# On the Erdős-Szekeres Problem for Convex Permutations and Orthogonally Convex Point Sets

Heather S. Blake [B] Stefan Felsner [F] Rimma Hämäläinen [H]
Marcin Witkowski [W]

A finite set of points is *generic* if no two points are on the same vertical or horizontal line. The set is *orthogonally convex* if every point has an empty quadrant. We study the smallest integer $N_o(n)$ such that every generic set of $N_o(n)$ points contains a orthogonally convex subset of size $n$. For even $n$ we know the exact value, and in the odd case we get close upper and lower bounds. Generic sets correspond to permutations in a canonical way. A permutation is *convex* if it is order isomorphic to a finite generic set of points in convex position. The value of $N_o(n)$ is also the smallest $N$ such that every permutation of $N$ contains a convex subpermutation of size $n$.

## 1 Introduction

In 1935, Erdős and Szekeres [2] proved that for each $n$ there there exists a smallest positive integer $N(n)$ such that every set of at least $N(n)$ points in the plane in general position contains a subset of $n$ points in convex position. They proved the upper bound of $N(n) \leq \binom{2n-4}{n-2} + 1$ and conjectured that $N(n) = 2^{n-2} + 1$. In 2017, Suk [10] almost settled the Erdős-Szekeres conjecture by showing that $N(n) \leq 2^{n+o(n)}$.

A finite set $X$ of points in the plane is in *orthogonally convex position* if every point $x \in X$ has a quadrant $Q_x$ such that the intersection of $Q_x$ with $X$ is just $x$. We precisely determine the smallest positive integer $N_o(n)$ such that every set of at least $N_o(n)$ points in the plane in *generic position*, i.e., no two points on a common horizontal or vertical line, contains a subset of $n$ points in convex position. In contrast to the situation in the classical Erdős and Szekeres problem, the growth of $N_o(n)$ is only quadratic.

[B] Department of Mathematics and Computer Science, Davidson College, Davidson, NC, USA
E-mail: hsblake@davidson.edu
[F] Discrete Mathematics Group, Institut für Mathematik, Technische Universität Berlin, Germany
E-mail: felsner@math.tu-berlin.de
[H] Work completed while affiliated with the Discrete Mathematics Group, Institut für Mathematik, Technische Universität Berlin, Germany
E-mail: rimmahamalainen231@gmail.com
[W] Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Poznan, Poland
E-mail: mw@amu.edu.pl

**Theorem 1.1.** *For each $n \geq 4$ there exists a smallest positive integer $N_o(n)$ such that each finite set of $N_o(n)$ points in generic position contains an orthogonally convex subset of size $n$. When $n$ is even, the value of $N_o(n)$ is given by*

$$N_o(n) = \binom{s+1}{2} + 1 = \frac{1}{8}(n^2 + 2n + 8), \text{ for } n = 2s.$$

*In the odd case with $n = 2s - 1$, we have the upper bound $N_o(n) \leq \binom{s+1}{2} - 1 = \frac{1}{8}(n^2 + 4n - 5)$ and two lower bounds: if $n = 4t - 1$, then $N_o(n) \geq 2t^2 + 1 = \frac{1}{8}(n^2 + 2n + 9)$, and if $n = 4t - 3$, then $N_o(n) \geq 2t^2 - 2t + 2 = \frac{1}{8}(n^2 + 2n + 13)$.*

The proof of the theorem in Section 3 depends on the Ferrers shape associated with an order. According to a theory of Greene and Kleitman this shape depends on the maximum sizes of families of chains and antichains.

The bridge from partial orders to orthogonal convexity is via permutations which can be interpreted as 2-dimensional partial orders or as point sets. In Section 2 we describe these connections. There we also introduce terminology and concepts from orthogonal convexity. Albert et al. [1] define *convex permutations* as permutations which are order isomorphic to a finite generic convex set, i.e., the points of the plot of the convex permutation can be displaced such that they are the corners of a convex polygon without changing their horizontal and vertical order. They consider the least $N_c(n)$ such that every permutation of length $N_c(n)$ contains a convex subpermutation of length $n$. Albert et al. prove that $n^2/8 < N_c(n) < n^2/4$. Since the Erdős and Szekeres problems for orthogonal convexity and permutation convexity are the same (Proposition 2.2) we also establish that $N_c(n)$ is slightly above $n^2/8 + n/4$.

