Coding and Counting Arrangements of Pseudolines

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Abstract. Arrangements of lines and pseudolines are important and appealing objects for research in discrete and computational geometry. We show that there are at most $2^{0.657\,n^2}$ simple arrangements of n pseudolines in the plane. This improves on previous work by Knuth who proved an upper bound of $3^{\binom{n}{2}} \cong 2^{0.792\,n^2}$ in 1992 and the first author who obtained $2^{0.697\,n^2}$ in 1997. The argument uses surprisingly little geometry. The main ingredient is a lemma that was already central to the argument given by Knuth.

1 Introduction

Arrangements of pseudolines are the topic of a chapter in the Handbook on Discrete and Computational Geometry [8]. The monograph [4] is another general reference. In most texts arrangements of pseudolines are defined with the real projective plane as ambient space. In contrast, we consider arrangements in the Euclidean plane.

A *pseudoline* in the Euclidean plane is a curve extending to infinity on both sides. An *arrangement of pseudolines* is a family of pseudolines with the property that each pair of pseudolines has a unique point of intersection where the two pseudolines cross. An arrangement is *simple* if no three pseudolines have a common point of intersection. Arrangements with a distinguished unbounded cell called the *north-cell* are known as *marked arrangements*. Note that if in an arrangement a north-cell c_n has been selected, then there is a unique unbounded cell separated from c_n by all pseudolines this cell is called the *south-cell*. Pseudolines in a marked arrangement have a natural orientation such the the north-cell is to left and the south-cell to the right of the oriented pseudoline.

pseudoline
arrangement of pseudolines
simple
marked arrangements

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Two arrangements are *isomorphic*, i.e., considered the same, if they can be mapped onto each other by a homeomorphism of the plane. In the case of marked arrangements it is required that an isomorphism respects the distinguished cell and preserves the induced orientations.

In this paper we are interested in the number B_n of marked simple arrangements of n pseudolines. It is known, e.g. from [3], that $B_n \in 2^{\Theta(n^2)}$. Our interest is in the multiplicative constant b hidden in the $\Theta(n^2)$, i.e, such that $B_n \in 2^{bn^2+o(n^2)}$. Knuth [9] considers the counting problem for several related classes of arrangements, e.g. arrangements without marking, projective arrangements or more abstractly reorientation classes of uniform oriented matroids of rank three. Their numbers only differ by lower order factors, more precisely, their number is also of the form $2^{bn^2+o(n^2)}$ with the same constant b.

We are going to study the growth of $b_n = \log_2(B_n)$. An easy lower bound construction is given in [11, sec. 6.2]; it yields $b_n > \frac{1}{9}n^2$. Knuth [9, page 37] shows $b_n > \frac{1}{6}n^2 - O(n)$. In Section 4 we use enumeration results for rhombic tilings to prove $b_n > 0.188\, n^2$.

The upper bound $B_n \leq 3^{\binom{n}{2}}$, i.e., $b_n \leq 0.7924\, n^2$, was shown by Knuth [9, page 39]. At the end of this monograph, Knuth [9, page 96] comments that an improved bound of $b_n \leq 0.7194\, n^2$ can be obtained from the the sharpest version of the zone theorem. Felsner [3] obtained the bound $b_n \leq 0.6974\, n^2$. In Section 2 we review the idea in Knuth's proof and add a new simple idea to get $b_n \leq 0.6609\, n^2$. In Section 3 we refine the analysis and prove the bound $b_n \leq 0.6571\, n^2$.

There are several nice representations and encodings of simple arrangements of pseudolines. We close the introduction by explaining three of them. First however we fix a labeling of the pseudolines. Given a marked arrangement $\mathcal{A}$ of n pseudolines we label the pseudolines with $1, \dots, n$ such that an oriented curve from the south-cell to the north-cell that has all crossings of pseudolines on the right intersects the pseudolines in increasing order, see Figure 1 (left).

Local sequences. Associate with pseudoline i the permutation α_i of $\{1, .., n\} \setminus i$ reporting the order from left to right in which the other pseudolines cross line i. The family $(\alpha_1, \alpha_2, \dots, \alpha_n)$ is called the family of *local sequences* of the arrangement. *local sequences*

Wiring diagrams. Goodman [7] introduced a class of drawings of arrangements called *wiring diagrams* to get well-arranged pictures of arrangements. The idea is to specify a set of n horizontal lines (wires) and confine the pseudolines to these wires except for positions where they cross each other. In the case of a simple arrangement crossings always involve two pseudolines from adjacent wires. Figure 1 shows an example.

