Using SAT to study plane Hamiltonian substructures in simple drawings*†

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Abstract

In a *simple drawing* of a complete graph, all edges are drawn as simple curves and every pair of edges intersects in at most one point which is either a common vertex or a proper crossing. Simple drawings generalize straight-line drawings in a similar vein as abstract order types generalize point sets. They have been studied for many years and have become a source for many open problems and conjectures.

We provide a SAT framework which allows enumerating simple drawings with specified properties or to decide that no such drawing exists. Using this framework we look at open problems on classes of simple drawings. Based on the data, we provide new strengthenings and modifications of existing conjectures. Some of these problems concern non-crossing substructures in simple drawings. The most prominent example may be the conjecture by Rafla (1988), which asserts that every simple drawing of the complete graph K_n contains a plane Hamiltonian cycle. Today, however, only the existence of plane paths of logarithmic size and plane matchings of size $\Omega(\sqrt{n})$ is known (Suk and Zeng 2022; Aichholzer et al. 2022). Here we present a proof for the existence of plane Hamiltonian subgraphs with $2n-3$ edges in *convex drawings* which are a rich subclass of simple drawings. Our proof comes with a polynomial time algorithm.

1 Introduction

In a *simple drawing* of a graph in the plane (resp. on the sphere), the vertices are mapped to distinct points, and edges are drawn as simple curves that connect the corresponding endpoints but do not contain other vertices. Moreover, every pair of edges intersects in at most one point, which is either a common vertex or a proper crossing (no touching), and no three edges cross at a common point. Figure 1 shows the obstructions to simple drawings. Throughout this article, we will only consider simple drawings of the complete graph K_n.

Problems related to simple drawings have attracted a lot of attention. One of the reasons is that simple drawings are closely related to interesting classes of drawings such as crossing-minimal drawings or straight-line drawings, a.k.a. *geometric drawings*. The focus of this article will be on the class of convex drawings, which was recently introduced by Arroyo

* Supplemental source code and data is available at [1]

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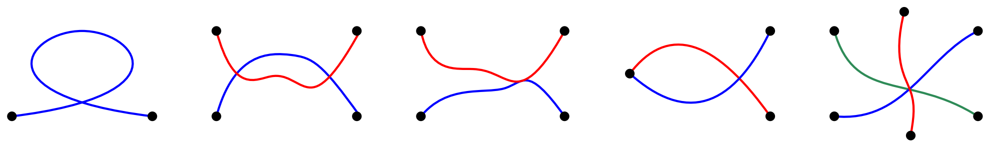


Figure 1 The five obstructions to simple drawings.

et al. [5]. Convex drawings are nested between simple drawings and geometric drawings, and a large subclass of convex drawings is in correspondence with acyclic oriented matroids of rank 3. For further aspects of the convexity hierarchy we refer the interested reader to [5, 4, 6].

The notion of convexity is based on the *triangles* of a drawing, i.e., the subdrawings induced by three vertices. Since the edges of a triangle do not cross in simple drawings, a triangle partitions the plane (resp. sphere) into exactly two connected components. The closures of these components are the two *sides* of the triangle. A side S of a triangle is *convex* if every edge that has its two vertices in S is completely drawn inside S. A simple drawing of the K_n is *convex* if every triangle has a convex side.

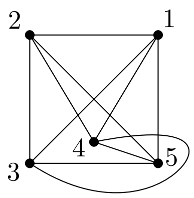
To study forced and forbidden substructures in general simple drawings and subclasses such as convex drawings, we develop a Python program which generates a Boolean formula in conjunctive normal form (CNF) that is satisfiable if and only if there exists a simple drawing of K_n (for a specified value of n) with prescribed properties. Moreover, the solutions of these instances are in one-to-one correspondence with non-isomorphic simple drawings with the prescribed properties. We then use the state of the art SAT solvers PicoSAT [7] and CaDiCaL [8] to decide whether a solution exists and to enumerate the solutions. Unsatisfiability results can be verified using the independent proof checking tool DRAT-trim [21].

In order to encode a simple drawing of the complete graph in terms of a CNF, it is important to note that combinatorial properties such as pairs of crossings are fully determined by the rotation system. For a given simple drawing D and a vertex v of D, the cyclic order π_v of incident edges in counterclockwise order around v is called the *rotation of v* in D. The collection of rotations of all vertices is called the *rotation system* of D. More specifically, the rotation system captures the combinatorial properties of a simple drawing on the sphere – the choice of the outer cell when stereographically projecting the drawing onto a plane has no effect on the rotation system.