## 2 Orthogonal convexity and related concepts

A finite set $X \subseteq \mathbb{R}^2$ in general position is in convex position if $X$ is the set of corners of its convex hull which is a convex polygon.

The definition of orthogonal convexity is as follows: A set $A \subseteq \mathbb{R}^2$ is *orthogonally convex* (*o-convex*) if and only if $A \cap L$ is connected whenever $L$ is a horizontal or a vertical straight line in $\mathbb{R}^2$. In particular, $A \cap L$ is empty, a single point, a line segment, a half-line or all of $L$. Note that orthogonally convex sets need not be connected. Various definitions of an *orthogonal convex hull* have been proposed in the literature, see Ottman et al. [9], and the more recent book about the more general concept of restricted-orientation convexity [4]. Some of the definitions of the orthogonal convex hull lead to a convex hull which is not unique and in all definitions the convex hull of a set of points may be disconnected.

We base our investigations on the notion of an extremal point of a set. This allows us to speak about point sets in orthogonally convex position without deciding on a definition of the orthogonal convex hull.

A finite set $X \subseteq \mathbb{R}^2$ in general position is in convex position if every point $x \in X$ is extremal in $X$, i.e., there is a halfplane $H_x$ such that $H_x \cap X = \{x\}$. In the orthogonal setting we replace the halfplane by a quadrant. A finite set $X \subseteq \mathbb{R}^2$ in generic position is in *o-convex position* if to every point $x \in X$ there is a quadrant $Q_x$ such that $Q_x \cap X = \{x\}$, i.e., every point of $X$ is o-extremal.

Let $X$ be a generic set of $n$ points in the plane. With $X$ we associate a permutation $\pi(X)$ as follows: The points of $X$ are labeled from 1 to $n$ by increasing $y$-coordinates. Then to obtain $\pi(X)$ we list the labels by increasing $x$-coordinate. Two generic point sets with identical permutations are called *order isomorphic*. With a permutation $\pi : [n] \to [n]$ we associate the point set $X_\pi = \{(i, \pi(i)) : i \in [n]\}$. This representation is the *plot* of the permutation. The plot is a canonical representative of the order isomorphism class of $\pi$.

A *convex permutation* is a permutation whose order isomorphism class contains a point set in convex position. Convex permutations were introduced by Albert et al. [1]. They study enumeration questions and the Erdős-Szekeres problem for convex subpermutations of permutations. With the following proposition they make convex permutations accessible for their study. The entry $\tau(j)$ is a *left-to-right maximum* of the permutation $\tau$ if $\tau(i) < \tau(j)$ for all $i < j$. We denote the set of left-to-right maxima of $\tau$ as $\mathsf{LRmax}(\tau)$. The sets $\mathsf{LRmin}(\tau)$ (left-to-right minima), $\mathsf{RLmin}(\tau)$ (right-to-left minima), and $\mathsf{RLmax}(\tau)$ (right-to-left maxima) are defined alike. An entry is *extremal* if it belongs to the union of the four sets.

**Proposition 2.1** (Albert et al. [1])**.** *A permutation is convex if and only if all its entries are extremal.*

Let $X$ be a point set in the order isomorphism class of $\tau$ and let $p$ be the point of $X$ corresponding to the entry $\tau(j)$ of $\tau$. Then $p$ is o-extremal in $X$ if and only if $\tau(j)$ is extremal in $\tau$. Indeed, the o-extremality of $p$ is witnessed by the $1^{\text{st}}$ quadrant if and only if $\tau(j)$ is a right-to-left maximum. Similarly points with a witnessing $2^{\text{nd}}$ quadrant correspond to left-to-right maxima, while points with a witnessing $3^{\text{rd}}$ quadrant correspond to left-to-right minima, and those with a witnessing $4^{\text{th}}$ quadrant to right-to-left minima. With these observations we obtain the following consequence of Proposition 2.1.

