



Plane Hamiltonian Cycles in Convex Drawings

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Abstract

A conjecture by Rafla from 1988 asserts that every simple drawing of the complete graph K_n admits a plane Hamiltonian cycle. It turned out that already the existence of much simpler non-crossing substructures in such drawings is hard to prove. Recent progress was made by Aichholzer et al. and by Suk and Zeng who proved the existence of a plane path of length $\Omega(\log n / \log \log n)$ and of a plane matching of size $\Omega(n^{1/2})$ in every simple drawing of K_n .

Instead of studying simpler substructures, we prove Rafla’s conjecture for the subclass of convex drawings, the most general class in the convexity hierarchy introduced by Arroyo et al. Moreover, we show that every convex drawing of K_n contains a plane Hamiltonian path between each pair of vertices (Hamiltonian connectivity) and a plane k -cycle for each $3 \leq k \leq n$ (pancyclicity), and present further results on maximal plane subdrawings.

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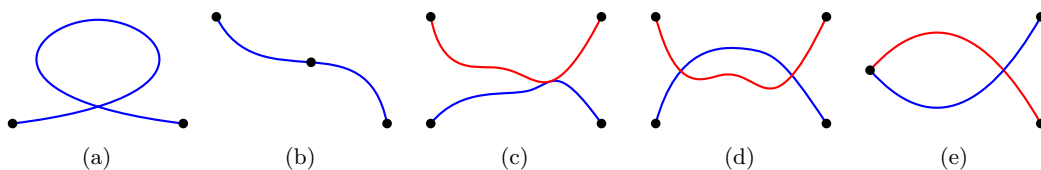
1 Introduction

Starting from Turán’s brick factory problem from the 1940’s, which initiated the study of crossing-minimal drawings, simple drawings gained a lot of attention. In a *simple drawing*¹ of a graph in the plane (resp. on the sphere), the vertices are mapped to distinct points, and edges are drawn as simple curves such that they connect the two corresponding end-vertices but do not contain other vertices. Moreover, two edges share at most one common point,

¹ In the literature, simple drawings are also called *good drawings* or *simple topological graphs*.



which is either a common vertex or a proper crossing. This in particular excludes touchings. Figure 1 shows these forbidden patterns.



■ **Figure 1** In a simple drawing, edges are not allowed to (a) cross themselves or (b) pass through vertices. (c) If two edges meet in their relative interior, they have to cross (no touchings). (d) Each pair of edges crosses at most once and (e) adjacent edges do not cross.

Instead of minimizing the number of crossings, we investigate plane structures. In particular we consider plane subdrawings that appear in every simple drawing of the complete graph K_n . One of the earliest statements in that direction was the following conjecture by Rafla, which is the starting point for this paper.

► **Conjecture 1.1** (Rafla [17]). *Every simple drawing of K_n on $n \geq 3$ vertices contains a plane Hamiltonian cycle.*

For *geometric drawings*, which correspond to point sets in the plane connected via straight-line segments, the existence of a plane Hamiltonian cycle can easily be shown. The study of plane structures gets significantly harder in the more general setting of simple drawings. As one of the first results on guaranteed plane structures in simple drawings of K_n , Pach, Solymosi, and Tóth [15] proved the existence of any fixed plane tree of size $\mathcal{O}((\log n)^{1/6})$. Subsequently, plane matchings and paths were investigated as a relaxation of plane cycles [16, 9, 19, 11, 10, 18]. The best bounds to date are by Aichholzer et al. [2], who showed that every simple drawing of K_n contains a plane matching of size $\Omega(n^{1/2})$ and a plane path of length $\Omega(\log n / \log \log n)$. The latter was also independently proven by Suk and Zeng [20].

Rafla's conjecture itself was verified for all simple drawings of K_n on $n \leq 9$ vertices [1]. In their paper on plane matchings, Aichholzer et al. [2] studied so-called generalized twisted drawings of K_n and proved that they always contain a plane Hamiltonian path and, for odd n , also a plane Hamiltonian cycle.

Aichholzer, Orthaber, and Vogtenhuber [5] recently introduced a variant of Rafla's conjecture, which asserts that simple drawings of K_n are *plane Hamiltonian-connected*, that is, there exists a plane Hamiltonian path between every pair of vertices.

► **Conjecture 1.2** (Aichholzer, Orthaber, Vogtenhuber [5]). *For each pair of vertices s, t in a simple drawing of K_n there exists a plane Hamiltonian path from s to t .*

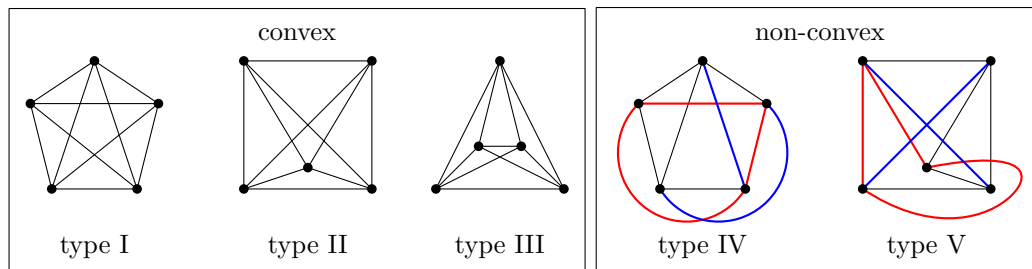
They proved Conjectures 1.1 and 1.2 for the subclasses of strongly c-monotone and cylindrical drawings², and showed that their conjecture is indeed a strengthening: If it was true for all simple drawings of K_n it would imply that also Rafla's conjecture holds in general.

Convex drawings. In this article, we investigate plane Hamiltonian paths and cycles in the subclass of convex drawings. Convex drawings were introduced by Arroyo et al. [6] as the most general class of the convexity hierarchy between geometric and simple drawings

² We are not aware of any relation between these subclasses and convex drawings. A schematic overview of known relations between various classes of simple drawings is given in [5, Figure 7].

of K_n . Arroyo, Richter, and Sunohara [7] already showed Rafla's conjecture for *h-convex* drawings, which are a subclass of convex drawings. The notion of convexity in simple drawings generalizes the classic convexity and is based on *triangles* of a drawing, i.e., the subdrawings induced by three vertices. Since in a simple drawing the edges of a triangle do not cross, a triangle partitions the plane (resp. the sphere) into exactly two connected components. The closures of these components are the two *sides* of the triangle. In particular, the three vertices forming the triangle lie in both sides of the triangle. A side S of a triangle is *convex* if every edge that has its two incident vertices in S is fully contained inside S . A simple drawing \mathcal{D} of K_n is *convex* if every triangle of \mathcal{D} has a convex side.

Arroyo et al. [6] showed that convex drawings of K_n can be characterized in terms of their induced subdrawings on 5 vertices. To define this, we call two simple drawings of the same graph *isomorphic*³ if they have the same pairs of crossing edges up to relabeling of the vertices. Figure 2 shows the five non-isomorphic simple drawings of K_5 , which are denoted as type I to V; cf. [1]. A simple drawing is convex if and only if every induced subdrawing on 5 vertices is isomorphic to a geometric drawing, i.e., of type I, II, or III [6]. The above mentioned *generalized twisted drawings*, for which a plane Hamiltonian path exists [2], are simple drawings where all 5-tuples are of type V [3].



■ **Figure 2** The five non-isomorphic ways to draw the complete graph K_5 . Type IV and type V are non-convex because the red triangles have no convex side, as witnessed by the blue edges.