Zonotopal tilings. A *simple zonotopal tiling* $\mathcal{T}$ is a tiling of a regular $2n$-gon with vertices $x_0, x_1, \dots, x_{2n-1}$ in clockwise order starting with the highest vertex x_0. The tiles of $\mathcal{T}$ are rhombi $R(i,j)$, $1 \leq i < j \leq n$, such that $R(i,j)$ has one side which is a translated copy of the segment $[x_{i-1}, x_i]$ and one side which is a translated copy of the segment $[x_{j-1}, x_j]$. The tiles are not allowed to be rotated. *simple zonotopal tiling*

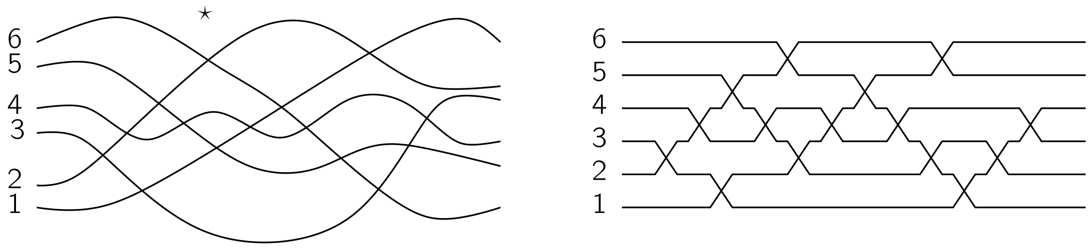

Figure 1: An arrangement $\mathcal{A}$ with a cell marked by a star and a wiring diagram of $\mathcal{A}$. The local sequences of this arrangement are: $\alpha_1 = 3, 5, 4, 6, 2$, $\alpha_2 = 3, 4, 5, 6, 1$, $\alpha_3 = 2, 1, 6, 5, 4$, $\alpha_4 = 2, 5, 1, 6, 3$, $\alpha_5 = 2, 4, 1, 6, 3$, $\alpha_6 = 2, 1, 4, 5, 3$.

Simple zonotopal tilings can be viewed as normalized drawings of the duals of marked simple arrangements. Figure 2 shows an example. For additional information on zonotopal tilings and their relation to arrangements see [4] and [2].

Proofs of equivalence of the three representations are detailed in [4]. The basic tool for the proof of equivalence is to sweep a representations, resp. the arrangement, from left to right to transform one representation into another.

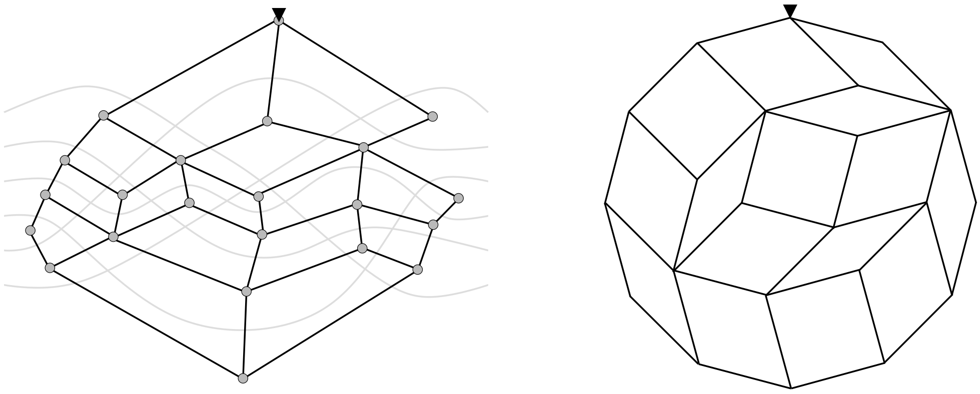

Figure 2: Arrangement $\mathcal{A}$ with its dual and the corresponding zonotopal tiling.

2 The upper bound

The upper bound for the number of simple Euclidean arrangements given in [3] was based on 'horizontal encodings' of arrangements. The first step was to replace the numbers in the local sequences α_i by single bits, a 1 for numbers j with $j < i$ and a 0 for $j > i$.

The proof of Knuth [9] takes a 'vertical' approach. Let $\mathcal{A}$ be an arrangement of $n+1$ pseudolines and consider pseudoline $n+1$ drawn into the wiring diagram of the arrangement $\mathcal{A}'$ induced by the first n pseudolines of $\mathcal{A}$. The *course* of pseudoline $n+1$ describes a *cutpath* descending from the north-cell to the south-cell of $\mathcal{A}$. Looking at the zonotopal *cutpath* tiling representation of $\mathcal{A}'$ as a graph a cutpath corresponds to a vertically decreasing path from the highest vertex x_0 to the lowest x_n. Figure 3 shows an example.