**Proposition 2.2.** *A permutation is convex if and only if the point sets in its order isomorphism class are o-convex.*

Permutations and their plots, or equivalently generic point sets, can be interpreted as 2-dimensional orders with a given realizer. This remark allows us to shift between notation and concepts from order theory and from the world of permutations. Most important to us is the equivalence of chains/antichains in point sets and increasing/decreasing subsequences in permutations. In the following we will think of $X$ as a point set, an order, and a permutation interchangeably.

## 3 The Ferrers shape of a point set

A partition is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers. With a partition $\lambda$, one associates its Ferrers diagram (also called Ferrers shape), which is a down-set of squares in the first quadrant with $\lambda_i$ squares in the $i$-th row. The Robinson-Schensted correspondence is a bijection between permutations and pairs $(\mathbf{P}, \mathbf{Q})$ of Young tableaux of the same shape. We denote the Ferrers shape of the tableaux associated with $\pi$ as $F(\pi)$.

A *$k$-chain* of an order $P = (X, <_P)$ is defined as a family of $k$ pairwise disjoint chains, and a *$\ell$-antichain* is a family of $\ell$ pairwise disjoint antichains. Interest in $k$-chains and $\ell$-antichains of orders goes back to Greene and Kleitman [8, 7] who discovered a rich duality between maximum $k$-chains and maximum $\ell$-antichains. The following theorem is part of the theory.

**Theorem 3.1.** *With an ordered set $P$ with $n$ elements there is an associated integer partition $\lambda$ of $n$ with Ferrers shape $F(P)$, such that the number of squares in the $k$ longest columns of*

*$F(P)$ equals the maximal number of elements covered by a $k$-chain of $P$, and the number of squares in the $\ell$ longest rows of $F(P)$ equals the maximal number of elements covered by an $\ell$-antichain of $P$.*

It is known that if $X$ is a generic point set in the order isomorphism class of $\pi$ and $P = (X, <)$ is the dominance order on $X$, then $F(\pi) = F(P)$. This follows from work of Greene [6] and from Viennot's [11] planarized version of the Robinson-Schensted correspondence, see also [3].

Figure 1 shows an example. In this case the Ferrers diagram $F$ of the point set corresponds to the partition $(5, 3, 3, 2, 2, 1)$. Several proofs of Theorem 3.1 are known. The approach taken by András Frank [5] is particularly elegant and provides additional insight into the interplay of maximum chain and antichain families. Following Frank we call a chain family $\mathscr{C}$ and an antichain family $\mathscr{A}$ of an order $P = (X, <)$ an *orthogonal pair* if the following two conditions hold:

$$(1) \quad X = \Big(\bigcup_{A\in\mathscr{A}} A\Big) \cup \Big(\bigcup_{C\in\mathscr{C}} C\Big), \qquad \text{and} \qquad (2) \quad |A \cap C| = 1 \quad \text{for all } A \in \mathscr{A},\ C \in \mathscr{C}.$$

Frank proved the existence of a sequence of orthogonal chain and antichain families. The sequence consists of an orthogonal pair for every point from the boundary of $F(P)$. With the point $(k, \ell)$ from the boundary of $F(P)$ we get an orthogonal pair $(\mathscr{C}, \mathscr{A})$ such that $\mathscr{C}$ is a $k$-chain and $\mathscr{A}$ is an $\ell$-antichain. (See Figure 1.)