Maximal plane subdrawings. The study of plane subdrawings raises the question about the maximal number of edges in a plane subdrawing. In this article, we investigate maximal plane subdrawings in convex drawings and show that they are in fact maximum plane. A *maximum* plane subdrawing is a plane subdrawing with the largest number of edges among all plane subdrawings. A plane subdrawing is called *maximal* if adding any further edge would result in a crossing. García, Pilz and Tejel [13] showed that it is NP-complete to determine a *maximum* plane subdrawing in a simple drawing. However, every simple drawing of K_n contains a plane subdrawing with at least $2n - 3$ edges [13, Corollary 3.4], which is best possible as witnessed by the geometric drawing of n points in convex position.

Empty k -cycles. Besides Hamiltonian cycles and maximum/maximal plane subdrawings, we also investigate smaller plane structures. We introduce empty k -cycles as a link between empty triangles ($k = 3$), which are known to exist in every simple drawing of K_n [14, 4], and plane Hamiltonian cycles ($k = n$). Similar as in the case of a triangle, a plane cycle

³ Since there are also other types of isomorphisms in the literature, this isomorphism is usually referred to as *weak isomorphism*.

partitions the plane (resp. sphere) into two connected components, which we call the *sides* of the cycle. A side S is *empty* if there are no vertices contained in the interior of S . An *empty k -cycle* is a plane cycle of length k such that at least one of its two sides is empty.

1.1 Our contribution

In Section 2 we prove Conjectures 1.1 and 1.2 for the class of convex drawings. More specifically, in Theorem 2.5 we show the existence of a plane Hamiltonian cycle and in Theorem 2.4 the existence of a plane Hamiltonian path connecting any pair of vertices.

In Section 3 we show that in a convex drawing of K_n all maximal plane subdrawings have the same number of edges (Theorem 3.1). This allows to find *maximum* plane subdrawings in a greedy fashion and, in particular, to extend every plane Hamiltonian cycle by at least $n - 3$ edges while preserving planarity (Corollary 3.3). Consequently, in every convex drawing of K_n there is a *plane Hamiltonian subdrawing* with $2n - 3$ edges, i.e., a plane subdrawing with $2n - 3$ edges that contains a Hamiltonian cycle.

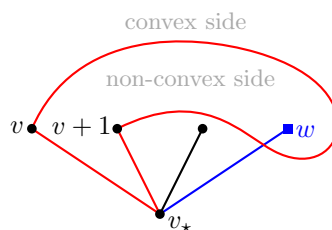
On top of that, we show in Section 4 that for every convex drawing and every fixed vertex v_* there exists a plane Hamiltonian subdrawing with a spanning star centered at v_* (Theorem 4.1). This structural result implies that in every convex drawing of K_n every edge is contained in at least one plane Hamiltonian path (Corollary 4.2). Moreover, we obtain the existence of an empty k -cycle for every integer k with $3 \leq k \leq n$ (Corollary 4.4).

2 Hamiltonian paths and cycles

In this section we prove the two conjectures for all convex drawings: In a first step we show that for each pair of vertices in a convex drawing there is a plane Hamiltonian path connecting them (Conjecture 1.2). From this we then derive the existence of a plane Hamiltonian cycle in every convex drawing (Conjecture 1.1).

For the following proofs we fix one vertex v_* , which we call the *star vertex*. By the properties of a simple drawing the edges incident to v_* , which we refer to as *star edges*, do not cross each other and hence build a plane spanning star. The cyclic order of the star edges in the drawing is the *rotation* of v_* . In this context we often identify the edges with their incident vertices that are different from v_* . Since convexity and the existence of a plane substructure are independent of the choice of the outer face of a drawing, in the following illustrations v_* will be on the outer face. Moreover the statements are independent from the labeling of the vertices. For convenience, we assume that $v_* = n$ and that the other vertices are labeled from 1 to $n - 1$ according to the rotation of v_* . To deal with the cyclic order of the vertices around v_* , we use arithmetics modulo $n - 1$.

For every non-star edge $e = \{u, v\}$, the triangle spanned by the two vertices u and v of e and the star vertex v_* is denoted as T_e . An edge e is *star-crossing* if it crosses at least one of



■ **Figure 3** A bad edge $b = \{v, v + 1\}$ with witness w (blue). The triangle T_b is highlighted in red.

the star edges. Particular focus will be on the edges $e = \{v, v + 1\}$ with $1 \leq v \leq n - 1$. We call such an edge *good* if it is not star-crossing. Otherwise, if edge $b = \{v, v + 1\}$ crosses a star edge $\{w, v_\star\}$, then we say that b is a *bad edge* with *witness* w ; see Figure 3. Note that the side of the triangle T_b containing the witness is not convex. However, by definition every triangle in a convex drawing has at least one convex side. Thus, the triangle T_b has exactly one convex side, which is the side not containing the witness. This shows that all witnesses are in the same side. Furthermore, since v and $v + 1$ are consecutive in the rotation of v_\star , all vertices in the interior of the non-convex side of T_b are witnesses of b .

► **Observation 2.1.** *Let $b = \{v, v + 1\}$ be a bad edge. Then the side of the triangle T_b not containing the witnesses is the unique convex side. Moreover, a vertex is a witness of b if and only if it is in the interior of the non-convex side of T_b .*

By the definition of convexity, an edge between two vertices in a convex side is contained in this convex side. In the following lemma, we show that a similar property holds for the non-convex side of the triangle T_b induced by a bad edge b .

► **Lemma 2.2.** *Let b be a bad edge and let e be a non-star edge connecting two vertices from the non-convex side of T_b . Then e does not cross the triangle T_b and is therefore contained in the non-convex side of T_b .*

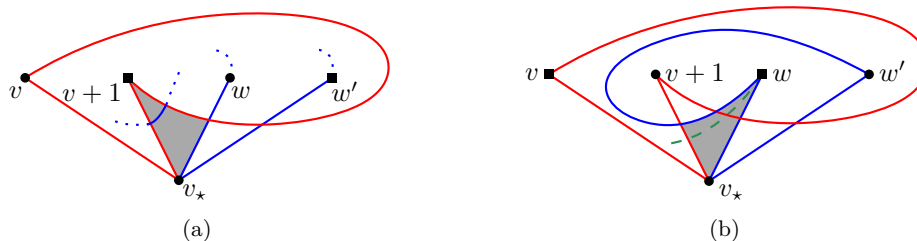
Proof. Clearly the edge $e = b$ is contained in the non-convex side of T_b .

Next, we consider an edge e from a vertex w in the interior of the non-convex side of T_b to a vertex of b . By Observation 2.1, w is a witness. Hence, in the subdrawing induced by the four vertices $v_\star, v, v + 1, w$ the bad edge b and the star edge $\{w, v_\star\}$ cross. Since a simple drawing of K_4 has at most one crossing, e cannot cross the triangle T_b .

Finally, we consider an edge $e = \{w, w'\}$ connecting two interior vertices of the non-convex side of T_b , i.e., e connects two witnesses of b . We assume, without loss of generality, that $v, v + 1, w, w'$ appear in exactly this order in the rotation of v_\star , because v and $v + 1$ are consecutive vertices in that rotation.

Assume towards a contradiction that e crosses T_b . Since both vertices of e lie in the same side of T_b , the edge e crosses T_b an even number of times. This implies that e crosses exactly two of the three edges of T_b and, in particular, at least one of its star edges. By symmetry, we assume that e crosses $\{v + 1, v_\star\}$. Then e passes through the region bounded by $\{v + 1, v_\star\}$, and by the parts of b and $\{w, v_\star\}$ up to their crossing; see Figure 4(a). Consequently e crosses b , which fixes all crossings of e with T_b .

