# Drawing $HV$-Restricted Planar Graphs

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**Abstract.** A strict orthogonal drawing of a graph $G = (V, E)$ in $\mathbb{R}^2$ is a drawing of $G$ such that each vertex is mapped to a distinct point and each edge is mapped to a horizontal or vertical line segment. A graph $G$ is $HV$-restricted if each of its edges is assigned a horizontal or vertical orientation. A strict orthogonal drawing of an $HV$-restricted graph $G$ is good if it is planar and respects the edge orientations of $G$. In this paper we give a polynomial-time algorithm to check whether a given $HV$-restricted plane graph (i.e., a planar graph with a fixed combinatorial embedding) admits a good orthogonal drawing preserving the input embedding, which settles an open question posed by Maňuch, Patterson, Poon and Thachuk (GD 2010). We then examine $HV$-restricted planar graphs (i.e., when the embedding is not fixed). Here we completely characterize the 2-connected maximum-degree-three $HV$-restricted outerplanar graphs that admit good orthogonal drawings.

## 1 Introduction

An *orthogonal drawing* $\Gamma$ of an undirected graph $G = (V, E)$ in $\mathbb{R}^2$ is a drawing of $G$ in the plane, where each vertex of $G$ is mapped to a distinct point and each edge of $G$ is mapped to an orthogonal polyline. $\Gamma$ is called *planar* if no two edges in $\Gamma$ cross, however, two edges can meet at their common endpoints. Otherwise, the drawing is a *non-planar orthogonal drawing*. Orthogonal drawings have been extensively studied over the last two decades [1, 3, 8, 14, 16] because of its applications in many practical fields such as VLSI floor-planning, circuit schematics, and entity relationship diagrams.

An orthogonal drawing is *strict* if every edge in the drawing is represented by a single vertical and horizontal line segment. In 1987, Tamassia [14] gave a polynomial-time algorithm to decide whether a plane graph (i.e., when the embedding is fixed) admits a strict orthogonal drawing preserving the input

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embedding. Later, Garg and Tamassia [6] proved that deciding strict orthogonal drawability is NP-hard for planar graphs (i.e., when the embedding is not fixed). However, polynomial time algorithms have been developed for some well-known subclasses of planar graphs. For example, Di Battista et al. [1] showed that the problem is polynomial-time solvable for series-parallel graphs and maximum-degree-three planar graphs. Nomura et al. [13] showed that every maximum-degree-three outerplanar graph admits a planar strict orthogonal drawing if and only if it contains no cycle of three vertices.

Many variants of strict orthogonal drawings impose constraints on how the edges of the input graph have to be drawn. One of these variants describes the input graph $G$ as an *LRDU-restricted graph* that associates each vertex-edge incidence of $G$ with an orientation, i.e., left(L), right(R), up(U), or down(D), and asks to find an orthogonal drawing of $G$ that respects the prescribed orientations. Another variant considers *HV-restricted graphs*, where the orientation of an edge is either horizontal(H), or vertical(V). By a *good orthogonal drawing* we denote a planar strict orthogonal drawing that preserves the input edge orientations.

In this paper we only examine strict orthogonal drawings of $HV$-restricted plane and planar graphs, and hence from now on we omit the term 'strict'.

**$HV$-Restricted Plane Graphs.** In 1985, Vijayan and Wigderson [15] gave an algorithm that can decide in linear time whether an $LRDU$-restricted plane graph admits a good orthogonal drawing, but takes $O(n^2)$ time to construct such a drawing when it exists. Later, Hoffmann and Kriegel [7] gave a linear-time construction. The task of characterizing $HV$-restricted plane graphs is more involved. The difficulty arises from the exponential number of choices for drawing $HV$-restricted paths, where the drawing of an $LRDU$-restricted path is unique, as illustrated in Figures 1(a)–(c). Recently, Maňuch et al. [10] examined several results on the non-planar orthogonal drawings of $LRDU$- and $HV$-restricted graphs. They proved that non-planar orthogonal drawability maintaining edge orientations can be decided in polynomial-time for $LRDU$-restricted graphs, but is NP-hard for $HV$-restricted graphs. An interesting open question in this context, as posed by Maňuch et al. [10], is to determine the complexity of deciding good orthogonal drawability of $HV$-restricted plane graphs. In Section 2 we settle this question by giving a polynomial-time algorithm to recognize $HV$-restricted plane graphs. Here we assume that a planar embedding of the input graph is given, and our algorithm decides whether there exists a solution that respects the input embedding.