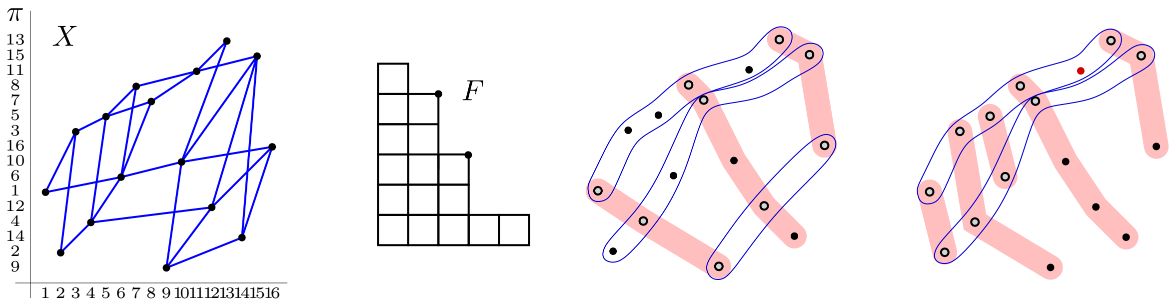


Figure 1: A point set $X$ where the blue segments are the edges of the diagram of $P = (X, <)$, the Ferrers shape $F = F(P)$ and two orthogonal pairs of $X$. The orthogonal pairs, with chains outlined in blue and antichains in pink, correspond to the boundary points $(3, 3)$ and $(2, 5)$ of $F$, . The elements of the corresponding rectangles are emphasized.

If $(\mathscr{C}, \mathscr{A})$ is an orthogonal $(k, \ell)$-pair in $X$, then there are exactly $k \cdot \ell$ points in $X$ which belong to a chain of $\mathscr{C}$ and an antichain of $\mathscr{A}$. We call such a set $R$ a $(k, \ell)$-rectangle of $X$, formally $R = \{C \cap A : C \in \mathscr{C} \text{ and } A \in \mathscr{A}\}$.

A rectangle $R$ is a point set in its own right. It can be seen as an order and it comes with a permutation.

**Lemma 3.2.** *If $R$ is a $(k, \ell)$-rectangle, then* $\mathsf{width}(R) = k$ *and* $\mathsf{height}(R) = \ell$.

*Proof.* Let $(\mathscr{C}, \mathscr{A})$ be the orthogonal $(k, \ell)$-pair defining $R$. Since $R$ can be covered by $k$ chains $\mathsf{width}(R) \le k$. Further, every antichain $A$ in $\mathscr{A}$ has a nonempty intersection with each of the $k$ disjoint chains of $\mathscr{C}$, hence, $|A| \ge k$. With $\mathsf{width}(R) = \max(|A| : A \text{ antichain in } R)$ we get $\mathsf{width}(R) = k$. The argument for $\mathsf{height}(R)$ is dual, by exchanging the role of chains and antichains and the letters $k$ and $\ell$. $\square$

The *canonical antichain partition* of a poset $P = (X, <)$ is constructed by recursively removing all minimal elements from $P$ and making them one of the antichains of the partition. More explicitely $A_1 = \mathsf{Min}(X)$ and $A_j = \mathsf{Min}\big(X \setminus \bigcup\{A_i : 1 \le i < j\}\big)$ for $j > 1$. Note that by definition for each element $y \in A_j$ with $j > 1$ there is some $x \in A_{j-1}$ with $x < y$. Due to this property there is a chain of $h$ elements in $P$ if the canonical antichain partition consists of $h$ non-empty antichains. This in essence is the dual of Dilworth's theorem, i.e., the statement: *the maximal size of a chain equals the minimal number of antichains partitioning the elements of $P$.*

The *transpose* of a point set $X$ is the set $X^\top = \{(y, x); (x, y) \in X\}$. Mapping the canonical antichain partition of $X^\top$ back to $X$ yields the *canonical chain partition* of $X$. The following lemma will be useful:

**Lemma 3.3.** *The canonical antichain partition and the canonical chain partition of a $(k, \ell)$-rectangle $R$ are an orthogonal pair of $R$.*

*Proof.* Let $A_1, \ldots, A_h$ be the canonical antichain partition. Since $\mathsf{height}(R) = \ell$ and the canonical antichain partition is a minimum antichain partition we have $h = \ell$. From $\bigcup_1^h A_i = R$, and $R = k \cdot \ell$, and $\mathsf{width}(R) = k$, we deduce that $|A_i| = k$ for all $i \in \{1, .., \ell\}$. The dual argument yields $|C_j| = \ell$ for every chain $C_j$ from the canonical chain partition. This also implies that $C_j$ must have a nonempty intersection with each $A_i$. $\square$