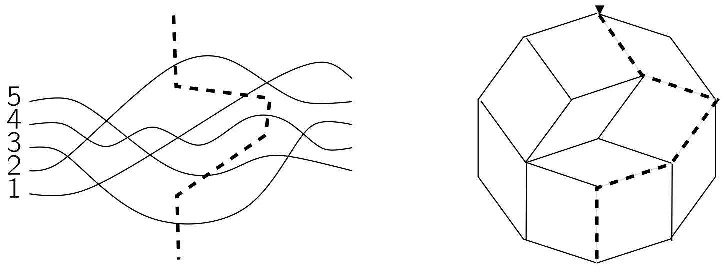

Figure 3: The cutpath corresponding to pseudoline 6 in the arrangement of pseudolines 1,2,3,4, and 5.

The number of arrangements $\mathcal{A}$ such that $\mathcal{A} \setminus \{n+1\}$ equals $\mathcal{A}'$ is exactly the number of different cutpaths of $\mathcal{A}'$. Define γ_n as the maximal number of cutpaths an arrangement of n pseudolines can have. It then follows that

$$B_{n+1} \leq \gamma_n \cdot B_n. \tag{1}$$

Knuth proves that $\gamma_n \leq 3^n$ and he also notes that the 'bubblesort arrangement' (see Figure 4) of size n has approximately $n\,2^{n-2}$ cutpaths. Knuth also conjectures that the bubblesort arrangement is the maximizing example. The bubblesort arrangement is a particular Euclidean arrangement corresponding to the projective cyclic arrangement, cf. [12].

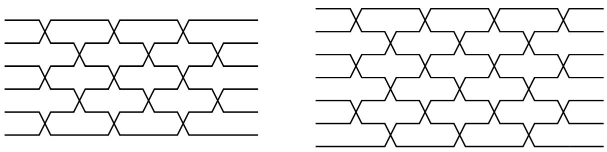

Figure 4: Wiring diagrams of the bubblesort arrangements of 6 and 7 lines.

In social choice theory a set T of permutations of $[n]$ is called an *acyclic set* if for all $i, j, k \in [n]$ at most two of ijk, jki, kij appear as a restriction of a permutation in T to $\{i, j, k\}$. The interest in acyclic sets comes from the fact that they avoid Condorcet cycles. That is, if voters are constrained to preference lists from an acyclic set T, then the majority digraph on the alternatives is acyclic. It has been shown in [6] that the set of cutpaths of an arrangement $\mathcal{A}$ is an acyclic set. Fishburn [5] introduced the *alternating scheme* as a large acyclic set. It turned out that the permutations in the alternating scheme correspond to the cutpaths of the bubblesort arrangement (Figure 4). Galambos and Reiner [6] gave a precise formula for the size of the alternating scheme and conjecture that this is the largest size of an acyclic set that can be obtained as the set of cuthpaths of an arrangement, i.e., in a different context they came up with the same conjecture as Knuth. Ondřej Bílka, a student of the second author, found a construction of arrangements with 2.076^n cutpaths [1]. This disproves the conjecture.

acyclic set

In the remainder of this section we present the main lemma of Knuth and show how to use it to bound the number γ_n of cutpaths of an arrangement.

Consider a cutpath p descending through the wiring diagram of $\mathcal{A}$. Having reached a cell c, the path has to continue by crossing the wire w bounding c from below. The cells that can be reached from c by crossing w are ordered from left to right as $c_1, c_2, .., c_d$. Let their number d be the *degree* of c. When $d \geq 2$ we let c_1 be the *left successor* of c, and c_d be the *right successor* of c. The other cells $c_2, \ldots, c_{d-1}$ are called *middle successors* of c. When $d = 1$ we let c_1 be the *unique successor* of c.

degree

middle

When the cutpath p of $\mathcal{A}$ traverses a cell c such that there is a middle successor cell c' of c separated from c by pseudoline j we say that p *sees a middle of color j at c*. If p descends from c to c' we say that p *has crossed pseudoline j as a middle*. Figure 5 illustrates the terminology. The *middles of p* denotes the set of all pseudolines crossed by p as a middle. Similarly, the *uniques of p* denotes the set of all pseudolines crossed by p when leaving cells of degree 1.