**$HV$-Restricted Planar Graphs.** A problem analogous to drawing $LRDU$-restricted graphs in $\mathbb{R}^2$ has been well studied in $\mathbb{R}^3$, but polynomial-time algorithms are known only for cycles [4] and theta graphs [5]. The exponential number of possible orthogonal embeddings in $\mathbb{R}^3$ makes the problem very difficult. Similarly, we find the problem of characterizing $HV$-restricted planar graphs that admit good orthogonal drawings in $\mathbb{R}^2$ nontrivial even for outerplanar graphs, where the difficulty arises from the exponential number of choices for plane embeddings of the input graph.

To further illustrate the challenge, here we prove that the $HV$-restricted outerplanar graph of Figure 1(d) does not admit a good orthogonal drawing. Suppose for a contradiction that $\Gamma$ is a good orthogonal drawing of $G$, and consider the drawing of the face $F = (a, b, ..., f)$ in $\Gamma$. Since the edges $(a, b)$ and $(c, d)$ are horizontally oriented and $(a, f)$ is vertically oriented, either $(a, b)$ lies above $(e, f)$, or $(e, f)$ lies above $(a, b)$ in $\Gamma$. If $(a, b)$ lies above $(e, f)$ as in Figure 1(e), then the drawing of cycle $a, b, i, j$ would create an edge crossing (irrespective of whether it lies inside or outside of $F$). Similarly, if $(e, f)$ lies above $(a, b)$ as in Figure 1(f), then the drawing of cycle $e, f, h, g$ would create an edge crossing. Drawing both of these cycles without crossing would imply a unique drawing of $F$, as shown in Figure 1(g). However, in this case we cannot draw the cycle $c, d, k, l$ without edge crossings. In Section 3 we characterize 2-connected maximum-degree-three outerplanar graphs that admit good orthogonal drawings. Our proof is constructive, i.e., given an $HV$-restricted 2-connected maximum-degree-three outerplanar graph $G$, in polynomial time we can decide whether $G$ admits a good orthogonal drawing, and find such a drawing if it exists. Note that the construction can choose any feasible embedding (i.e., the embedding is not fixed), and the output is not necessarily outerplanar.

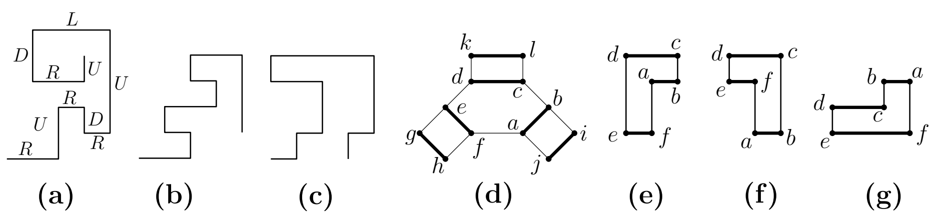



**Fig. 1.** (a) Drawing of an $LRDU$-restricted path. (b)–(c) Two different drawings of an $HV$-restricted path. (d) An $HV$-restricted outerplanar graph $G$ with maximum degree three, where the horizontal and vertical orientations are shown in black and gray, respectively. (e)–(g) Drawing of the face $F$.

## 2 Drawing *HV*-Restricted Plane Graphs

In this section we give a polynomial-time algorithm that checks whether a given $HV$-restricted plane graph admits a good orthogonal drawing that preserves the input embedding. If the answer is affirmative, the algorithm certifies its answer by constructing a good orthogonal drawing.

We will first identify some necessary conditions and later show that they are also sufficient for the existence of the good drawing. The first condition is that every vertex has most two incident edges with label $H$ and at most two with label $V$, and if the degree is four, the labels alternate. This condition is easily checked and from now on we assume it to be satisfied by the input.

Assume that a good drawing exists and consider a face $f$ in the drawing. The face is represented by a polygon, hence, if $f$ has $k$ corners, then the sum of all interior angles of $f$ must be $(k-2)\pi$ (the outer face makes an exception, here the angles sum to $(k+2)\pi$). Since $f$ is an orthogonal polygon, the angle contributed by each corner is a multiple of $\pi/2$. From the given edge orientations we can infer the angle of some corners precisely: if a corner has two incident edges with the same label, then it contributes an angle of $\pi$, and if a corner corresponds to a vertex of degree one, it contributes $2\pi$. The interesting corners are those where the incident edges have different labels, these corners contribute either $\pi/2$ or $3\pi/2$. Dual to the angle condition for faces we also have the obvious condition for vertices: around each vertex the sum of angles is $2\pi$.