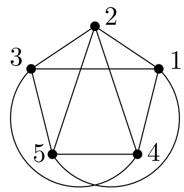
Π_4^o :	$\Pi_{5,1}^o$:	$\Pi_{5,2}^o$:
π_1 : 2 3 4	π_1 : 2 3 4 5	π_1 : 2 3 4 5
π_2 : 1 3 4	π_2 : 1 3 4 5	π_2 : 1 3 5 4
π_3 : 1 2 4	π_3 : 1 4 2 5	π_3 : 1 4 2 5
π_4 : 1 3 2	π_4 : 1 5 3 2	π_4 : 1 5 3 2
	π_5 : 1 4 2 3	π_5 : 1 2 4 3

Figure 2 The three obstructions Π_4^o, $\Pi_{5,1}^o$, and $\Pi_{5,2}^o$ for rotation systems.

The SAT encoding uses Boolean variables and clauses to encode the rotations of vertices. To assert that the prescribed permutations for the vertices (which we refer to as *pre-rotation system*) can be realized by a simple drawing, we use Boolean formulas to forbid obstructions. A computational result of Ábrego et al. [2] together with a result of Kynčl [14, Theorem 1.1] characterizes drawability in terms of induced 4- and 5-tuples: A pre-rotation system on n elements is drawable if and only if it does not contain Π_4^o, $\Pi_{5,1}^o$ and $\Pi_{5,2}^o$ (Figure 2) as a subconfiguration. To encode convex drawings, we use of result by Arroyo et al. [5] that characterizes convex drawings in terms of 5-tuples: A simple drawing is *convex* if and only if it does not contain $\Pi^{\text{oc}}_{5,1}$ or $\Pi^{\text{oc}}_{5,2}$ (depicted in Figure 3) as a subconfiguration.



$\Pi^{\text{oc}}_{5,1}$:
π_1 : 2 3 4 5
π_2 : 1 3 4 5
π_3 : 1 2 4 5
π_4 : 1 2 5 3
π_5 : 1 2 4 3



$\Pi^{\text{oc}}_{5,2}$:
π_1 : 2 3 4 5
π_2 : 1 3 5 4
π_3 : 1 2 4 5
π_4 : 1 2 5 3
π_5 : 1 4 2 3

Figure 3 The two obstructions $\Pi^{\text{oc}}_{5,1}$ and $\Pi^{\text{oc}}_{5,2}$ for convex drawings.

We discuss questions regarding plane substructures in drawings of the complete graph. In particular we focus on variants of Rafla's conjecture [17], which asserts that every simple drawing of the complete graph contains a plane Hamiltonian cycle.

▶ **Conjecture 1.1** ([17]). *Every simple drawing of K_n contains a plane Hamiltonian cycle.*

Rafla verified the conjecture for $n \leq 7$. Later Ábrego et al. [2] did a complete enumeration of all rotation systems for $n \leq 9$ and verified the conjecture for $n \leq 9$. Furthermore the conjecture was proven for some particular subclasses such as geometric, monotone and cylindrical drawings. With the SAT framework we are able to verify Conjecture 1.1 for all $n \leq 10$.

Recently, Suk and Zeng [20] and Aichholzer et al. [3] independently showed that simple drawings contain a plane path of length $(\log n)^{1-o(1)}$. Suk and Zeng also showed that every simple drawing of K_n contains a plane copy of every tree on $(\log n)^{1/4-o(1)}$ vertices. Aichholzer et al. moreover showed the existence of a plane matching of size $\Omega(n^{1/2})$, which improves older bounds from a series of papers [15, 16, 9, 19, 11, 10, 18].

Fulek and Ruiz–Vargas [11, Lemma 2.1] showed that every simple drawing of K_n contains a plane subdrawing with $2n-3$ edges. Note that this bound is best-possible because every plane subgraph of the straight-line drawing of K_n on a point set in convex position (see Figure 4(left)) is outerplanar and thus has at most $2n-3$ edges. In general, it is NP-complete to determine the size of the largest plane subdrawing [12]. Based on the data for small n, we conjecture that indeed every simple drawing contains a plane Hamiltonian subgraph on $2n-3$ edges. We verified this conjecture for $n \leq 8$.

▶ **Conjecture 1.2.** *Every simple drawing of K_n with $n \geq 3$ contains a plane Hamiltonian subdrawing on $2n-3$ edges.*

For the class of convex drawings, we succeeded in proving a strengthened version of the conjecture where the plane Hamiltonian subgraph contains a spanning star.