Now consider the triangle T_e . The side of T_e containing $v + 1$ is not convex because $\{v + 1, v_\star\}$ crosses e . Since b crosses all three edges of T_e , v lies in the unique convex side



■ **Figure 4** (a) If an edge $e = \{w, w'\}$ (dotted blue) between two witnesses crosses a star edge of T_b , then it crosses b . (b) Moreover, in this case the edge $\{w, v\}$ (dashed green) crosses its adjacent edge b .

of T_e . Therefore $\{w, v\}$ lies in this convex side. Starting at w , $\{w, v\}$ passes through the region bounded by $\{w, v_\star\}$, and by parts of e and $\{v+1, v_\star\}$, leaving through $\{v+1, v_\star\}$; see Figure 4(b). Also b passes through this region, crossing e and $\{w, v_\star\}$. Thus, $\{w, v\}$ crosses its adjacent edge b ; a contradiction.

This shows that in all cases e cannot cross T_b and hence stays in the non-convex side. ◀

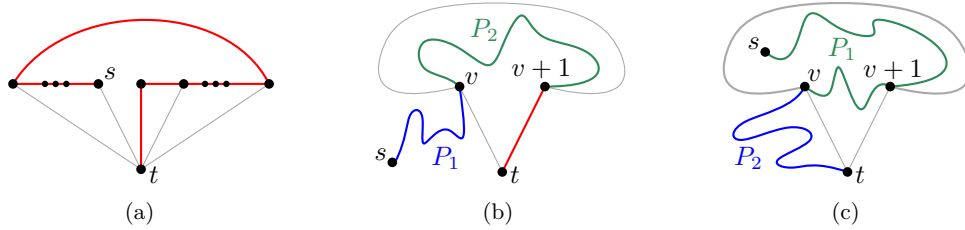
Lemma 2.2 is the key tool to show Conjecture 1.2 for convex drawings. It will also be important that for two non-star edges e and e' that do not cross star edges, the rotation of v_\star determines whether e and e' cross.

► **Observation 2.3.** *Let $e = \{u, v\}$ and $e' = \{u', v'\}$ be two non-star edges that do not cross any star edge. If u, v, u', v' appear in this cyclic order around v_\star , then e and e' do not cross.*

The main idea to show the existence of a Hamiltonian path between each pair of vertices is to divide the drawing into two parts, one part consisting of the vertices in the convex side of a triangle T_b and the other one with the vertices from the non-convex side.

► **Theorem 2.4.** *For each pair of vertices s, t in a convex drawing of K_n there exists a plane Hamiltonian path from s to t .*

Proof. We prove this statement by induction on n . The base cases $n = 2$ and $n = 3$ are trivial. For the inductive step, we pick t as the star vertex v_\star . If there is no bad edge, we start at s , traverse all other vertices in cyclic order around t via good edges, and finally take a star edge to terminate in t ; see Figure 5(a). This Hamiltonian path is plane by Observation 2.3.



■ **Figure 5** (a) illustrates a plane Hamiltonian s - t -path with good edges. (b) and (c) illustrate the two cases to construct a plane Hamiltonian s - t -path if a bad edge exists.

Now assume that there is a bad edge $b = \{v, v+1\}$. The triangle T_b partitions the remaining vertices into two parts: The vertices V_C in the interior of its convex side S_C , which might be an empty set, and the vertices V_N in the interior of its non-convex side S_N , which contains at least one witness of b . Note that V_C and V_N do not contain vertices of the triangle T_b . By convexity, for each pair of vertices from $V_C \cup \{v, v+1, t\}$ the connecting edge is contained in S_C . By Lemma 2.2, for each pair of vertices from $V_N \cup \{v, v+1\}$ the connecting edge is contained in S_N . In particular, those edges do not cross the edges of the triangle T_b . We distinguish the following two cases depending on the position of s .

Case 1: $s \in V_C$ (s is in the interior of the convex side). By induction there exists a plane path P_1 in the subdrawing induced by $V_C \cup \{v\}$ from s to v , which traverses all vertices in $V_C \cup \{v\}$ and is contained in S_C . Similarly, by induction there exists a plane path P_2 in the subdrawing induced by $V_N \cup \{v, v+1\}$ from v to $v+1$, which traverses all vertices in $V_N \cup \{v, v+1\}$ and is contained in S_N . The concatenation of P_1 , P_2 , and the star edge $\{v+1, t\}$ yields the desired plane Hamiltonian path from s to t ; see Figure 5(b).

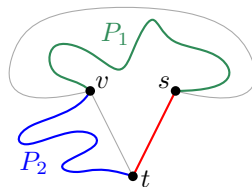
Case 2: $s \notin V_C$ (s is in the non-convex side). By symmetry, we assume without loss of generality that $s \neq v$. By induction there exists a plane path P_1 in the subdrawing induced

by $V_N \cup \{v, v+1\}$ from s to v , which traverses all vertices of $V_N \cup \{v, v+1\}$ and is contained in S_N . Note that this in particular includes the case $s = v+1$. Similarly, by induction there exists a plane path P_2 in the subdrawing induced by $V_C \cup \{t, v\}$ from v to t , which traverses all vertices of $V_C \cup \{t, v\}$ and is contained in S_C . The concatenation of these two paths P_1 and P_2 yields the desired plane Hamiltonian path from s to t ; see Figure 5(c). ◀

► **Theorem 2.5.** *Every convex drawing of K_n with $n \geq 3$ contains a plane Hamiltonian cycle.*

Proof. We choose an arbitrary vertex t as the star vertex. If there is no bad edge, we can easily construct a plane Hamiltonian cycle by traversing all but one good edges according to the rotation of t and close that path with two consecutive star edges.

Otherwise, we pick a bad edge $b = \{v, v+1\}$ and construct a Hamiltonian path from $s = v+1$ to t as in Case 2 of the proof of Theorem 2.4. The path consists of two parts, P_1 and P_2 , which exist by applying Theorem 2.4 to the vertices in the non-convex side and the convex side of T_b , respectively. Path P_1 connects s to v and is contained in the non-convex side S_N , and P_2 connects v to t and is contained in the convex side S_C . Since none of the path edges crosses the star edge $\{t, s\}$, closing the path with the edge $\{t, s\}$ yields the desired plane Hamiltonian cycle; see Figure 6. ◀



■ **Figure 6** Constructing a plane Hamiltonian cycle in a convex drawing.

3 Maximal plane subdrawings

In the previous section we have seen that convex drawings of K_n admit a plane Hamiltonian cycle. Moreover, for each pair of vertices there is a plane Hamiltonian path connecting them. In this section we investigate how many edges we can add to a Hamiltonian cycle in a convex drawing while staying plane. In other words, we ask for maximal plane Hamiltonian substructures. As a first step, we omit the Hamiltonicity and show that all maximal plane subdrawings of a convex drawing have the same number of edges. This is very different from the general case of simple drawings where it is NP-complete to decide whether there exists a plane subdrawing with a given number of edges [13].

► **Theorem 3.1.** *Every maximal plane subdrawing of a convex drawing \mathcal{D} of K_n is maximum plane.*

Proof. Let \mathcal{D}' be an arbitrary maximal plane subdrawing of \mathcal{D} . By [13, Theorem 3.1], \mathcal{D}' is 2-connected and spanning. Clearly, in a 2-connected plane drawing, every face is bounded by a cycle. We show that, for every $r \geq 4$, the r -faces are the same in every maximal plane subdrawing of \mathcal{D} , where a face is an r -face if it is bounded by exactly r edges.