Associate a variable $x_c$ with each corner $c$ of the plane graph. The above conditions can all be written as linear equations in these variables. This yields a linear system $Ax = b$ and the unified necessary condition that the system has a solution $\bar{x}$ where each component $\bar{x}_c$ is in $\{1, 2, 3, 4\}$. Such a solution is called a *global admissible angle assignment*. Similar quests for global angle assignments have been studied in rectangular drawing problems, where Miura et al. [11] reduced the problem to perfect matching, and in the context of orthogonal drawing with bends, where Tamassia [14] modeled an angle assignment problem with minimum-cost maximum-flow.

Instead of directly using the linear system stated above, we use the fact that the value of some variables $x_c$ is prescribed by the input. The value for the remaining variables and hence a global admissible angle assignment can be determined using a maximum-flow problem.

To construct the flow network start with the angle graph $A(G)$ of the plane graph $G$. The vertex set is $V_{A(G)} = V_G \cup F_G$, i.e., the vertices of $A(G)$ are the vertices and faces of $G$ or stated in just another way: the vertices of $A(G)$ are the vertices of $G$ together with the vertices of the dual $G^*$. The edges of $A(G)$ correspond to the corners of $G$: if $v \in V_G$ and $f \in V_F$ are incident at a corner $c$ then there is an edge $e_c = (v, f)$ in $E_{A(G)}$.

Next step is to remove an edge $e_c = (v, f)$ from $A(G)$ when the value of the variable $x_c$ is prescribed by the input, i.e., in the following situations:

(a) If the two edges of a corner have the same orientation and the edges are distinct, then the corner is assigned a $\pi$ angle, i.e., $x_c = 2$.
(b) If the vertex corresponding to a corner is of degree one, then the corner is assigned a $2\pi$ angle, i.e., $x_c = 4$.
(c) If the two edges of a corner have different orientations and the vertex is of degree three or more, then the corner is assigned a $\pi/2$ angle, i.e., $x_c = 1$.

Let $A^\star(G)$ be the graph after removing all these edges. Since $A(G)$ is a plane graph the same is true for $A^\star(G)$. Figure 2(a) shows an example of a graph $G$ together with the network $A^\star(G)$.

Since we want to use a fast maximum-flow algorithm, we describe the flow-problem using a planar flow network with multiple sources and sinks. It only remains to decide for some vertices of degree two in $G$ which of its corners is of

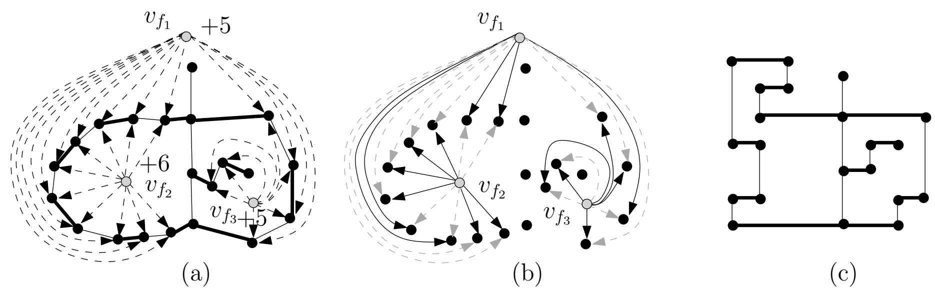



**Fig. 2.** (a) An $HV$-restricted plane graph $G$ (induced by solid edges), and its corresponding flow network $A^{\star}(G)$ (induced by dotted edges). The edges with horizontal (respectively, vertical) orientations in $G$ are bold (respectively, thin). (b) A feasible flow in $A^{\star}(G)$, where each solid edge correspond to one unit of flow. (c) A corresponding orthogonal drawing of $G$.

size $\pi/2$ and which is of size $3\pi/2$. We model a $\pi/2$ corner with a flow of one unit entering the corresponding vertex.

An original vertex $v \in V_G$ is incident to an edge in $A^{\star}(G)$ if and only if $v$ is a vertex of degree two in $G$. With these vertices we assign a demand of 1. The capacities of all the edges are also restricted to 1. Finally, we have to set the excess of all $f \in F_G$. We know the total angle sum of $f$ and the angles that have been assigned in the reduction step from $A(G)$ to $A^{\star}(G)$. Since all the remaining angles are of size $\pi/2$ or $3\pi/2$, we can compute how many of size $3\pi/2$ are needed, this number $z_f$ is the excess of $f$. (Note that if the computation yields a $z_f$ that is not an integer, then $G$ does not admit a good orthogonal realization). Similarly, we can also compute the number $z'_f$ of $\pi/2$ angles that we need. For example, for the face $f_2$ in Figure 2(a), we consider $3z_{f_2} + z'_{f_2} = 18$ and $z_{f_2} + z'_{f_2} = 10$, which solves to $(z'_{f_2}, z_{f_2}) = (4, 6)$. Since all edges $e_c \in E_{A^{\star}(G)}$ connect a source $f$ to a sink $v$, we may think of them as directed edges $f \to v$. Figure 2(b) illustrates a maximum flow for the flow-network of Figure 2(a).