sees a middle of color j at c

has crossed pseudoline j as a middle

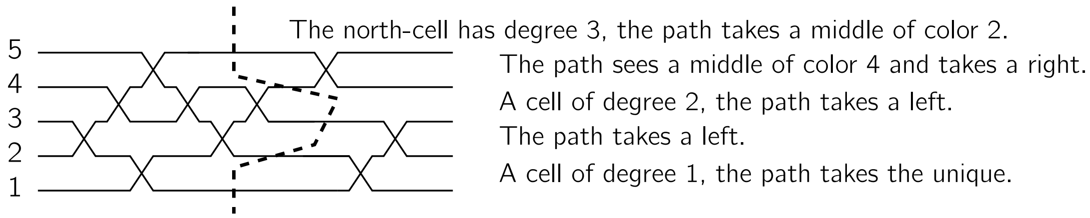

Figure 5: A cutpath through a wiring diagram.

Lemma 1 (Knuth) *For every pseudoline j and every cutpath p it holds: p sees a middle of color j at most once.*

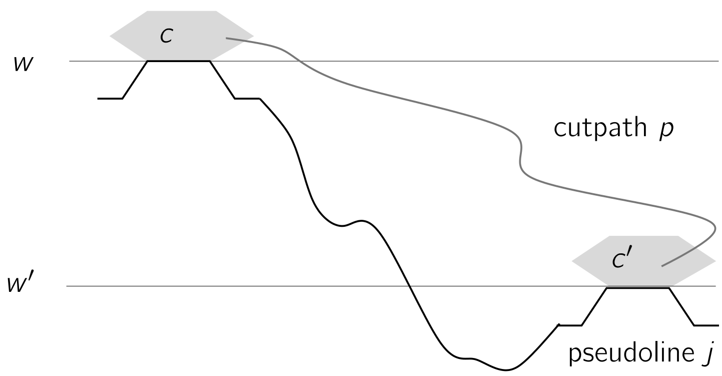

Figure 6: Illustrating the proof of Lemma 1

Proof. Suppose a cutpath p sees a middle of color j at different cells c and c'. Assuming that p visits c before c' we have a situation as sketched in Figure 6. Let t be the number of wires strictly between w and w'. Between the visits of the borders of c and c', pseudoline j has to change at least $(t+1)+2$ times from a wire to another, i.e., pseudoline j has at least

$t+3$ crossings in this range. Every pair of pseudolines has only one crossing. Therefore, every pseudoline crossing j between c and c' also has to be traversed by the cutpath p on its way from c to c'. The cutpath p only intersects $t+1$ wires on its way from c to c', a contradiction. $\square$

We use the lemma to encode cutpaths of an arrangement $\mathcal{A}$. With a cutpath p we associate two combinatorial objects:

- A set $M_p \subset [n]$ consisting of all j such that pseudoline j is crossed by p as a middle.
- A binary vector $\beta_p = (b_p(0), b_p(1), \ldots, b_p(n-1))$ such that $b_p(i) = 1$ only if p proceeds to the left successor of the cell c enclosed between wire i and wire $i+1$.

The cutpath p from Figure 5 has $M_p = \{2\}$, $\beta_p(1) = 0$ and $\beta_p(2) = \beta_p(3) = 1$, the values of $\beta_p(0) = \beta_p(4) = 0$ are irrelevant since p is taking a middle in the first step and a unique in the last.

Claim I. The mapping $p \to (M_p, \beta_p)$ is injective from cutpaths of $\mathcal{A}$ to pairs consisting of a subset M of $[n]$ and a binary vector of length n.

Proof. Given (M_p, β_p) the cutpath can uniquely be reconstructed: Assume an initial piece of p up to some cell c has been constructed. If there is only one successor c' of c, i.e., if the degree of c is one, then p has to continue to c'. If there is a $j \in M_p$ such that c has a middle of color j, then p has to cross pseudoline j when leaving c (here we use Lemma 1). Otherwise c has to continue to the left or the right successor of c and a lookup at the corresponding position of β_p reveals which is to be taken. $\square$

From the claim it immediately follows that $\gamma_n \le 2^n\, 2^n = 4^n$. To improve the bound we use two simple observations:

- Every j taken as a middle forces that some entry of β_p is irrelevant, i.e., not needed for the encoding of the cutpath.
- The lookups of entries of β_p are done in increasing order of indices.

It follows that we can take β_p to be a binary string of length $n - |M_p|$ and agree that lookups are always taken at the first unused position of β. This improved encoding yields:

$$\gamma_n \;\le\; \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} \;=\; 2^n\,(1+\frac{1}{2})^n \;=\; 3^n. \tag{2}$$

This is the upper bound of Knuth, only the arithmetic in our derivation is simpler.

