. Syst. Sci., 2 (2003), pp. 365–380.

[18] M. SCHAEFER AND D. ŠTEFANKOVIČ, *Decidability of string graphs*, J. Comput. Syst. Sci, 68 (2004), pp. 319–334.

[19] E. R. SCHEINERMAN, *Intersection classes and multiple intersection parameters of graphs*, PhD thesis, Princeton University, 1984.