We claim that a flow satisfying all the constraints (demand/excess/capacity) exists if and only if $G$ admits a good orthogonal drawing preserving the input embedding. If a flow $y \in \{0,1\}^{E_{A^{\star}(G)}}$ exists, then we get a solution vector for the linear system by defining $x_c = 3 - 2y_c$ for all $e_c \in E_{A^{\star}(G)}$. Together with the variables defined by conditions (a) – (c) we obtain a global admissible angle assignment which by definition satisfies:

1. The sum of angles around each vertex $v$ in $G$ is $2\pi$.
2. For every edge $(u, v)$ in $G$, the angle assignment at the corners of $u$ and $v$ is consistent with respect to the two faces that are incident to $(u, v)$.
3. The total assigned angle of every face $f$ is the angle sum required for polygons with that many corners. All angles are multiples of $\pi/2$, i.e., the induced representation is orthogonal.

These conditions on an angle assignment are sufficient to construct a plane orthogonal representation that respects the input embedding [14]. In fact the orthogonal drawing can be computed in linear time. Figure 2(c) shows an orthogonal representation corresponding to the flow of Figure 2(b).

For the converse, if $G$ admits a good orthogonal drawing $\Gamma$ respecting the input embedding, then the angles at the degree two vertices readily imply a flow in the network satisfying the constraints. We thus obtain the following theorem.

**Theorem 1.** *Given an $HV$-restricted plane graph $G$ with $n$ vertices, one can check in $T(n)$ time whether $G$ admits a good orthogonal drawing preserving the input embedding, and construct such a drawing if it exists. Here $T(n)$ is the time to find maximum flows in multiple-source multiple-sink directed planar graphs.*

Since the maximum flow problem for a multiple-source and multiple-sink directed planar graph can be solved in $O(n \log^3 n)$-time [2], one can check whether a given $HV$-restricted plane graph that admits a good orthogonal drawing preserving the input embedding in $O(n \log^3 n)$ time. Note that we precisely know the production or demand of each node in the flow network, and hence we are actually finding a feasible flow. There are faster algorithms in such cases, e.g., Klein et al. [9] gave an algorithm to find a feasible integral flow in $O(n \log^2 n)$-time. Later, Mozes and Wulff-Nilsen [12] improved the running time to $O(n \log^2 n / \log \log n)$.

## 3 Drawing 2-Connected Outerplanar Graphs with $\Delta = 3$

In this section we give a polynomial-time algorithm to determine whether an arbitrary 2-connected $HV$-restricted outerplanar graph with maximum degree three admits a good orthogonal drawing, and construct such a drawing if it exists. Note that the good orthogonal drawing we produce is not necessarily an outerplanar embedding. We first introduce some notation.

Let $G$ be an $HV$-restricted planar graph. By a *segment* of $G$, we denote a maximal path in $G$ such that all the edges on that path have the same orientation. A graph is *outerplanar* if it admits a planar drawing with all its vertices on the outer face. Let $G$ be a 2-connected $HV$-restricted embedded outerplanar graph with $\Delta = 3$, where $\Delta$ is the maximum degree of $G$. Let $e$ be an edge of $G$. Then by $\lambda_e$ we denote the orientation of $e$ in $G$. Let $F$ be an inner face of $G$. Note that $G$ is an embedded graph. Thus any edge of $G$ is an *inner edge* if it does not lie on the boundary of the outer face of $G$, and all the remaining edges of $G$ are the *outer edges*. An inner edge $e$ of $G$ on the boundary of $F$ is called *critical* if the two edges preceding and following $e$ have the same orientation that is different from $\lambda_e$. For example, in Figure 1(d), the edge $(a, b)$ is a critical edge of the inner face $F = (a, b, ..., f)$. An edge $e$ is *h-critical* (respectively, *v-critical*) if it is a critical edge and $\lambda_e = H$ (respectively, $\lambda_e = V$). For some inner face $F$ in $G$, let $E_v(F)$ and $E_h(F)$ be the number of distinct edges of $F$ with vertical and horizontal orientations, respectively. By $C_v(F)$ and $C_h(F)$ we denote the number of $v$-critical and $h$-critical edges of $F$.