**Proposition 3.4.** *Let $R$ be a $(k, \ell)$-rectangle with canonical chain and antichain partitions $C_1, \ldots, C_k$ and $A_1, \ldots, A_\ell$. All the points in $A_1$, $A_\ell$, $C_1$, and $C_k$ are o-extremal, moreover, if $k > 1$ and $\ell > 1$, then $|A_1 \cup A_\ell \cup C_1 \cup C_k| = 2(k + \ell) - 4$.*

*Proof.* Let $\rho$ be the permutation corresponding to $R$. Since $A_1 = \mathsf{Min}(R)$, elements of $A_1$ have an empty 3-rd quadrant, so they are the elements of $\mathsf{LRmin}(\rho)$. Similarly $A_\ell = \mathsf{Max}(R) = \mathsf{RLmax}(\rho)$ and the elements of $C_1$ and $C_k$ correspond to $\mathsf{LRmax}(\rho)$ and $\mathsf{RLmin}(\rho)$. Hence, they are all o-extremal. By construction the two chains and the two antichains are disjoint. A chain and an antichain can share at most one point and they do, hence, $|A_1 \cup A_\ell \cup C_1 \cup C_k| = 2(k + \ell) - 4$. $\square$

We now know that a large rectangle in $X$ contributes a large o-convex set in $X$. To obtain a sharp bound, however, we need a second type of o-convex set. A subset $T$ of $X$ is called *2-thin* if either $\mathsf{width}(T) \le 2$ or $\mathsf{height}(T) \le 2$.

**Proposition 3.5.** *A 2-thin subset $T$ of a point set $X$ is o-convex.*

*Proof.* If $\mathsf{height}(T) \le 2$, then $T = \mathsf{Min}(T) \cup \mathsf{Max}(T)$. Hence, every element of $T$ has at least one empty quadrant. The case of $\mathsf{width}(T) \le 2$ is dual. $\square$

Now let $X$ be a point set and suppose that $X$ contains no o-convex set of size $m$. Let $F$ be the Ferrers shape of $X$. The boundary of $F$ is strictly below the line $x + y = \frac{m+4}{2}$, otherwise $X$ would contain a $(k, \ell)$-rectangle with $2(k + \ell) - 4 \ge m$. This contradicts the assumption. Hence $x + y \le \frac{m+4}{2} - \frac{1}{2} = \frac{m+3}{2}$.

The largest Ferrers shape $F$ below this line belongs to the partition $(\frac{m+1}{2}, \frac{m-1}{2}, \ldots, 3, 2, 1)$. This triangular shape has $\frac{1}{8}(m^2 + 4m + 3)$ cells. However, if $X$ has the shape of this partition, then it contains a 2-chain and a 2-antichain, each of size $\frac{m+1}{2} + \frac{m-1}{2} = m$, which are 2-thin o-convex sets. It follows that we have to take off the extremal cells in the first row and the first column. The remaining shape is shown in Figure 2 is has $\frac{1}{8}(m^2+4m+3)-2$ cells. Adding a cell to this shape makes an o-convex subset of size $m$ unavoidable.

Note that we tacitly used that $m+1$ is even, i.e., $m$ is odd. This yields the first upper bound on $N_o(m)$: if $m=2s-1$ and $X$ has at least $\frac{1}{8}(m^2+4m+3)-1=\binom{s+1}{2}-1$ elements, then $X$ contains an o-convex subset of size $m$.

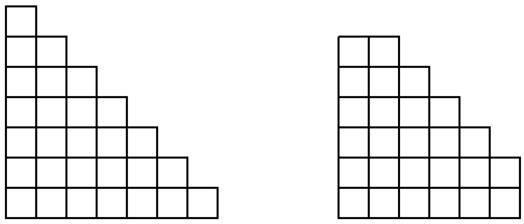

Figure 2: The triangular shape with $\binom{s+1}{2}$ cells is extremal for $n=2s$. The extremal shape for odd $n=2s-1$ is the truncated triangular shape with $\binom{s+1}{2}-2$ cells. The depicted shapes have $s=7$.