Note that our estimate for the length of β_p does not yet take into account that some cells may have degree one. Define $\Gamma_{\mathcal{A}}(k,r)$ as the set of cutpaths in $\mathcal{A}$ that leave k cells trough a middles and visit r bounded cells of degree one. From the above considerations we immediately have

$$|\Gamma(k,r)| \;\le\; \binom{n}{k} 2^{n-k-r} \tag{3}$$

With the next lemma we show how to make use of this.

Lemma 2 $$|\Gamma(k,r)| \;\leq\; \min\left\{\binom{n}{k},\binom{n}{r}\right\}2^{n-k-r}.$$

Proof. Paths in $\Gamma_{\mathcal{A}}(k,r)$ can also be encoded as cutpaths in the arrangement $\hat{\mathcal{A}}$ obtained from $\mathcal{A}$ via a 180° rotation of the plane. A cutpath p of $\mathcal{A}$ takes a middle to change from c to c' exactly if the rotated cutpath $\hat{p}$ of $\hat{\mathcal{A}}$ is reaching cell c as the unique successor of c', see Figure 7. In other words cells that are left through a middle by p and bounded cells that are left through a unique by $\hat{p}$ are in bijection as well as middles of $\hat{p}$ and bounded uniques of p. This yields $\Gamma_{\mathcal{A}}(k,r) = \Gamma_{\hat{\mathcal{A}}}(r,k)$ and the lemma follows from formula (3). $\square$

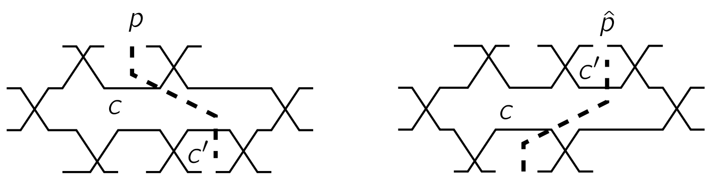

Figure 7: Illustrating the symmetry between middles and uniques used in Lemma 2

Using the lemma we get

$$\begin{aligned}
\gamma_n &\leq \sum_{k,r}|\Gamma_{\mathcal{A}}(k,r)| \;\leq\; \sum_{k,r}\min\left\{\binom{n}{k},\binom{n}{r}\right\}2^{n-k-r}\\
&\leq 2^n\sum_{k=0}^{n}\binom{n}{k}2^{-k}\sum_{r\geq k}2^{-r} \;+\; 2^n\sum_{r=0}^{n}\binom{n}{r}2^{-r}\sum_{k\geq r}2^{-k}\\
&= 2\cdot 2^n\sum_{k=0}^{n}\binom{n}{k}2^{-k}\sum_{r\geq k}2^{-r} \;=\; 2^{n+1}\sum_{k=0}^{n}\binom{n}{k}2^{-2k}\sum_{j\geq 0}2^{-j}\\
&= 2^{n+2}\left(1+\frac{1}{4}\right)^n \;=\; 4\left(\frac{5}{2}\right)^n. \qquad (4)
\end{aligned}$$

Combining this with (1) we get:

Theorem 1 *The number B_n of arrangements of n pseudolines is at most $4^{n-1}\left(\frac{5}{2}\right)^{\binom{n}{2}}$, hence for n large enough $b_n \leq 0.6609\, n^2$.*

3 The upper bound, refined

In this section we show that a careful analysis of the distribution of middles along cutpaths yields an improved bound on the size of $\Gamma(k,r)$.

With an arrangement $\mathcal{A}$ associate the directed dual graph $G^*_{\mathcal{A}}$. An example of the underlying undirected dual is shown in Figure 2. The vertices of $G^*_{\mathcal{A}}$ are the cells of $\mathcal{A}$ and the orientation is from north to south. This means that if $\{x,y\}$ is an edge dual to pseudoline p such that x and the north-cell z_n are in the same halfplane of p and consequently y and the south-cell z_s are in the other halfplane of p, then the edge is oriented as (x, y). Cutpaths of $\mathcal{A}$ are in bijection to directed z_n to z_s paths in $G^*_{\mathcal{A}}$; henceforth we will use the same name *cutpath* for these paths in $G^*_{\mathcal{A}}$. Edges of $G^*_{\mathcal{A}}$ are classified as *left*, *right*, *middle* or *unique* and they are colored with the label i of their dual pseudoline.