Let $pqrs$ be a rectangle, and let $a$ and $b$ be two points in the proper interior of $qr$ and $rs$, respectively, as shown in Figures 3(a) and (b). Construct a rectangle $sbcd$, where $c$ and $d$ lie outside of the rectangle $pqrs$. Then the region consisting of the rectangles $pqrs$ and $sbcd$ is called a *flag*. A flag includes all the segments on its boundary except the segment $aq$. The rectangles $pqrs$ and $sbcd$ are called the *banner* and *post*, respectively. The segments $ar$ and $br$ are called the *borders* of the flag.

### 3.1 Necessary and Sufficient Conditions

Throughout this section, $G$ denotes an arbitrary 2-connected $HV$-restricted embedded outerplanar graph with $\Delta = 3$; see Figure 3(c) for an example. We now prove the following theorem, which is the main result of this section.

**Theorem 2.** *Let $G$ be a 2-connected $HV$-restricted embedded outerplanar graph with maximum degree three. Then $G$ admits a good planar orthogonal drawing if and only if the following three conditions hold.*

- *($C_1$) For every inner face $f$, the sequence of orientations of the edges in clockwise order contains $HVHV$ as a subsequence .*
- *($C_2$) For every inner face $f$, if $C_v(f) = E_v(f)$, then $C_v(f)$ is even. Similarly, if $C_h(f) = E_h(f)$, then $C_h(f)$ is even.*
- *($C_3$) Every vertex of $G$ has at most two edges of the same orientation.*

### 3.2 Necessity

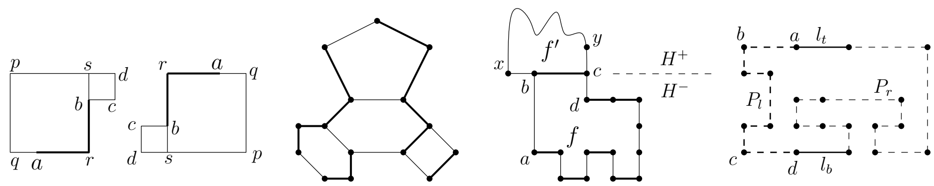
We first show that Conditions ($C_1$)–($C_3$) are necessary for $G$ to admit a good planar orthogonal drawing. We use the following two lemmas.

**Lemma 1.** *Let $\Gamma$ be a good orthogonal drawing of $G$, and let $(b, c)$ be an inner edge of some face $f = (a, b, c, d, \ldots, a)$. Figure 3(d) illustrates an example. Since $(b, c)$ is an inner edge, there is another face $f' = (b, x, \ldots, y, c, b)$ that does not contain any edge of $f$ except $(b, c)$. Let $H^+$ and $H^-$ be the two half-planes determined by the straight line through $(b, c)$. If $(b, c)$ is a critical edge in $f$, then either both $(a, b)$ and $(c, d)$ lie in $H^+$, or both lie in $H^-$.*

*Proof.* Without loss of generality assume that $\lambda_{bc} = H$. Since $(b, c)$ is a critical edge, $\lambda_{bc} \neq \lambda_{ab}$ and $\lambda_{ab} = \lambda_{cd}$. If $(a, b)$ and $(c, d)$ lie in $H^+$ and $H^-$, respectively, then one of $x$ and $y$ must lie interior to $f$ and the other must lie exterior to $f$. Therefore, the path $b, x, ..., y, c$ must create an edge crossing with $f$, which contradicts that $\Gamma$ is a good orthogonal drawing. □

Let $x(v)$ and $y(v)$ denote the $x$- and $y$-coordinates of a vertex $v$. We now use Lemma 1 to prove the following.

**Lemma 2.** *Let $\Gamma$ be good orthogonal drawing of $G$. Let $f$ be an inner face in $\Gamma$, and let $(a, b)$ and $(c, d)$ be two edges on $f$ (without loss of generality assume that $(a, b)$ is above $(c, d)$), where $\lambda_{ab} = \lambda_{cd} = H$, $x(a) > x(b)$ and $x(d) > x(c)$. Let $P = (a, b, ..., c, d)$ be a path on the boundary of $f$ in anticlockwise order, e.g., see the path $P_l$ in Figure 3(e). If all the vertically oriented edges of $P$ are critical, then the number of such critical edges on $P$ must be odd. This property holds symmetrically for $P = (b, a, ..., d, c)$.*




(a) (b) (c) (d) (e)

**Fig. 3.** (a)–(b) Two flags, where the borders are shown in bold. (c) An outerplanar graph $G$ with $\Delta = 3$. (d) Illustration for Lemma 1. (e) Illustration for $P_l$ and $P_r$, where $P_l$ contains three $v$-critical edges and $P_r$ contains five $v$-critical edges.