Let F be an r -face in \mathcal{D}' with $r \geq 4$ and denote its bounding r -cycle by C_r . We call an edge of \mathcal{D} that connects two non-consecutive vertices of C_r a *chord* of the cycle. Since \mathcal{D}' is maximal plane, no chord of C_r lies entirely in F . Consequently, as shown by García, Pilz, and Tejel [13, Lemma 3.6], all chords of C_r lie entirely in the complement of F . In particular,

in \mathcal{D} the vertices of C_r induce a drawing of K_r isomorphic to the geometric drawing of r points in convex position. Moreover, since \mathcal{D}' is spanning and F is a face of \mathcal{D}' , no vertex of \mathcal{D} lies inside of F . Since \mathcal{D} is convex we use a result of Arroyo et al. [6, Lemma 3.5] to conclude that all edges of \mathcal{D} are in the complement of F . Thus C_r is completely uncrossed in \mathcal{D} . Hence, C_r and F are part of every maximal plane subdrawing of \mathcal{D} .

This fixes, for every $r \geq 4$ and every maximal plane subdrawing \mathcal{D}' of \mathcal{D} , the number of r -faces in \mathcal{D}' . Hence, Euler's formula implies $f_3 = e + c$, where f_3 is the number of 3-faces, e is the number of edges, and c is a constant depending on the number of vertices and faces of size at least four. On the other hand, double counting the number of edges via incident faces gives $2e = 3f_3 + c'$ for some constant c' depending on the number of faces of size at least four. Since there is a unique solution to these two equations, the number of 3-faces and edges in every maximal plane subdrawing is fixed. Thus, every maximal plane subdrawing of \mathcal{D} is maximum plane. ◀

García, Pilz, and Tejel [13, Corollary 3.4] showed that every simple drawing of K_n has a plane subdrawing with at least $2n - 3$ edges. Together with Theorem 3.1 this shows that all maximal plane subdrawings of a convex drawing of K_n have at least $2n - 3$ edges. In general, this bound is best possible as every triangulation of n points in convex position has exactly $2n - 3$ edges.

► **Corollary 3.2.** *Every maximal plane subdrawing of a convex drawing of K_n has at least $2n - 3$ edges.*

In contrast, for general simple drawings it is known that all maximal plane subdrawings have at least $\frac{3n}{2}$ edges, which is tight as shown by García, Pilz, and Tejel [13, Proposition 3.3]. They give a construction of a (non-convex) simple drawing \mathcal{D} of K_n having a maximal plane subdrawing \mathcal{D}' with $\frac{3n}{2}$ edges. In addition, \mathcal{D}' has a Hamiltonian cycle that crosses all edges of $\mathcal{D} \setminus \mathcal{D}'$. This shows that there are plane Hamiltonian cycles in simple drawings that cannot be extended by more than $\frac{n}{2}$ edges while staying plane. For convex drawings, combining Theorems 2.5 and 3.1, we obtain that every plane Hamiltonian cycle can be extended by $n - 3$ edges in a greedy fashion to obtain a plane Hamiltonian subdrawing with $2n - 3$ edges.

► **Corollary 3.3.** *Every Hamiltonian cycle in a convex drawing of K_n can be extended to a plane Hamiltonian subdrawing on $2n - 3$ edges.*

4 Stars and cycles

As we have seen in the last section, in convex drawings we can extend every plane Hamiltonian cycle to a plane subdrawing with $2n - 3$ edges. In this section we investigate the structure of such plane Hamiltonian subdrawings. In particular, we show that for every vertex v_\star there exists a plane Hamiltonian subdrawing with $2n - 3$ edges that contains all edges incident to v_\star . We present a sketch of the proof in Section 4.1. The full proof including the runtime analysis is deferred to Appendix A.

► **Theorem 4.1.** *For every convex drawing \mathcal{D} of K_n with $n \geq 3$ and every vertex v_\star in \mathcal{D} , there exists a plane Hamiltonian cycle that does not cross any edge incident to v_\star . Such a Hamiltonian cycle can be computed in $\mathcal{O}(n^2)$ time.*

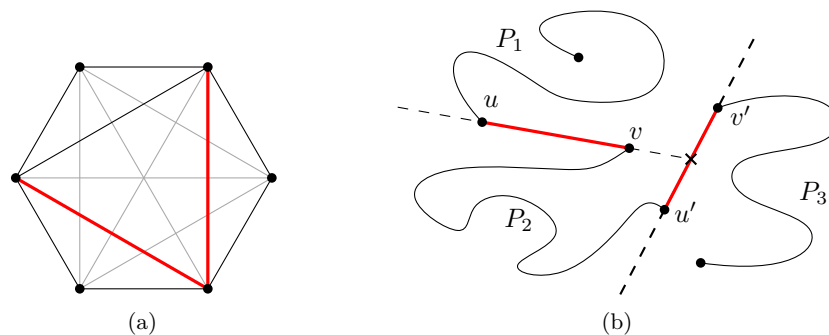
This theorem provides a lot of structure for plane Hamiltonian subdrawings, which makes it possible to investigate further properties. For example, already in geometric drawings there are edges that are not contained in any plane Hamiltonian cycle (see one of the red

edges in Figure 7(a)). However, replacing this property with Hamiltonian paths, we show that the plane Hamiltonian paths of a given convex drawing cover all edges.

► **Corollary 4.2.** *For every convex drawing \mathcal{D} of K_n and every edge e in \mathcal{D} , there exists a plane Hamiltonian path containing e .*

Proof. Let $e = \{u, v\}$ be an arbitrary edge of \mathcal{D} . We consider u as the star vertex. By Theorem 4.1 there exists a plane Hamiltonian subdrawing \mathcal{D}' of \mathcal{D} that contains all edges incident to u . Let the plane Hamiltonian cycle C in \mathcal{D}' traverse u, x_1, \dots, x_{n-1} in this order with $v = x_i$ for some index i . If $i = 1$ or $i = n - 1$, then C contains the edge $\{u, v\}$ and therefore yields a Hamiltonian path that fulfills the desired property. Otherwise $x_{i-1}, x_{i-2}, \dots, x_1, u, v, x_{i+1}, \dots, x_{n-1}$ is a Hamiltonian path containing the edge e . It is plane because all its edges belong to the plane drawing \mathcal{D}' . ◀

Given this property of prescribing an edge for a Hamiltonian path, a natural generalization is to prescribe more than one edge. To obtain a plane substructure, the prescribed edges cannot induce a crossing in the drawing. Moreover, as illustrated in Figure 7(a), prescribing two adjacent edges or longer subpaths is not possible in general. It is however possible to prescribe two non-crossing independent edges for a plane Hamiltonian path in a geometric drawing of K_n .



■ **Figure 7** (a) Two adjacent edges in a geometric drawing of K_6 that cannot be extended to a plane Hamiltonian path. (b) An illustration of Proposition 4.3: extending two independent edges in a geometric drawing of K_n to a plane Hamiltonian path.

► **Proposition 4.3.** *For every geometric drawing \mathcal{D} of K_n and every pair of non-crossing independent edges e and e' in \mathcal{D} , there exists a plane Hamiltonian path containing e and e' .*

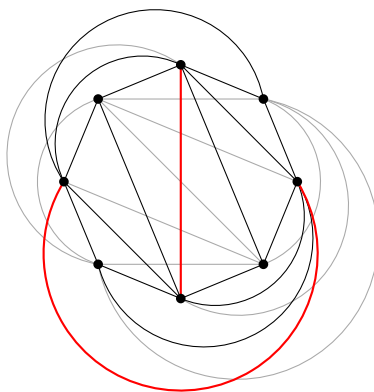
Proof. Let $e = \{u, v\}$ and $e' = \{u', v'\}$ be the two independent edges that do not cross in the geometric drawing \mathcal{D} . Since we are in the geometric setting, we can extend the two straight-line segments e and e' to lines ℓ and ℓ' , respectively. And because the segments e and e' do not cross, we may assume that e has no crossing with ℓ' as otherwise we exchange the roles of e and e' . So ℓ' partitions the plane into two regions with e being contained in one of them. We further partition the region containing e using the line ℓ and obtain three regions: R_1 and R_2 contain e on its boundary, and R_3 has e' on its boundary; see Figure 7(b) for an illustration. Moreover, we may assume u' lies on the boundary of R_2 as otherwise we exchange the roles of R_1 and R_2 . By Theorem 2.4⁴ there is a plane path P_1 in R_1 that

⁴ We remark that for geometric drawings of K_n , an s - t -path can be constructed directly from the rotation around s .