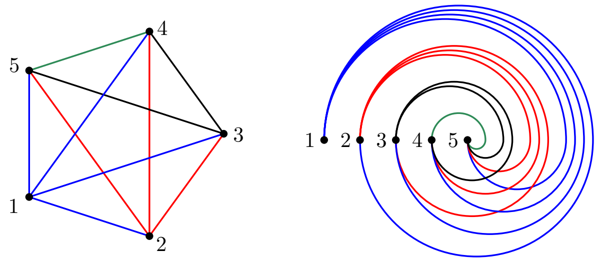


Figure 4 (left) the perfect convex C_5 and (right) the perfect twisted drawing T_5.

▶ **Theorem 1.3.** *Let D be a convex drawing of K_n with $n \geq 3$ and let $v_\star$ be a vertex of D. Then D contains a plane Hamiltonian cycle C which does not cross any edge incident to $v_\star$. This Hamiltonian cycle can be computed in $O(n^2)$ time.*

Proof sketch. We show that convex drawings have a layering structure and reduce the problem of finding a Hamiltonian cycle to finding a Hamiltonian path for each layer independently. To simplify the proof and to reduce the number of cases that have to be considered, we make use of the SAT framework.

For a convex drawing D of the complete graph K_n and a fixed vertex $v_\star$, we give an algorithm that computes a plane Hamiltonian cycle which does not cross edges incident to $v_\star$. We assume that $v_\star = n$ and that the other vertices are labeled from 1 to $n-1$ in cyclic order around $v_\star$. Moreover, by applying suitable stereographic projections, we can assume that $v_\star$ belongs to the outer face. We denote vertex $v_\star = n$ as the *star vertex* and edges incident to $v_\star$ as *star edges*. We call an edge *star-crossing* if it crosses a star edge. For the Hamiltonian cycle we only use edges which are not star-crossing. We particularly focus on the edges $e = \{v, v+1\}$ between consecutive neighbors in the cyclic order around $v_\star$, that is, $1 \leq v < n$ and $v+1$ is considered modulo $n-1$. An edge $e = \{v, v+1\}$ is called *good* if it is not star-crossing. Otherwise, if the edge $b = \{v, v+1\}$ crosses a star edge $\{w, v_\star\}$, then we say that b is a *bad edge* and w is a *witness* for b.

If there is at most one bad edge $\{v, v+1\}$, then the $n-2$ good edges together with the two star edges $\{v, v_\star\}$ and $\{v+1, v_\star\}$ form a Hamiltonian cycle. Hence it remains to deal with the case of two or more bad edges. Firstly, using the properties of a convex drawing we prove that every pair of bad edges appears in a nested structure as illustrated in Figure 5 where for every witness w of a bad edge $\{v, v+1\}$, we have $w < v$. For this we may have to relabel the vertices cyclically and choose a different outer face. Next, the nesting property implies that we can label the bad edges as $b_1, \ldots, b_m$ for some $m \geq 2$, such that if $b_i = \{v_i, v_i + 1\}$, then $1 < v_1 < v_2 < \ldots < v_m = n-2$. Moreover, let w_i^L and w_i^R be the leftmost and the rightmost witnesses of the bad edge b_i, and by $L_i = \{x \in [n-1] : w_{i+1}^R < x < w_i^L\}$ and $R_i = \{x \in [n-1] : v_i + 1 \leq x \leq v_{i+1}\}$ be the left and the right blocks of vertices between two consecutive bad edges b_i and b_{i+1}; see Figure 5.

To construct the desired plane Hamiltonian cycle of non-star-crossing edges, we begin with the edge $\{v_\star, v_1\}$ and then iteratively add a plane path from v_i to v_{i+1} that includes all previously unvisited vertices from L_{i-1}, all the vertices of R_i and some vertices of L_i. When reaching the vertex $v_m + 1 = n-1$, we close the plane Hamiltonian path by adding edge $\{v_m + 1, v_\star\}$. Figure 6 illustrates the simple case, where all L_i's are empty.

In general, however, the L_i's are not empty and the procedure is more involved. The path from v_i to v_{i+1} consists of a subpath from v_i to $v_i + 1$ and a subpath from $v_i + 1$ to v_{i+1}.

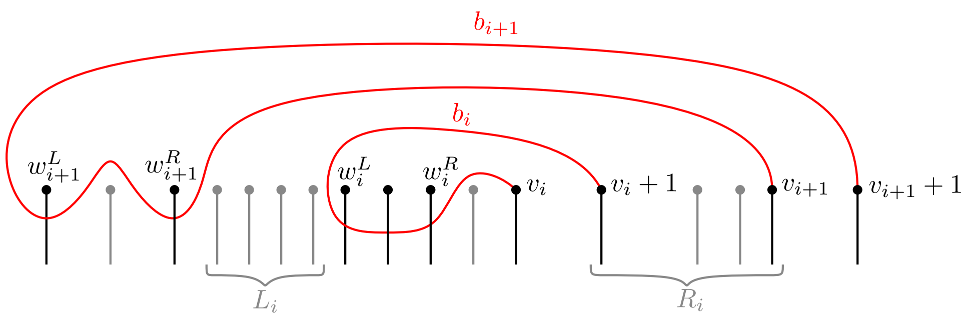


Figure 5 An illustration of nesting of two bad edges, where bad edges are depicted in red.