*Proof.* Consider a traversal of the edges of $P$ starting at $a$. Let $e$ be a $v$-critical edge on $P$, and let $e'$ and $e''$ be the edges preceding and following $e$, respectively. By Lemma 1, $e'$ and $e''$ must lie on the same side of $e$ in $\Gamma$. Therefore, if we traverse $e'$ from left to right, then we have to traverse $e''$ from right to left, and vice versa. In other words, every $v$-critical edge reverses the direction of traversal. Since we traverse $(a, b)$ and $(c, d)$ from opposite directions and all the vertically oriented edges of $P$ are critical, we need an odd number of $v$-critical edges on $P$ to complete the traversal. □

We are now ready to prove the necessity part of Theorem 2.

If $(C_1)$ does not hold for some $f$, then the face $f$ does not admit a planar orthogonal drawing. Because, drawing $f$ would require the sum of the interior angles of the corresponding polygon to be at least $2\pi$.

If $(C_2)$ does not hold, then without loss of generality assume that for some $f$, $C_v(f) = E_v(f)$ and $C_v(f)$ is odd. Let $\Gamma_f$ be a drawing of $f$ such that $l_t$ and $l_b$ are topmost and bottommost horizontal edges in $\Gamma_f$. Then we can find two disjoint paths $P_l$ and $P_r$ by traversing $f$ anticlockwise and clockwise from $l_t$ to $l_b$, respectively, as shown in Figure 3(e). Since $C_v(f)$ is odd, either $P_l$ or $P_r$ must contain an even number of $v$-critical edges, which contradicts Lemma 2.

If $(C_3)$ does not hold at some vertex $v$, then the drawing of its incident edges would contain edge overlapping.

### 3.3 Sufficiency

To prove the sufficiency we assume that $G$ satisfies $(C_1)$–$(C_3)$, and then construct a good orthogonal drawing of $G$. The idea is to first draw an arbitrary inner face $f$ of $G$, and then the other faces of $G$ by a depth first search on the faces of $G$ starting at $f$.

Let $f = (v_1, v_2, \ldots, v_r, \ldots, v_s, \ldots, v_t(= v_1))$ be the vertices of $f$ in clockwise order. Let $P = (v_r, \ldots, v_s, \ldots, v_t)$ be a maximal path on $f$ such that all the edges on path $P_v = (v_r, \ldots, v_s)$ (respectively, $P_h = (v_s, \ldots, v_t)$) have vertical (respectively, horizontal) orientation. The maximality of $P$ ensures that $\lambda_{v_1 v_2} = V$ and $\lambda_{v_{r-1} v_r} = H$. An example of such a path $P$ in the face of Figure 4(a) is $a(= v_r), b, c(= v_s), d, e(= v_t)$. Observe that $\lambda_{v_1 v_2} = \lambda_{eg} = V$ and $\lambda_{v_{r-1} v_r} = \lambda_{ia} = H$. We now have the following lemma.

**Lemma 3.** *Given an inner face $f$ of $G$ that satisfies conditions $(C_1)$–$(C_3)$, and a drawing of two consecutive segments $P_h$ and $P_v$ of $f$. One can find a good orthogonal drawing $\Gamma_f$ of $f$ that satisfies the following properties.*

- *Lemma 1 holds for every critical edge $e$ in $\Gamma_f$, i.e., the two edges preceding and following $e$ lie in the same side of $e$.*
- *$\Gamma_f$ is contained in a flag $F$ with borders $P_h$ and $P_v$.*
- *If $P_h$ is a critical edge, then the post of $F$ (if exists) is incident to $P_v$. Similarly, if $P_v$ is a critical edge, then the post of $\Gamma_f$ (if exists) is incident to $P_h$. (Note that since $\Delta = 3$, both $P_h$ and $P_v$ cannot be critical).*

*Proof.* Due to space constraints, here we only sketch the steps of the proof.

We first prove that if $f$ satisfies Conditions $(C_1)$–$(C_3)$, then $f$ admits a good orthogonal drawing such that Lemma 1 holds for every critical edge of $f$. Our proof is constructive. We construct two drawings $\Gamma_{f_1}$ and $\Gamma_{f_2}$ of $f$, and prove that one of these two drawings satisfies the lemma. Since $f$ satisfies $(C_1)$, $P$ must contain at least three vertices. We first draw the path $P$ maintaining edge orientations.Let the drawing be $\Gamma_P$. We next draw $P' = (v_1, v_2, \ldots, v_r)$ in two different ways that give the drawings $\Gamma_{f_1}$ and $\Gamma_{f_2}$, as follows.