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traverses all interior vertices from R_1 and ends in vertex u . Similarly, there is a plane path P_2 in R_2 starting at vertex v , traversing all interior vertices from R_2 , and ending in vertex u' . In the same manner, there is a plane path P_3 in R_3 starting at vertex v' and traversing all interior vertices from R_3 . By concatenating P_1 , e , P_2 , e' , and P_3 we obtain the desired Hamiltonian path, which is plane because the three regions are linearly separated. ◀

Even though we used the geometric notion of convexity in the argument above, the result does not generalize to convex drawings because we cannot extend edges in that setting. And indeed prescribing two edges is in general not possible in convex drawings: Figure 8 shows a drawing of K_8 with two prescribed edges e and e' (highlighted red) which are not contained in a plane Hamiltonian path. Let S be the set of four vertices which are not incident to any red edge. Each vertex in S is incident to only three edges that are not crossed by a red edge and these three edges are all adjacent to a red edge. Hence, in a plane Hamiltonian path containing the red edges, each vertex of S is an end-vertex of the path or it is the unique vertex of S between the two red edges. This is a contradiction because $|S| = 4$.



■ **Figure 8** A convex drawing of K_8 . The two prescribed edges, highlighted red, are not contained in any common plane Hamiltonian path.

Besides prescribing edges, Theorem 4.1 implies that for every $3 \leq k \leq n$ every convex drawing admits a plane cycle of length k , as we argue below. In abstract graphs the existence of cycles of arbitrary length is often referred to as *pancyclicity*. Recall that a plane cycle has two sides. In addition to cycles of arbitrary length, we find cycles such that one side does not contain vertices in its interior, i.e., *empty k -cycles*. For $k = 3$ those are known as *empty triangles*, a concept which is considered often in the study of simple drawings. For simple drawings of K_n with $n \geq 3$, Harborth [14] proved that there are at least two empty triangles and conjectured that the minimum among all simple drawings of K_n is $2n - 4$. Today it is known that there are at least n empty triangles [4], and that all generalized twisted drawings have exactly $2n - 4$ empty triangles [12].

Note that every plane Hamiltonian cycle is an empty n -cycle as both of its sides are empty. Moreover, from the properties of the plane Hamiltonian cycle constructed in Theorem 4.1, we derive the existence of empty k -cycles for all k .

► **Corollary 4.4.** *For every convex drawing \mathcal{D} of K_n and every integer $3 \leq k \leq n$ there exists an empty k -cycle in \mathcal{D} .*

Proof. For a fixed star vertex v_* , the Hamiltonian cycle constructed in Theorem 4.1 consists of two star edges incident to v_* and a plane Hamiltonian path P on the remaining vertices

$1, \dots, n-1$. The crucial property is that P does not cross any star edge. The plane path P starts at a specific vertex $v \in \{1, \dots, n-1\}$ which is determined by the choice of the star vertex, and for every $1 \leq k \leq n-1$, the indices of the first k vertices of P determine an integer interval $\{i, i+1, \dots, j-1, j\}$. Hence the next vertex v' of P is either $i-1$ or $j+1$. Inductively it follows that closing the path after visiting k vertices with the two star edges $\{v, v_\star\}$ and $\{v', v_\star\}$ gives the desired empty k -cycle. \blacktriangleleft

4.1 Proof of Theorem 4.1 (Sketch)

In this section we give a sketch of the algorithm to compute, for a given convex drawing of K_n and a vertex v_\star , a plane Hamiltonian cycle that does not cross edges incident to v_\star . The full proof and runtime analysis is deferred to Appendix A.

As in the proof of Rafla's conjecture for convex drawings (Theorem 2.5) we consider v_\star as the star vertex, assume $v_\star = n$ and label the remaining vertices with $1, \dots, n-1$ corresponding to the rotation of v_\star . Similar to Section 2, bad edges will play an important role in this proof. For drawings with at most one bad edge, we can concatenate $n-2$ good edges around v_\star to obtain a plane path through all non-star vertices, which does not cross any star edges. To obtain the desired plane Hamiltonian cycle, we add the two star edges connecting the two end-vertices of the path to v_\star . Hence in the following we consider the case that there are at least two bad edges. Recall that for a bad edge b , by Observation 2.1, the unique convex side of T_b is the side not containing the witnesses. To deal with multiple bad edges, we investigate their structure in the following lemma, which extends Lemma 2.2.

► **Lemma 4.5.** *Let $b = \{v, v+1\}$ and $b' = \{v', v'+1\}$ be two distinct bad edges with witnesses w and w' , respectively. Then the following four statements hold:*

- (i) *The edge b does not cross any star edge of $T_{b'}$ or b' does not cross any star edge of T_b .*
- (ii) *The triangle T_b is contained in the convex side of $T_{b'}$ and vice versa.*
- (iii) *$w \neq w'$.*
- (iv) *w', w, v, v' appear in this or the reversed cyclic order in the rotation of v_\star .*

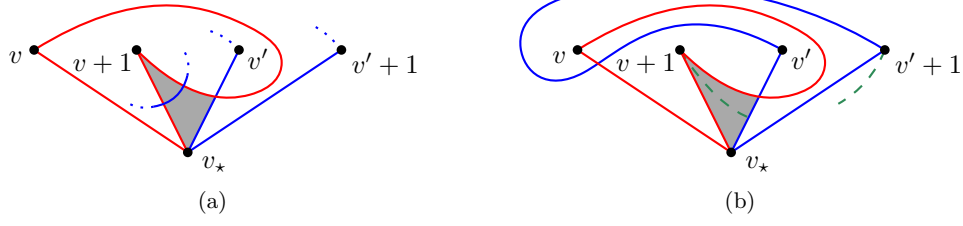
Proof. To prove property (i), we first consider the case that b and b' share a vertex. We assume without loss of generality that $v+1 = v'$. The subdrawing induced by the four vertices $v, v', v'+1, v_\star$ is a simple drawing of K_4 and has therefore at most one crossing. If both b and b' cross a star edge of the other triangle, there would be two crossings on that K_4 . Hence the property holds in this case.

For the remaining case in which b and b' are independent, consider the subdrawing induced by the five distinct vertices $v, v+1, v', v'+1, v_\star$. If b crosses both star edges of $T_{b'}$, then v' and $v'+1$ are both in the non-convex side of T_b . Thus, by Lemma 2.2, b' does not cross any star edge of T_b and the property follows. An equivalent argument holds if b' crosses both star edges of T_b .

Hence, from now on we assume that b crosses exactly one star edge of $T_{b'}$ and b' crosses exactly one star edge of T_b . Without loss of generality assume that b crosses $\{v', v_\star\}$, the other case is symmetric. Since b' crosses exactly one star edge of T_b and the two vertices v' and $v'+1$ are on different sides of T_b , the edge b' crosses T_b exactly once. Similarly to the arguments in the proof of Lemma 2.2, a crossing of b' and $\{v+1, v_\star\}$ forces an additional crossing with b . For an illustration see Figure 9(a). Hence, b' crosses the star edge $\{v, v_\star\}$.