Fix a bitstring β and consider the set Λ_β of all cutpaths that can be constructed by using lookup in β for the left-right decisions. The paths in $\Lambda_\beta(\mathcal{A})$ naturally define a (directed) rooted tree $\mathcal{T}_\beta$: A node c of the tree corresponds to all cutpaths from Λ_β that share a given initial subpath. All edges in $\mathcal{T}_\beta$ are oriented away from the root. Corresponding to the tree node c there is a vertex v_c in $G^*_{\mathcal{A}}$. If v_c only has one successor, then tree node c has a *unique* successor. If v_c has more successors, then there is a single *sided* successor of *sided*
c representing the cell that is reached with a lookup in β and there is a *middle* successor of c for every middle edge leaving v_c. Leaves of $\mathcal{T}_\beta$ are in bijection to the cutpaths in Λ_β.

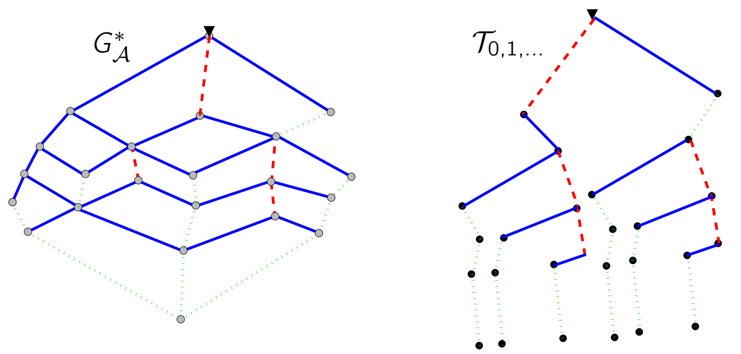

Figure 8: The graph $G^*_{\mathcal{A}}$ and the tree $\mathcal{T}_\beta$ with $\beta = 0, 1, \ldots$, i.e., right, left, ... Red edges are dashed and green edges are pointed.

Consider the subtree $\mathcal{T}_\beta(r, k)$ of $\mathcal{T}_\beta$ consisting of cutpaths in $\Gamma_\beta(k, r) = \Lambda_\beta \cap \Gamma(k, r)$, i.e., cutpaths using k middle and r unique edges. Let $\mathcal{T}^*_\beta(r, k)$ be obtained from $\mathcal{T}_\beta(r, k)$ by contracting all *unique* edges. The contractions does not change the number of leaves. Color the sided edge of tree nodes blue and all middle edges red. Note that $\mathcal{T}^*_\beta(r, k)$ has the following properties:

(a) Every non-leaf node has exactly one blue out-edge.

(b) Every path from the root to a leaf has length $h = n - r$.

(c) Every path from the root to a leaf uses exactly k red edges.

(d) The nodes along any path from the root to a leaf have altogether at most n red out-edges (Lemma 1).

Definition 1 *For $n \geq h \geq k$, let $T(n, h, k)$ be the maximum number of leaves of a rooted tree with red and blue edges and properties (a)–(d).*

To determine this quantity we first study a different maximization problem. A k-transversal of a partition $\Pi = (B_1, \ldots, B_h)$ of $[n]$ is a k-element subset A of $[n]$ such that $|A \cap B_i| \leq 1$ for each $i \in \{1, \ldots, h\}$.

Definition 2 *For $n \geq h \geq k$, let $P(n, h, k)$ be the maximum number of k-transversals a partition $\Pi = \{B_1, \ldots, B_h\}$ of $[n]$ with h blocks can have.*

Lemma 3 *If $n = d \cdot h + r$ with $0 \leq r < h$, then*

$$P(n, h, k) = \sum_{\ell=0}^{r} \binom{r}{\ell} (d+1)^{\ell} \binom{h-r}{k-\ell} d^{k-\ell}.$$

Proof. Let Π^* be a partition of $[n]$ into r blocks of size $d+1$ and $h-r$ blocks of size d. It is an easy exercise to show that the number of k-transversals of Π^* is equal to the right hand side of the equation in the lemma.

Now consider a partition Π of $[n]$ into h blocks and suppose it is not equivalent to Π^*. Then there are blocks B and B' in Π whose size differs by at least two. Let $|B| = s$ and $|B'| = s + t$ with $t \geq 2$. Now let $\hat{\Pi}$ be obtained by moving one element from B' to B so that $\hat{\Pi}$ has blocks $\hat{B}$ and $\hat{B}'$ with $|\hat{B}| = s + 1$ and $|\hat{B}'| = s + t - 1$. Note that the k-transversals of Π and $\hat{\Pi}$ whose intersection with $B \cup B'$ is at most one are the same. Now let M be the number of $(k-2)$-transversals of $\Pi \setminus \{B, B'\}$. The number of k-transversals whose intersection with $B \cup B'$ is two is $s\,(s+t)\,M$ for Π and $(s+1)\,(s+t-1)\,M$ for $\hat{\Pi}$. Since $(s+1)\,(s+t-1) > s\,(s+t)$ it follows that $\hat{\Pi}$ has more k-transversals than Π. $\square$