For $m=2s$ the triangular partition $(s,s-1,\dots,2,1)$ is extremal. Rectangles in this shape yield o-convex sets of size $2s-2$ and the maximum 2-thin subsets have size $2s-1$. The addition of a single cell to this shape, however, yields a rectangle which contributes an o-convex set of size $2s$. This gives the upper bound on $N_o(m)$ in the even case: if $m=2s$ and $X$ has at least $\binom{s+1}{2}+1=\frac{1}{8}(m^2+2m+8)$ elements, then $X$ contains an o-convex subset of size $m$.

## Lower bound examples

From the analysis leading to the upper bound we know that matching lower bound examples have to be point sets whose Ferrers shapes are the triangular and truncated triangular shapes shown in Figure 2. Our examples will be weak orders, i.e., they are obtained by substituting the elements of a chain by antichains. Albert et. al [1] used similar examples for their lower bound and they called them *layered permutations.* We write $X=W[a_1,\dots,a_s]$ if $X$ is obtained from a chain $C$ of size $s$ by substituting the $i$-th element of $C$ with an antichain $A_i$ of size $a_i$. Figure 3 shows $W[1,3,5,7,5,3,1]$ and $W[1,3,5,7,6,4,2]$ and $W[2,4,6,8,6,4,2]$.

**Lemma 3.6.** *Let $\beta_1=(a,a+2,\dots,a+2(j_1-1),a+2j_1)$ and $\beta_2=(b+2j_2,b+2(j_2-1),\dots,b+2,b)$ be an increasing and a decreasing sequence and let $\alpha=\beta_1\beta_2$ be their concatenation. The largest o-convex subset of $W[\alpha]$ is of size $a+b+2(j_1+j_2)$.*

*Proof.* Let $K$ be an o-convex subset of $W[a_1,\dots,a_s]$ and suppose that $i$ is the least index with $K\cap A_i\neq\emptyset$ and $j$ is the largest index with $K\cap A_j\neq\emptyset$. Then $|K|\leq a_i+a_j+2(j-i-1)$ because $\mathsf{Min}(K)\subseteq A_i$, $\mathsf{Max}(K)\subseteq A_j$, and every antichain between $A_i$ and $A_j$ can contribute at most two elements to $K$.

In $\beta_1$ and $\beta_2$ the step size is 2. Suppose $\mathsf{Min}(K)$ is an antichain $A_i$ from $\beta_1$. If $K'$ is obtained from $K$ by removing $A_i$ from $K$ and adding $A_{i+1}$, then the size of $K$ and $K'$ is upper bounded by the same value. This value is equal to the sum of the cardinalities of the largest antichain coming from $\beta_1$ and the largest of $\beta_2$, i.e., $K\leq a+b+2(j_1+j_2)$. □

Applying the bound of the lemma to appropriate point sets $W[\alpha]$ we obtain the lower bounds on $N_o(n)$ stated in Theorem 1.1.

- The set $W[1,3,5,\ldots,2t-1,2t,2t-2,\ldots,4,2]$ contains no o-convex set of size $4t$. Since the size of the set is $\binom{2t+1}{2}$ this shows that if $n=2s=4t$, then $N_o(n) \geq \binom{s+1}{2}+1 = \frac{1}{8}(n^2+2n+8)$.
- The set $W[1,3,5,\ldots,2t+1,2t,2t-2,\ldots,4,2]$ contains no o-convex set of size $4t+2$. Since the size of the set is $\binom{2t+2}{2}$, this shows that if $n=2s=4t+2$, then $N_o(n) \geq \binom{s+1}{2}+1 = \frac{1}{8}(n^2+2n+8)$.
- The set $W[1,3,\ldots,2t-3,2t-1,2t-3,\ldots,3,1]$ contains no o-convex set of size $4t-3$. Since the size of the set is $t^2+(t-1)^2$ this shows that if $n=4t-3$, then $N_o(n) \geq 2t^2-2t+2 = \frac{1}{8}(n^2+2n+13)$.
- The set $W[2,4,\ldots,2t-2,2t,2t-2,\ldots,4,2]$ contains no o-convex set of size $4t-1$. Since the size of the set is $2t^2$ this shows that if $n=4t-1$, then $N_o(n) \geq 2t^2+1 = \frac{1}{8}(n^2+2n+9)$.