We now consider the edge $\{v+1, v'+1\}$. Its vertices $v+1$ and $v'+1$ are both in the convex side of both triangles T_b and $T_{b'}$. To stay in the convex side of T_b , starting at $v+1$, the edge $\{v+1, v'+1\}$ is forced to cross $\{v', v_\star\}$, see Figure 9(b). This contradicts $\{v+1, v'+1\}$ staying in the convex side of $T_{b'}$, which completes the proof of property (i).

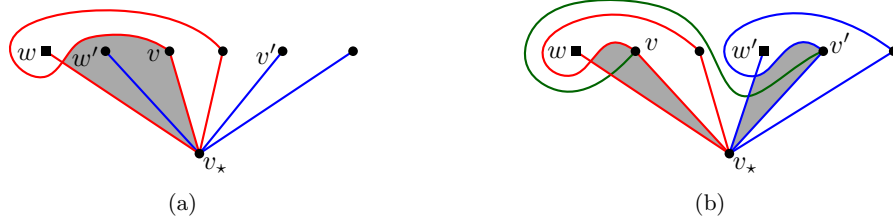
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■ **Figure 9** Illustrations of the case where b and b' cross exactly one star edge of the other triangle.

Next we derive property (ii) from property (i). Assume without loss of generality that b does not cross any star edge of $T_{b'}$. Then, by definition, v' and $v' + 1$ are not witnesses for b and hence by Observation 2.1 lie in the convex side of T_b . Since v_* lies in the convex side of T_b , by convexity, the triangle $T_{b'}$ is contained in the convex side of T_b . In particular, b' does not cross any star edge of T_b . With the same arguments as for b , T_b is contained in the convex side of $T_{b'}$, implying property (ii).

To show property (iii), recall that, by Observation 2.1, the witnesses w and w' lie in the interior of the non-convex side of T_b and $T_{b'}$, respectively. Now property (ii) implies that these non-convex sides are interiorly disjoint, and therefore b and b' do not have any common witness.



■ **Figure 10** Illustration that the two shown cyclic orders cannot appear in a convex drawing.

It remains to show property (iv), which concerns the cyclic order of vertices around v_* . Up to symmetries there are three possibilities, namely w, w', v, v' or w, v, w', v' or the claimed case w', w, v, v' . In the first case, it is not possible that b' crosses $\{w', v_*\}$ without crossing T_b or $\{w, v_*\}$ because $\{w', v_*\}$ is contained in a region bounded by $\{v, v_*\}$ and parts of $\{w, v_*\}$ and b . For an illustration see Figure 10(a).

In the second case, we consider the edge $\{v, v'\}$; see Figure 10(b). Since v is in the convex side of $T_{b'}$ and v' is in the convex side of T_b , $\{v, v'\}$ is in the intersection of both convex sides. In particular, it crosses $\{w, v_*\}$ and $\{w', v_*\}$. Moreover, the triangle $T_{\{v, v'\}}$ (spanned by the edge $\{v, v'\}$ and v_*) has w on one side and w' on the other. Hence $T_{\{v, v'\}}$ has no convex side; a contradiction. This completes the proof of property (iv). ◀

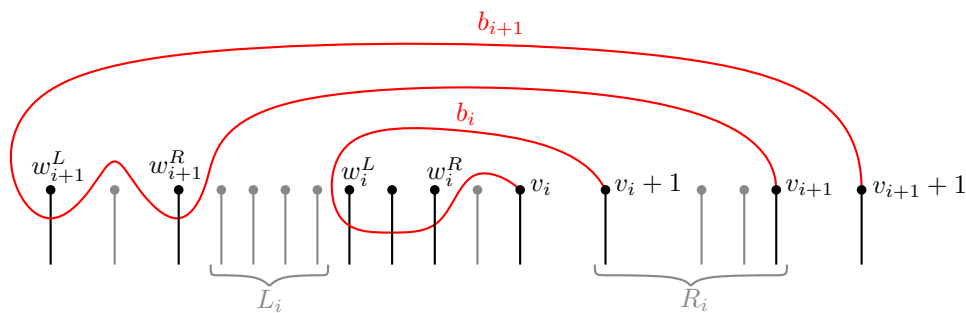
Property (iv) implies that the vertices $1, \dots, n - 1$ in the rotation of v_* can be partitioned into two blocks of consecutive vertices such that one block contains the vertices of all bad edges and the other block contains all the witnesses. In particular, if b is the bad edge whose vertices are last in the clockwise order of its block, we can cyclically relabel the vertices such that b becomes $\{n - 2, n - 1\}$. This makes the labels of all witnesses smaller than the labels of vertices of bad edges. We then have the following two properties:

(sidedness) If $\{v, v + 1\}$ is a bad edge with witness w , then $w < v$.

(*nestedness*) If $b = \{v, v + 1\}$ and $b' = \{v', v' + 1\}$ are bad edges with respective witnesses w and w' and if $v < v'$, then $w > w'$.

In addition we can choose the outer face such that the vertex v_* and the initial parts of the edges $\{v_*, 1\}$ and $\{v_*, n - 1\}$ are incident to it.

The nesting property implies that we can label the bad edges as b_1, \dots, b_m for some $m \geq 2$, such that if $b_i = \{v_i, v_i + 1\}$, then $1 < v_1 < v_2 < \dots < v_m = n - 2$. Moreover, let w_i^L and w_i^R denote the leftmost (smallest index) and the rightmost (largest index) witness of the bad edge b_i , respectively. Then $1 \leq w_m^L \leq w_m^R < w_{m-1}^L \leq w_{m-1}^R < \dots < w_1^L \leq w_1^R$. Sidedness additionally implies $w_i^R < v_i$ for all $i = 1, \dots, m$. Figure 11 shows the situation for two bad edges b_i and b_{i+1} . Note that $v_i + 1 = v_{i+1}$ is possible. Moreover, for $i = 1, \dots, m - 1$ let $L_i = \{x \mid w_{i+1}^R < x < w_i^L\}$ and $R_i = \{x \mid v_i + 1 \leq x \leq v_{i+1}\}$ denote the left and the right blocks of vertices between two consecutive bad edges b_i and b_{i+1} ; see Figure 11.

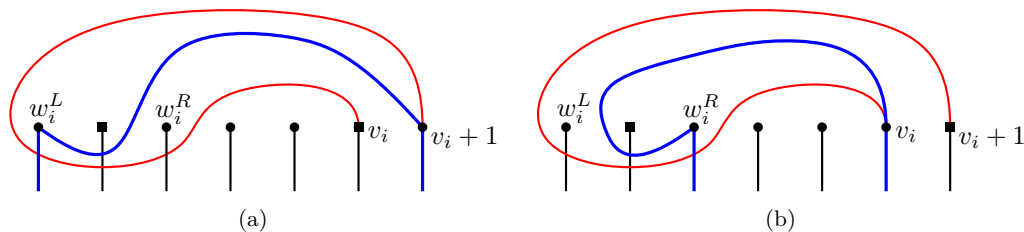


■ **Figure 11** Illustration of sidedness and nestedness for two bad edges. All vertical edges are incident to the star vertex v_* .

In a first step, we consider edges between one of the two vertices of a bad edge b_i and a witness of b_i . In particular, we show that $\{w_i^L, v_i + 1\}, \{w_i^R, v_i\}$ are not star-crossing.

► **Lemma 4.6.** For all $i = 1, \dots, m$, neither $\{w_i^L, v_i + 1\}$ nor $\{w_i^R, v_i\}$ is star-crossing.