Lemma 4 *For all $k \leq h \leq n$,*

$$P(n, h, k) \leq \binom{h}{k} \left(\frac{n}{h}\right)^{k}.$$

Proof. Let Π be a partition of $[n]$ into h blocks whose number of k-transversals is exactly $P(n, h, k)$. From Π we obtain a partition Π^+ of $[hn]$ into h blocks by substituting single elements by sets of h elements, e.g., $i \to (i-1)h + 1, \ldots, ih$.

The number of k-transverals of of Π^+ is $h^k\, P(n, h, k)$. With Lemma 3 we get the result: $h^k\, P(n, h, k) \leq P(hn, h, k) = \binom{h}{k} n^k$. $\square$

Lemma 5 $T(n, h, k) \leq P(n, h, k)$.

Proof. The proof is by induction on h. Since $k \leq h$ we get a start by verifying $P(n, 1, 0) = T(n, 1, 0) = 1$.

For the inductive step consider a tree realizing $T(n, h+1, k)$. The root of the tree has some degree $s+1$. It follows from the defining properties (a)–(d) that the subtree reached from the root through the blue edge contains at most $T(n-s, h, k)$ leaves and each of the s subtrees reached from the root through a red edge contains at most $T(n-s, h, k-1)$ leaves. Therefore,

$$T(n, h+1, k) \leq \max_{0 \leq s \leq n-k+1} \Big(T(n-s, h, k) + s\, T(n-s, h, k-1) \Big). \tag{5}$$

Now let s be the value where the maximum is attained. By induction

$$T(n-s, h, k) + s\, T(n-s, h, k-1) \leq P(n-s, h, k) + s\, P(n-s, h, k-1). \tag{6}$$

Consider partitions Π_1 and Π_2 maximizing $P(n-s, h, k)$ and $P(n-s, h, k-1)$. A byproduct of the proof of Lemma 3 is that for $k \geq 2$ the structure of the maximizing partitions is independent of k. For $k = 0, 1$ all partitions with h blocks are maximizing. Hence, we can assume that $\Pi_1 = \Pi_2$. Let Π be the partition obtained by adding a new block B with $|B| = s$ to Π_1 and note that the right hand side of (6) is exactly the number of k-transversals of Π. Since Π is a partition of an n-element set into $h + 1$ blocks we conclude:

$$P(n-s, h, k) + s\, P(n-s, h, k-1) \le P(n, h+1, k). \tag{7}$$

This completes the proof. $\square$

We mention without proof that the inequality in Lemma 5 actually holds with equality.

To bound the total number γ_n of cutpaths in an arrangement $\mathcal{A}$, we use the ideas developed in this section in a series of inequalities:

$$\begin{aligned}
\gamma_n &\le \sum_{k,r} |\Gamma_{\mathcal{A}}(k,r)| \;=\; \sum_{k,r}\sum_{\beta} |\Lambda_\beta \cap \Gamma_{\mathcal{A}}(k,r)| \\
&= \sum_{k,r}\sum_{\beta} \#\text{ leaves of } \mathcal{T}_\beta(k,r) \;\le\; \sum_{k,r}\sum_{\beta} T(n, n-r, k) \\
&= \sum_{k,r} T(n, n-r, k)\, 2^{n-r-k} \;\le\; \sum_{k,r} P(n, n-r, k)\, 2^{n-r-k}.
\end{aligned} \tag{8}$$

Recall from Lemma 2 that the freedom of encoding cutpaths forward or backward yields a symmetry in the parameters k and r. Using this and the observation that $h > h'$ implies $P(n, h, k) > P(n, h', k)$ we get:

$$\begin{aligned}
\gamma_n &\le \sum_{k,r} \min\Big\{P(n, n-r, k), P(n, n-k, r)\Big\} 2^{n-r-k} \\
&\le \sum_{k\le r} 2\, P(n, n-r, k)\, 2^{n-r-k} \;\le\; \sum_{k} 2\, P(n, n-k, k)\, 2^{n-2k} \sum_{j\ge 0} 2^{-j} \\
&= 4 \sum_{k} \binom{n-k}{k} \left(\frac{n}{n-k}\right)^{k} 2^{n-2k}.
\end{aligned} \tag{9}$$

The last equality follows from Lemma 4.