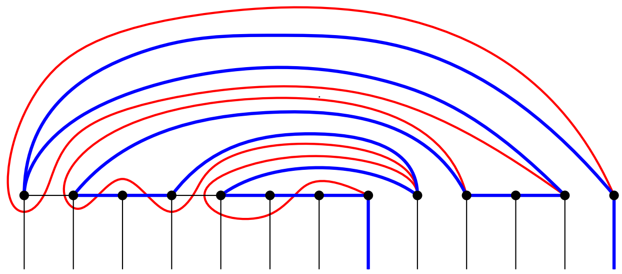


Figure 6 Example of a plane Hamiltonian cycle (blue) in the case where $L_i = \emptyset$ for all i.

For the former path (from v_i to $v_i + 1$), we go from v_i to w_i^R via the previously unvisited vertices from L_{i-1} and then one by one with decreasing labels to w_i^L. As long as there are vertices in L_i from which we can return to $v_i + 1$ via a non-star-crossing edge, we traverse these vertices and then return to $v_i + 1$. The latter path (from $v_i + 1$ to v_{i+1}) traverses all the vertices of R_i and some further vertices of L_i, and lies entirely in the region between the two bad edges b_i and b_{i+1}. ◀

2 Variations

We also studied variations of plane Hamiltonian structures which we briefly mention below.

Extending Hamiltonian cycles

Another way to interpret Theorem 1.3 is the following: given a spanning star in a convex drawing of K_n, we can extend this star to a plane Hamiltonian subdrawing of size $2n-3$. As a variant of this formulation, we tested whether the other direction is true, i.e., whether any given plane Hamiltonian cycle can be extended to a plane subdrawing with $2n-3$ edges. We verified that such an extension is possible for $n \leq 10$.

▶ **Conjecture 2.1.** *Let D be a convex drawing of K_n. Then every plane Hamiltonian cycle can be extended to a plane Hamiltonian subgraph on $2n-3$ edges.*

Hamiltonian cycles avoiding a matching

Instead of a prescribed spanning star, we can also prescribe a matching and ask whether there is a Hamiltonian cycle that together with the matching builds a plane Hamiltonian substructure, i.e., the edges of the matching are not crossed by the Hamiltonian cycle. Hoffman and Tóth [13] investigated the geometric setting. They showed that for every plane perfect matching M in a geometric drawing of K_n there exists a plane Hamiltonian cycle that (possibly contains edges of M but) does not cross any edge from M. We believe that the same holds for convex drawings as well, and have verfied it for $n \leq 11$.

▶ **Conjecture 2.2.** *For every plane (not necessarily perfect) matching M in a convex drawing of K_n there exists a plane Hamiltonian cycle that (possibly contains edges of M but) does not cross any edge from M.*

Hamiltonian paths with a prescribed edge

In general, it is not possible to find a Hamiltonian cycle containing a prescribed edge in simple drawings. Even if we only ask for a Hamiltonian path containing a prescribed edge there is an easy example: consider the edge $\{1,5\}$ in the perfect twisted T_5 depicted in Figure 4. However, the relaxation to Hamiltonian paths seems to be true for convex drawings. We have verified it for $n \leq 11$.

▶ **Conjecture 2.3.** *Let D be a convex drawing of K_n and let e be an edge of D. Then D has a plane Hamiltonian path containing the edge e.*

3 Discussion

We remark that our SAT-based investigation of substructures certainly surpasses the enumerative approach from [2]. While their approach required the enumeration and testing of 7 billion rotation systems on 9 vertices, our framework allows us to make investigations for up to $n = 20$ vertices with reasonably small resources. For example, most computations for $n = 9$ only took about 1 CPU hour and 200MB RAM. Even though exact computation times are not given in [2], the fact that the database for $n \leq 9$ covers about 1TB of disk space shows that our approach works with significantly less resources. Moreover, an enumerative approach for $n = 10$ (e.g. to test the existence of plane Hamiltonian cycles) would not be possible in reasonable time with contemporary computers because there are way too many rotation systems to be enumerated and tested.

Using our framework, we also studied uncrossed edges, crossing families, and empty triangles in simple drawings of the complete graph and believe that SAT-based investigations can be useful to make advancements in the study of simple drawings.

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