Proof. For every fixed i , by Lemma 2.2, both edges are contained in the non-convex side of the triangle T_{b_i} . Assume $\{w_i^L, v_i + 1\}$ crosses a star edge $\{x, v_*\}$. Then x is a witness of b_i with $w_i^L < x$. However, the side of the triangle $\{w_i^L, v_i + 1, v_*\}$ that contains x is not convex due to the edge $\{x, v_*\}$. Additionally, the other side is not convex due to the edge $\{v_i, v_i + 1\}$; a contradiction. A similar argument holds for the edge $\{w_i^R, v_i\}$. The two situations are depicted in Figures 12(a) and 12(b). ◀



■ **Figure 12** The edges $\{w_i^L, v_i + 1\}$ and $\{w_i^R, v_i\}$ cannot be star-crossing because otherwise the blue triangle has no convex side, as witnessed by edges incident to the vertices marked with squares.

The general idea to construct a plane Hamiltonian cycle that is not star-crossing is as following: Starting at v_1 (the leftmost vertex incident to a bad edge), we add the next

unvisited vertex to the left (smaller index) or to the right (larger index) using non-star-crossing edges. This yields a path through all vertices except v_* , which is plane by Observation 2.3. By adding the two star edges incident to the end-vertices of this path, we obtain the desired Hamiltonian cycle.

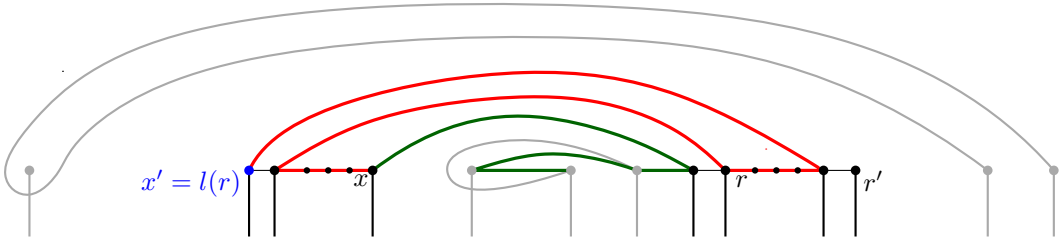
To determine which non-star-crossing edges are suitable for our path, we relate vertices from the right block $R = \{v_1 + 1, \dots, n - 2\}$ with vertices from the left block $L = \{1, \dots, v_1\}$. For each $i = 1, \dots, m - 1$ and each vertex $r \in R_i$, we determine a corresponding vertex $l(r) \in L_i \cup \{w_{i+1}^R\}$. With a few exceptions we get two edges incident to $l(r)$ which are not star-crossing.

► **Lemma 4.7.** *For all $r \in R_i$, if $l(r) > w_{i+1}^R$ the edge $\{l(r), r\}$ is not star-crossing. If $l(r) + 1 < w_i^L$ the edge $\{l(r) + 1, r\}$ is not star-crossing.*

Moreover, for increasing index of $r \in \bigcup R_i = R$ the values $l(r) \in L$ are decreasing. Both proofs are deferred to Appendix A.

► **Lemma 4.8.** *For $r, r' \in R_i$ with $r < r'$, we have $l(r) \geq l(r')$.*

Using this notion, we can now explicitly describe the Hamiltonian cycle: We initialize $x := v_1$ and $r := v_1 + 1$. By repeating the following procedure, we iteratively extend the plane path visiting all vertices $\{x, \dots, r - 1\}$ to a plane path visiting all vertices of $\{x', \dots, r' - 1\}$ with $x' < x$ and $r' > r$. An example is shown in Figure 13.



■ **Figure 13** Illustration of an extension step of the plane path construction. The previous path visiting $\{x, \dots, r - 1\}$ is depicted green. The extension visiting $\{x', \dots, r' - 1\}$ (highlighted in red) consists of good edges and non-star-crossing edges from Lemma 4.7.

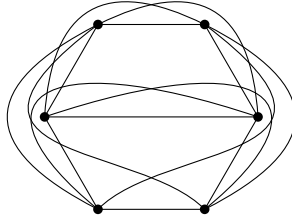
We set $x' := l(r)$ and $r' := \min(\{\tilde{r} \in R : r < \tilde{r}, l(r) \neq l(\tilde{r})\} \cup \{n - 1\})$, that is, r' is the first vertex to the right of r where $l(r') \neq l(r)$. We traverse $x, x - 1, \dots, x' + 1$ via good edges, use the non-star-crossing edge $\{x' + 1, r\}$ to reach r , traverse $r, r + 1, \dots, r' - 1$ via good edges, and use the non-star-crossing edge $\{r' - 1, x'\}$ to reach x' . We repeat this step with x' and r' in the roles of x and r , respectively, until we have included vertex $n - 2$ to the path. Then we traverse all remaining non-star vertices $x', x' - 1, \dots, 1, n - 1$ in this order via good edges, and close the Hamiltonian cycle via the two star edges $\{n - 1, v_*\}$ and $\{v_*, v_1\}$.

5 Conclusion

In this article, we investigated Rafla's conjecture on plane Hamiltonian cycles and variations of it in the restricted setting of convex drawings. Since our proofs use specific properties, which do not necessarily apply in the more general setting of simple drawings, not all shown statements are true for simple drawings.

A non-convex simple drawing for which some structural results do not hold is the type V, which is also known as the *twisted drawing* of K_5 and depicted in Figure 2. In general, the

twisted drawing of K_n contains exactly one edge e that crosses all its independent edges. Hence e cannot be contained in any plane Hamiltonian path (cf. Corollary 4.2). Starting with a star at one end-vertex of e , there is no possibility to obtain a Hamiltonian path on the remaining vertices not crossing the star edges. Moreover, the drawing of K_6 depicted in Figure 14 has not a single plane subdrawing consisting of a spanning star and a plane Hamiltonian cycle (cf. Theorem 4.1).



■ **Figure 14** A non-convex drawing of K_6 in which no vertex admits a plane Hamiltonian cycle not crossing the star edges.

It remains open whether Rafla's conjecture (Conjecture 1.1) and Conjecture 1.2 hold in simple drawings. Moreover, our computational data suggests that Corollary 4.4 and a weakening of Corollary 3.3 might apply to the general case of simple drawings; see [8] for more details. Hence, we conclude this article with the following two strengthenings of Rafla's conjecture.

- **Conjecture 5.1.** *For every simple drawing \mathcal{D} of K_n and every integer $3 \leq k \leq n$ there exists an empty k -cycle in \mathcal{D} .*
- **Conjecture 5.2.** *Every simple drawing of K_n with $n \geq 3$ contains a plane Hamiltonian subdrawing on $2n - 3$ edges.*

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A Plane Hamiltonian subdrawings in convex drawings

In this section we give the full proof of Theorem 4.1 even though parts of it have been shown in Section 4.1. We prove the existence of the Hamiltonian cycle in a constructive way and present an $\mathcal{O}(n^2)$ -time algorithm that, for a given convex drawing \mathcal{D} of K_n and a fixed vertex v_\star in \mathcal{D} , computes a plane Hamiltonian cycle that does not cross edges incident to v_\star .

As in the proof of Rafla's conjecture for convex drawings (Theorem 2.5) we consider v_\star as the star vertex, assume $v_\star = n$ and label the remaining vertices with $1, \dots, n-1$ corresponding to the rotation of v_\star . Similar to Section 2, bad edges will play an important role in this proof. For drawings with at most one bad edge, we can concatenate $n-2$ good edges around v_\star to obtain a plane path through all non-star vertices, which does not cross any star edges. To obtain the desired plane Hamiltonian cycle, we add the two star edges connecting the two end-vertices of the path to v_\star .

► **Observation A.1.** *If there is at most one bad edge, then \mathcal{D} contains a plane Hamiltonian cycle that does not cross any star edges and visits the non-star vertices in the cyclic order around the star vertex v_\star .*

Hence in the following we consider the case that there are at least two bad edges. Recall that, for a bad edge b , by Observation 2.1 the unique convex side of T_b is the side not containing the witnesses. To deal with multiple bad edges, we investigate their structure in the following lemma, which extends Lemma 2.2.