**Construction of $\Gamma_{f_1}$.** We construct $\Gamma_{f_1}$ in three steps. At Step 1, we draw $P'$ starting at $v_1$ such that every $v$-critical edge $e$ of $P'$ satisfies Lemma 1. However, the position of $v_r$ in the drawing of $P'$ may not coincide with its position in $\Gamma_P$. Let the resulting drawing of $P'$ be $\Gamma_{P'}$. At Step 2, we modify $\Gamma_{P'}$ such that Lemma 1 holds for every $h$-critical edge, except possibly $(v_{r-1}, v_r)$. While modifying $\Gamma_{P'}$, we ensure that the $v$-critical edges still satisfy Lemma 1. Therefore, after Step 2, the resulting drawing $\Gamma'_{P'}$ has all its critical edges, except possibly $(v_{r-1}, v_r)$, satisfying Lemma 1. At Step 3, we modify the drawing such that the positions of $v_r$ in $\Gamma'_{P'}$ and $\Gamma_P$ coincide. Thus after Step 3, we obtain a drawing $\Gamma_{f_1}$ of $f$ that respects all the edge orientations, furthermore, all the critical edges, except possibly $(v_{r-1}, v_r)$, satisfy Lemma 1.

**Construction of $\Gamma_{f_2}$.** To construct $\Gamma_{f_2}$, we start drawing $P'$ at $v_r$ of $\Gamma_P$, and then the construction is symmetric, i.e., here we treat the horizontal (respectively, vertical) orientations as the vertical (respectively, horizontal) orientations.

**Either $\Gamma_{f_1}$ or $\Gamma_{f_2}$ satisfies Lemma 3.** We first prove that one of $\Gamma_{f_1}$ and $\Gamma_{f_2}$ is a good orthogonal drawing and Lemma 1 holds for each of its critical edge. The idea of the proof is as follows. We first prove that both $\Gamma_{f_1}$ and $\Gamma_{f_2}$ are good. We next prove that if Lemma 1 does not hold for the critical edges in $\Gamma_{f_1}$, then $P'$ cannot contain any $v$-critical edge and $P_h$ cannot be an $h$-critical edge. We show that in such a scenario, Lemma 1 must hold for every critical edge in $\Gamma_{f_2}$. As a byproduct of our construction, we obtain the remaining two properties of $\Gamma_f$, i.e., $\Gamma_f$ is contained in a flag $F$ with borders $P_h$ and $P_v$, and if $P_h$ (respectively, $P_v$) is a critical edge, then the post of $F$ (if exists) is incident to $P_v$ (respectively, $P_h$). □

We are now ready to describe the drawing of $G$. We first construct the drawing $\Gamma_f$ for some inter face $f$ of $G$. We then draw the other inner faces of $G$ by a depth first search on the faces of $G$, such that after adding a new inner face, the resulting drawing remains

($P_1$) a good orthogonal drawing, and
($P_2$) each critical edge respects Lemma 1.

Let $\Gamma_k$ be a drawing of the set of inner faces $f_1(= f), f_2, \ldots, f_k$ that we have already constructed. Let $f_{k+1}$ be an inner face of $G$ that has not been drawn yet, but has an edge $(b, c)$ in common with some face $f_j$, where $1 \leq j \leq k$. Without loss of generality assume that $\lambda_{bc} = V$ in $\Gamma_k$. Furthermore, since $G$ is outerplanar, $f_{k+1}$ cannot have any edge other than $(b, c)$ in common with $f_j$. Let $l_v$ be a segment of $f_{k+1}$ that contains $(b, c)$, and let $l_h$ be another segment of $f_{k+1}$ incident to $l_v$. We now construct $\Gamma_{k+1}$ considering the following cases.

**Case 1 (None of $b$ and $c$ is an end vertex of $l_v$):** In this case none of the end vertices of the path formed by $l_v$ and $l_h$ belongs to $\Gamma_k$. Since $G$ satisfies Condition ($C_3$), the edges of $f_j$ that are incident to $b$ and $c$ must be horizontal, i.e., $(b, c)$ must be a $v$-critical edge of $f_j$. Since $\Gamma_k$ is a good orthogonal drawing, there is enough space to create a flag $F$ with borders $l_v$ and $l_h$ such that the banner and post of $F$ do not create any edge crossing. Figure 4(c) illustrates such an example. By Lemma 3, we can draw $f_{k+1}$ inside $F$ maintaining Properties ($P_1$) and ($P_2$). Thus the resulting drawing $\Gamma_{k+1}$ satisfy ($P_1$)–($P_2$).

