Henneberg Steps for Triangle Representations*

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Abstract. Which plane graphs admit a straight line representation such that all faces have the shape of a triangle? In previous work we have studied necessary and sufficient conditions based on flat angle assignments, i.e., selections of angles of the graph that have size π in the representation. A flat angle assignment that fullfills these conditions is called good. The complexity for checking whether a graph has a good flat angle assignment remains unknown.

In this paper we deal with extensions of good flat angle assignments. We show that if G has a good flat angle assignment and G^+ is obtained via a planar Henneberg step of type 2, then G^+ also admits a good flat angle assignment. A similar result holds for certain combinations of Henneberg type 1 steps followed by a type 2 step. As a consequence we obtain a large class of pseudo-triangulations that admit drawings such that all faces have the shape of a triangle. In particular, every 3-connected, plane generic circuit admits a good flat angle assignment.

1 Introduction

In this paper we study a representation of planar graphs in the classical setting, i.e., vertices are presented as points in the Euclidean plane and edges as straight line segments. We are interested in the class of planar graphs that admit a representation in which all faces are triangles. Note that in such a representation each face f has exactly $\deg(f) - 3$ incident vertices that have an angle of size π in f. Conversely each vertex has at most one angle of size π. In [2] we have studied necessary and sufficient conditions based on flat angle assignments, i.e., selections of angles of the graph that have size π in the representation. Flat angle assignments that fullfill these conditions are called *good*. The complexity for checking whether a graph has a good flat angle assignment remains unknown.

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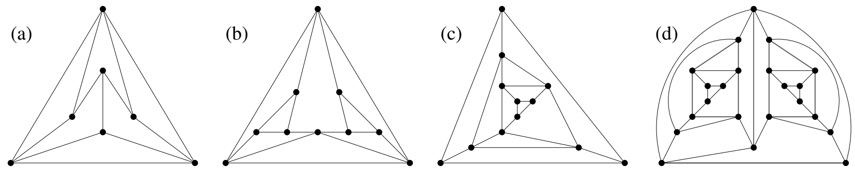

Figure 1: (a) A pseudo-triangulation that does not induce an SLTR, (b) a Laman graph that has an SLTR but can not be constructed using only the two steps we give in Section 2, (c) a planar Laman graph that has no SLTR for this embedding, (d) a planar Laman graph that has no SLTR.

A *pseudo-triangle* is a simple polygon with precisely three convex angles, all other vertices of the polygon admit a concave angle at the interior of the polygon. A *pseudo-triangulation* (PT) is a planar graph with a drawing such that all faces are pseudo-triangles. An example of a PT is given in Fig. 1 (a).

A pseudo-triangulation is *pointed* if each vertex has an angle of size $> \pi$. A pointed pseudotriangulation with n vertices must have exactly $2n-3$ edges. Indeed pointed pseudotriangulations have the Laman property: they have $2n-3$ edges, and subgraphs induced by k vertices have at most $2k-3$ edges. Laman graphs, and hence also pointed pseudotriangulations, are minimally rigid graphs. A detailed survey on pseudo-triangulations has been given by Rote et al. [8].

Pseudotriangulations induce an assignment of big angles to vertices. This assignment is closely related to a flat angle assignment.

A **Straight Line Triangle Representation** (SLTR) of a graph G is a plane drawing of G such that all edges are straight line segments and all faces are triangles (e.g. Fig. 1 (b)). Throughout this paper $G = (V, E)$ will be a plane, internally 3-connected graph. Three vertices which are the corners of the outer face in an SLT Representation of the graph are given and we call these vertices suspensions. A plane graph G with suspensions s_1, s_2, s_3 is said to be *internally 3-connected* when the addition of a new vertex v_∞ in the outer face, that is made adjacent to the three suspension vertices, yields a 3-connected plane graph.

A *flat angle assignment* (FAA) of a graph is a mapping from a subset U of the non-suspension vertices to faces such that, the vertex is incident to the face and,

[C$_v$] Every vertex of U is assigned to at most one face,

[C$_f$] For every face f, precisely $|f|-3$ vertices are assigned to f.

An FAA is called *good* (GFAA) when it induces an SLTR. In [2] we have shown that an FAA is good if and only if it induces a contact family of pseudosegments Σ which has the following property:

[C$_P$] Every subset S of Σ with $|S| \geq 2$ has at least three free points.