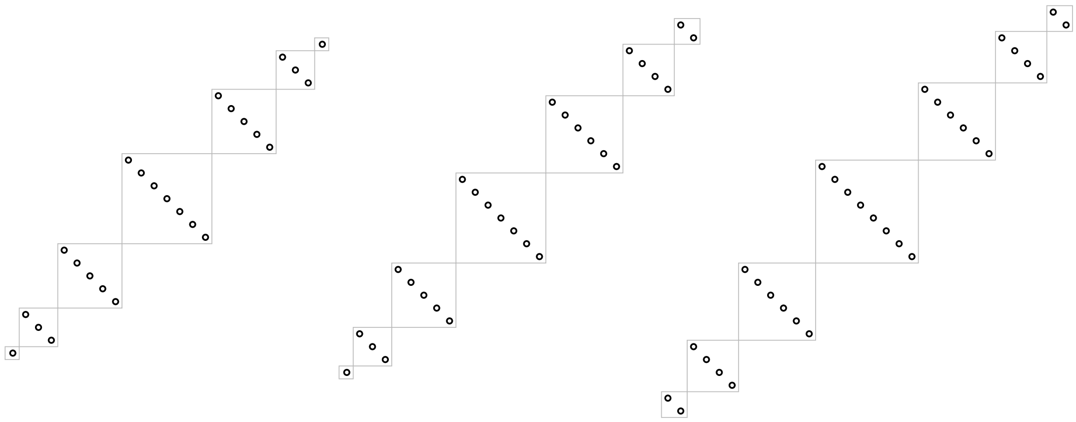

Figure 3: $W[1,3,5,7,5,3,1]$ and $W[1,3,5,7,6,4,2]$ and $W[2,4,6,8,6,4,2]$, they are conjectured to be maximum point sets without o-convex subset of size 13, 14, and 15 respectively. Hence $N_o(13) \geq 26$, $N_o(14) \geq 29$, $N_o(15) \geq 33$. We also know $N_o(13) \leq 27$, $N_o(14) \leq 29$, $N_o(15) \leq 35$.

# 4 Open Problems and Future Directions

For odd values of $n$ we have not yet been able to precisely determine the value of $N_o(n)$. We believe that the lower bound is tight. Since our lower bound examples are weak orders they have the property that they admit a chain partition $\mathscr{C}$ and an antichain partition $\mathscr{A}$ such that for all $(k,\ell)$ the $k$ longest chains of $\mathscr{C}$ together with the $\ell$ largest antichains of $\mathscr{A}$ form an orthogonal pair. (In the terminology of West [12] these partitions are completely saturated). It is easy to show that restricted to point sets with the above property the lower bound is tight. The crucial property is that in this case we find an o-convex subset of size $2(k+\ell)-1$ whenever $(k,\ell)$ is a concave corner on the boundary of the Ferrers shape, i.e., $(k+1,\ell)$ and $(k,\ell+1)$ are points of the boundary as well.

To close the gap between upper and lower bound in the odd case we either need lower bound examples with a more complex structure or additional techniques for constructing large o-convex subsets.

Another interesting problem, in addition to studying subsets in o-convex position, is the study of *o-convex holes*, namely, finding o-convex subsets whose interior is empty. Let $H_o(n)$ be the least integer such that every generic point set of size at least $H_o(n)$ contains an o-convex hole of size $n$. Clearly $H_o(n) \geq N_o(n) > \frac{1}{8}n^2$. Since every chain and antichain is an o-convex hole the lemma of Erdős-Szekeres yields the upper bound $H_o(n) \leq (n-1)^2 + 1$.

## 5 Acknowledgment

This work was initiated at the Workshop "Order and Geometry" 2022 in Ciążeń. Results (with a too strong claim) were presented at Permutation Patterns 2023. We thank Jean Cardinal for proposing the problem and for all the participants of the workshop for creating a friendly and productive atmosphere.

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