**Case 2 (Exactly one of $b$ and $c$ is an end vertex of $l_v$):** If $b$ (respectively, $c$) is an end vertex of $l_v$, then we choose $l_h$ such that it contains $b$ (respectively, $c$). Therefore, none of the end vertices of the path formed by $l_v$ and $l_h$ belongs to $\Gamma_k$. Figure 4(d) illustrates such an example. Similar to Case 1, we now draw $\Gamma_{k+1}$ satisfying ($P_1$)–($P_2$).

**Case 3 (Both $b$ and $c$ are end vertices of $l_v$):** Observe that in this case $l_v = (b, c)$. Let $a, b, c, d$ be a path of $f_{k+1}$. Since $l_v = (b, c)$ is a maximal set of edges with vertical orientation, we have $\lambda_{ab} = \lambda_{bc} = H$. Thus $l_v = (b, c)$ is a $v$-critical edge of $f_{k+1}$. We now create a flag $F$ with borders $l_v$ and $l_h$ such that the post of the flag is incident to $l_h$. Note that since $l_v$ is critical, by

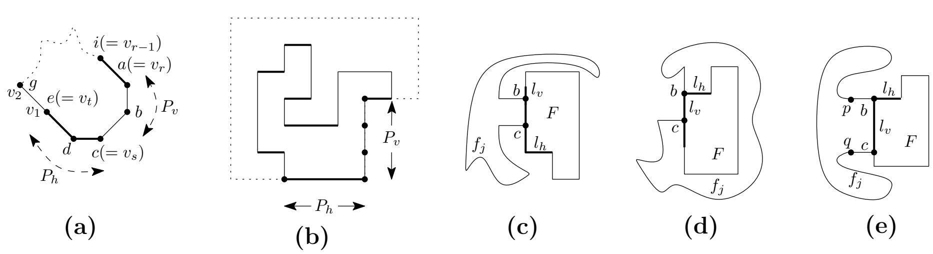



**Fig. 4.** (a) An inner face of $G$. (b) Illustration for $\Gamma_f$. (c)–(e) Illustration for the construction of $\Gamma_{k+1}$.

Lemma 3, we do not require a flag with its post incident to $l_v$. We now can draw $f_{k+1}$ inside $F$ maintaining $(P_2)$ and $(P_3)$. Figure 4(e) illustrates such an example. It may initially appear from the figure that drawing of $f_{k+1}$ inside $F$ may overlap the boundary of $f_j$, i.e., consider the Figure 4(e) with $\lambda_{cq} = V$. However, by definition of a flag, $F$ does not contain the part of its boundary that overlaps $f_j$, and hence drawing $f_{k+1}$ would not create any edge overlapping.

## 4 Conclusion

In Section 2 we have developed a polynomial-time algorithm to decide good orthogonal drawability of $HV$-restricted plane graphs. An interesting open question in this context, as Maňuch et al. [10] asked, is to determine the complexity of deciding good orthogonal drawability for $HV$-restricted planar graphs.

**Problem 1.** What is the time complexity of deciding whether an arbitrary $HV$-restricted planar graph admits a planar orthogonal drawing preserving the given edge orientations?

In Section 3 we have characterized $HV$-restricted 2-connected maximum-degree-three outerplanar graphs that admit good orthogonal drawings. If we relax the 2-connected constraint, then our characterization no longer holds. For example, the $HV$-restricted outerplanar graph $G$ of Figure 5(b) satisfies Conditions $(C_1)$-$(C_3)$ of Theorem 2, but does not admit any good orthogonal drawing.

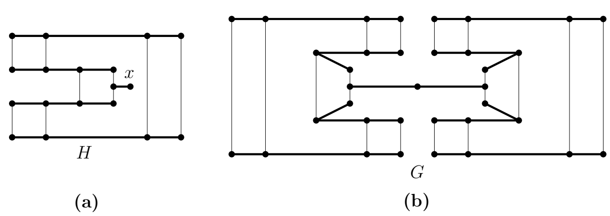



**Fig. 5.** Illustration for the graphs (a) $H$ and (b) $G$.

Observe that $G$ is constructed from two copies of the graph $H$ of Figure 5(a), where the vertices with label $x$ are identified. Since in any good orthogonal drawing of $H$ the vertex $x$ lies in some inner face, any orthogonal drawing of $G$ preserving edge orientations must contain edge crossing. Hence a natural open question is to extend our result for arbitrary outerplanar graphs.

**Problem 2.** Characterize the class of $HV$-restricted outerplanar graphs that admit planar orthogonal drawings preserving the given edge orientations.

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