Informally, pseudosegments arise from merging the edges that are incident to an assigned angle of a vertex, the vertex will be an interior point of the pseudosegment. Since a vertex is assigned at most once, the pseudosegments do not cross. The pseudosegments will be stretched to straight line segments to obtain an SLTR. Let p be an endpoint of a pseudosegment in $S \subseteq \Sigma$. If p is a suspension vertex, then p is a free point for S. When p is not a suspension then it is a free point for S if: p is incident to the unbounded region of S, it has at least one neighbor not in S and it is not an interior point for a pseudosegment in S.

The drawback of this characterization is that we are not aware of an efficient way to test whether a given graph has an FAA that is good.

A *combinatorial pseudo-triangulation* (CPT) is an assignment of the labels *big* and *small* to the angles around each vertex. Each vertex has at most one angle labeled *big* and each inner face has precisely three incident angles labeled *small*, the outer face has all angles labeled *big*. For an interior angle labeled *big*, let the incident vertex be assigned to the incident face, and a vertex is not assigned if it has no angle labeled *big*. Three vertices of the outer face are chosen to be the suspensions, the other vertices are assigned to the outer face. Hence a CPT induces an FAA and the similarly an FAA induces a CPT.

A CPT does not always induce a PT, Orden et al. have shown that the *generalized Laman condition* is necessary and sufficient for a CPT to induce a PT [7].

Lemma 1.1 (Generalized Laman Condition)**.** Let G be the graph of a pseudo-triangulation of a planar point set in general position. Every subset of x not assigned vertices plus y assigned vertices of G, with $x+y \geq 2$ spans a subgraph with at most $3x + 2y - 3$ edges.

Proposition 1.2. A GFAA of an internally 3-connected plane graph satisfies the generalized Laman Condition.

An FAA that is *not* good may also satisfy the generalized Laman Condition (e.g. Fig. 1 (a)), therefore this condition is necessary but not sufficient. Every Laman graph can be constructed from an edge by Henneberg steps [6, 9]. A graph $G = (V, E)$, with $|E| = 2|V| - 3$ that has an SLTR, has a CPT that induces a PT by Prop. 1.2 and by the result of Haas et al. it must be a Laman graph. Therefore it must have a Henneberg construction. In the next section we will investigate how to use this construction such that a GFAA can be extended along the steps.

2 Construction Steps

It has been shown that planar Laman graphs admit a planar Henneberg construction [5]. Since we consider plane graphs (with a given set of suspension vertices), we consider the Henneberg steps in a plane setting. Recall that the graph must be internally 3-connected.

Henneberg Type 2 Step

Given a graph G and a GFAA ψ of G. A Henneberg Type 2 step (HEN_2) subdivides an edge uv and connects the new vertex x to a third vertex w (see Fig. 2). The face f, incident to uv and w is splitted into f_u (the face incident to u) and

f_v. The other face incident to uv is denoted with f_x. The resulting graph is denoted G^+. We will construct an assignment ψ^+ for G^+ and proof that ψ^+ is a GFAA.

There are three vertices not assigned to f under ψ, we will call them *corners* of f. We consider two cases, firstly f_u is incident to all corners of f, secondly, f_u is incident to precisely two corners of f. Note that if w is a corner of f it will be a corner for both f_u and f_v. The vertices different from u, v, w, x, that are assigned to f under ψ, will be assigned in the trivial way under ψ^+, i.e., such a vertex is assigned to f_u resp. f_v, if in G^+ it is incident to f_u resp. f_v.

Case 1: *f_u is incident to all corners of f.* If u or w is assigned to f under ψ, it is assigned to f_u under ψ^+. The vertex v is assigned to f_x and x to f_u under ψ^+.

Case 2: *f_u is incident to precisely two corners of f.* If u or w is assigned to f under ψ, it is assigned to f_u under ψ^+, if v was assigned to f it is assigned to f_v under ψ^+ and x is assigned to f_x.

This yields an assignment ψ^+ for G^+.

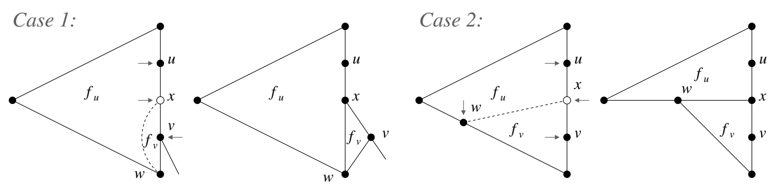

Figure 2: Updating the assignment after a HEN_2 step. The white vertex and dashed edge represent the step.

Theorem 2.1. Given a 3-connected, plane graph G with a GFAA ψ. Let G^+ be the result of a HEN_2 step applied to G and let ψ^+ be the updated assignment. Then ψ^+ is a GFAA and G^+ admits an SLTR.

Proof. It is easy to see that ψ^+ satisfies C_v and C_f and hence is an FAA.