Now we are interested in the summand of (9) whose contribution is asymptotically dominant. Using Stirling's approximation and parametrizing $k = an$, the summands can be estimated as

$$\binom{(1-a)n}{an} \left(\frac{n}{(1-a)n}\right)^{an} 2^{(1-2a)n} \;\approx\; \left[2\left(\frac{1-a}{1-2a}\right)^{(1-2a)} \left(\frac{1}{4a}\right)^{a}\right]^{n}. \tag{10}$$

At this point we started `Maple` and found that presumably the maximum of (10) is attained at $a \approx 0.186691$ and has the value of at least 2.486976, so that $\gamma_n \le 4n2.486976^n$. This yields our main theorem:

Theorem 2 *Let B_n be the number of arrangements of n pseudolines and let $b_n = \log_2 B_n$. For n large enough, $b_n \le 0.6571\, n^2$.*

4 A lower bound

Given three numbers i, j and k we consider the set of $i+j+k$ pseudolines $1, 2, \dots, i+j+k$ partitioned into the following three parts: $\{1, \dots, i\}$, $\{i+1, .., i+j\}$, and $\{i+j+1, .., i+j+k\}$. A partial arrangement on this set is called *consistent* if any two pseudolines from different parts cross while any two pseudolines from the same part do not cross. The zonotopal duals of consistent partial arrangements are rhombic tilings of the centrally symmetric hexagon $H(i,j,k)$ with side lengths i, j and k; Figure 9 shows an example.

consistent

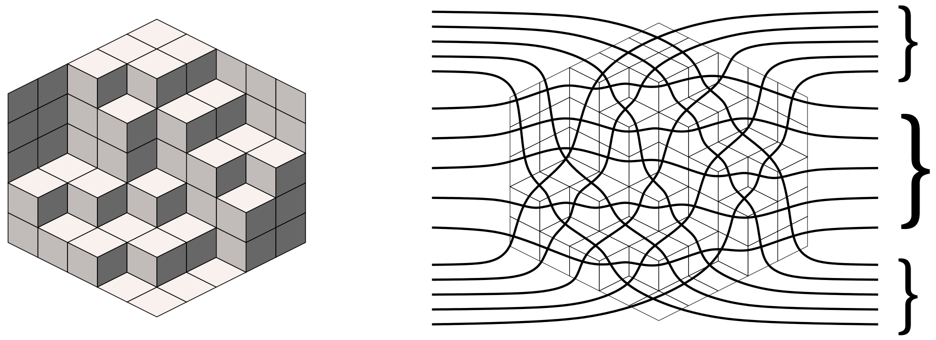

Figure 9: The hexagon $H(5,5,5)$ with one of its rhombic tilings and a consistent partial arrangement corresponding to the tiling.

The enumeration of rhombic tilings of $H(i,j,k)$ is a classical combinatorial problem solved by MacMahon [10]. There are

$$PP(i,j,k) = \prod_{a=0}^{i-1} \prod_{b=0}^{j-1} \prod_{c=0}^{k-1} \frac{a+b+c+2}{a+b+c+1} \tag{11}$$

such tilings.

Consider a consistent partial arrangement with three parts of size n. Such a partial arrangement can be completed to a 'full' arrangement of $3n$ pseudolines by adding any arrangement of n lines for each of the three parts. E.g. in Figure 9 the addition can be done by glueing three arrangements of 5 pseudolines each to the picture where the braces are. This construction shows that

$$B_{3n} \geq PP(n,n,n)\, {B_n}^3. \tag{12}$$

To find the growth rate of $PP(n,n,n)$ we first note that $PP(n,n,n) = T(n)/T(0)$ where $T(k) = \prod_{a=0}^{n-1} \prod_{b=0}^{n-1} (a+b+k+1)$. Let $t(k) = \ln T(k)$ and approximate $t(k)$ by an integral:

$$t(k) = \ln T(k) = \sum_{a=0}^{n-1} \sum_{b=0}^{n-1} \ln(a+b+k+1) \approx \int_{x=0}^{n} \int_{y=0}^{n} \ln(x+y+k+1)\, dy\, dx. \tag{13}$$

From this approximation it can be concluded that

$$\ln PP(n,n,n) = t(n) - t(0) \approx \big(\frac{9}{2}\ln(3) - 6\ln(2)\big)\, n^2. \tag{14}$$

Combining this with Formula (12) we get:

Proposition 1 *The number B_n of arrangements of n pseudolines is at least $2^{0.1887\, n^2}$.*

Computations were mainly done with `Maple`.

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