We consider the induced families of pseudosegments, Σ and Σ^+ of ψ and ψ^+ respectively. Since ψ is a Good FAA, we know that every subset S of Σ has at least three free points or S has cardinality at most one. Let $S \subseteq \Sigma^+$ of cardinality at least two, and for every pseudosegment p of Σ^+ which is not a pseudosegment of Σ consider $p \in S$ and show that S has at least three free points by using that there is an equivalent set under Σ which has at least three free points.

In both cases there are three pseudosegments that have changed, we will only discuss Case 1 here. Let s_x resp. s_v be the pseudosegment that has x resp. v as interior point and let s_w the pseudosegment containing the edge vw.

- If $s_x \in S$ then replace s_x by the pseudosegment s'_x of Σ that has u and v as interior points.

- If $s_v \in S$ then replace s_v by the pseudosegment s'_v of Σ that ends in v and contains all the edges, except vx, of s_v.
- If $s_w \in S$ then delete s_w.

Now we have a set $S' \in \Sigma$, thus S' has three free points unless $|S'| = 1$.
- If $s_x \in S$ then s_x contributes the same free points to S as s'_x to S'.
- If $s_v \in S$ then if v was a free point for S', x is for S. Hence s_v contributes the same number of free points to S as s'_v to S'.
- If $s_w \in S$ then if $|S'| = 1$, s_w contributes at least one free point to S and it covers no other points, thus S has three free points, or, if $|S'| > 1$ then S' has at least three free points, adding s_w does not cover any of them and therefore S has at least three free points.

We conclude that in Case 1 every set S of cardinality at least two has at least three free points. The argumentation for Case 2 is similar and it follows that ψ^+ is a GFAA. □

A graph $G = (V, E)$ is a *generic circuit* if $|E| = 2|V| - 2$ and subgraphs induced by k vertices have at most $2k - 3$ edges. The generic circuit with the least number of vertices is K_4.

Theorem 2.2. Every 3-connected, plane, generic circuit admits an SLTR.

Proof. A 3-connected, generic circuit can be constructed with HEN_2 steps from K_4 (Berg and Jordán [3]) and K_4 admits an SLTR. Every plane 3-connected generic circuit can be constructed with HEN_2 steps from K_4 such that all intermediate graphs are plane. By Thm. 2.1 we have that every 3-connected, plane generic circuit admits an SLTR. □

Henneberg Combination Step

For connectivity reasons single HEN_1 steps are not compatible with SLTRs. However certain sequences of HEN_1 steps followed by a HEN_2 step allow for the extension of a GFAA. Next we will describe such a combination step denoted HEN_{1^n2}.

Let f be a face with $n + 1$ vertices. The first HEN_1 step stacks a new vertex v_0 over an edge of f. The vertices $v_0, \dots, v_{n-1}$ are introduced by the n HEN_1 steps in such a way that $v_0, \dots, v_{n-1}$ is a path and each vertex of f is a neighbor of some v_i. The final HEN_2 step subdivides an edge which is incident to some v_i (not v_{n-1}) and connects the new vertex v_n to v_{n-1}. Of course the construction has to maintain planarity.

Theorem 2.3. Given a 3-connected, plane graph G with a GFAA ψ. Let G_n be the result of a HEN_{1^n2} step applied to G and ψ_n be the updated assignment. Then ψ_n is a GFAA and G_n admits an SLTR.

Due to the lack of space we have left out the algorithm that decides how to update the assignment, this and the proof of Thm. 2.3 can be found in the full version.

3 Conclusion and Open Problems

We have given two construction steps such that a GFAA can be extended along these steps and the extended assignment is also a GFAA. However, this does not define the class of Laman graphs that have an SLTR. Therefore the problem: Is the recognition of graphs that have an SLTR (GFAA) in P? is still open, even for graphs in which all non-suspension vertices have to be assigned.

The class of 3-connected quadrangulations is well-defined, e.g. Brinkmann et al. give a characterization using two expansion steps [4]. Adding a diagonal edge in the outer face of a plane, 3-connected quadrangulation yields a Laman graph. One of the expansion steps (denoted P_3 in [4]) is a Henneberg Combination step, hence a GFAA can be extended along this step. It would be interesting to know if a GFAA could also be extended along the other expansion step (denoted P_1 in [4]).

Adding an edge in a plane graph that has a GFAA requires only minor changes to the GFAA of the original graph to obtain a GFAA for the resulting graph. An interesting question arises: Does every graph that admits an SLTR in which not every non-suspension vertex admits a straight angle, have a spanning Laman subgraph that admits an SLTR?

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