► **Lemma 4.5.** *Let $b = \{v, v+1\}$ and $b' = \{v', v'+1\}$ be two distinct bad edges with witnesses w and w' , respectively. Then the following four statements hold:*

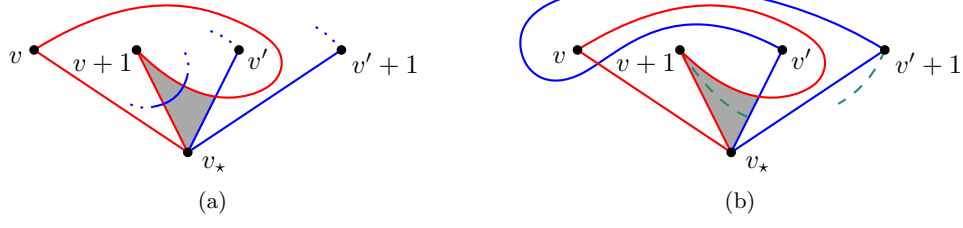
- (i) *The edge b does not cross any star edge of $T_{b'}$ or b' does not cross any star edge of T_b .*
- (ii) *The triangle T_b is contained in the convex side of $T_{b'}$ and vice versa.*
- (iii) *$w \neq w'$.*
- (iv) *w', w, v, v' appear in this or the reversed cyclic order in the rotation of v_\star .*

Proof. To prove property (i), we first consider the case that b and b' share a vertex. We assume without loss of generality that $v+1 = v'$. The subdrawing induced by the four vertices $v, v', v'+1, v_\star$ is a simple drawing of K_4 and has therefore at most one crossing. If both b and b' cross a star edge of the other triangle, there would be two crossings on that K_4 . Hence the property holds in this case.

For the remaining case in which b and b' are independent, consider the subdrawing induced by the five distinct vertices $v, v+1, v', v'+1, v_\star$. If b crosses both star edges of $T_{b'}$, then v' and $v'+1$ are both in the non-convex side of T_b . Thus, by Lemma 2.2, b' does not cross any star edge of T_b and the property follows. An equivalent argument holds if b' crosses both star edges of T_b .

Hence, from now on we assume that b crosses exactly one star edge of $T_{b'}$ and b' crosses exactly one star edge of T_b . Without loss of generality assume that b crosses $\{v', v_\star\}$, the other case is symmetric. Since b' crosses exactly one star edge of T_b and the two vertices v' and $v'+1$ are on different sides of T_b , the edge b' crosses T_b exactly once. Similarly to the arguments in the proof of Lemma 2.2, a crossing of b' and $\{v+1, v_\star\}$ forces an additional crossing with b . For an illustration see Figure 15(a). Hence, b' crosses the star edge $\{v, v_\star\}$.

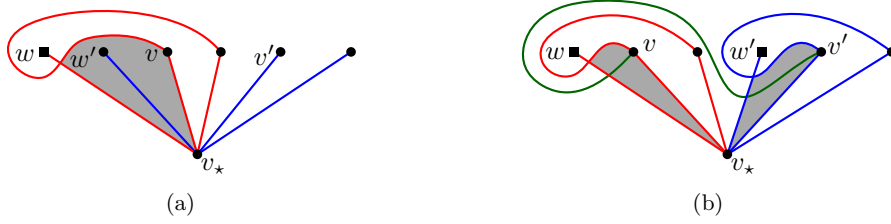
We now consider the edge $\{v+1, v'+1\}$. Its vertices $v+1$ and $v'+1$ are both in the convex side of both triangles T_b and $T_{b'}$. To stay in the convex side of T_b , starting at $v+1$, the edge $\{v+1, v'+1\}$ is forced to cross $\{v', v_\star\}$, see Figure 15(b). This contradicts $\{v+1, v'+1\}$ staying in the convex side of $T_{b'}$, which completes the proof of property (i).



■ **Figure 15** Illustrations of the case where b and b' cross exactly one star edge of the other triangle.

Next we derive property (ii) from property (i). Assume without loss of generality that b does not cross any star edge of $T_{b'}$. Then, by definition, v' and $v' + 1$ are not witnesses for b and hence by Observation 2.1 lie in the convex side of T_b . Since v_* lies in the convex side of T_b , by convexity, the triangle $T_{b'}$ is contained in the convex side of T_b . In particular, b' does not cross any star edge of T_b . With the same arguments as for b , T_b is contained in the convex side of $T_{b'}$, implying property (ii).

To show property (iii), recall that, by Observation 2.1, the witnesses w and w' lie in the interior of the non-convex side of T_b and $T_{b'}$, respectively. Now property (ii) implies that these non-convex sides are interiorly disjoint, and therefore b and b' do not have any common witness.



■ **Figure 16** Illustration that the two shown cyclic orders cannot appear in a convex drawing.

It remains to show property (iv), which concerns the cyclic order of vertices around v_* . Up to symmetries there are three possibilities, namely w, w', v, v' or w, v, w', v' or the claimed case w', w, v, v' . In the first case, it is not possible that b' crosses $\{w', v_*\}$ without crossing T_b or $\{w, v_*\}$ because $\{w', v_*\}$ is contained in a region bounded by $\{v, v_*\}$ and parts of $\{w, v_*\}$ and b . For an illustration see Figure 16(a).

In the second case, we consider the edge $\{v, v'\}$; see Figure 16(b). Since v is in the convex side of $T_{b'}$ and v' is in the convex side of T_b , $\{v, v'\}$ is in the intersection of both convex sides. In particular, it crosses $\{w, v_*\}$ and $\{w', v_*\}$. Moreover, the triangle $T_{\{v, v'\}}$ (spanned by the edge $\{v, v'\}$ and v_*) has w on one side and w' on the other. Hence $T_{\{v, v'\}}$ has no convex side; a contradiction. This completes the proof of property (iv). ◀

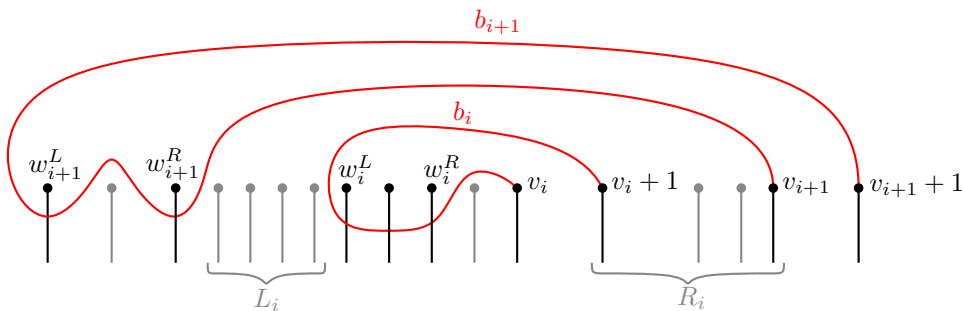
Property (iv) implies that the vertices $1, \dots, n - 1$ in the rotation of v_* can be partitioned into two blocks of consecutive vertices such that one block contains all vertices of all bad edges and the other block contains all the witnesses. In particular, if b is the bad edge whose vertices are last in the clockwise order of its block, we can cyclically relabel the vertices such that b becomes $\{n - 2, n - 1\}$. This makes the labels of all witnesses smaller than the labels of vertices of bad edges. We have the following two properties:

(sidedness) If $\{v, v + 1\}$ is a bad edge with witness w , then $w < v$.

(*nestedness*) If $b = \{v, v + 1\}$ and $b' = \{v', v' + 1\}$ are bad edges with respective witnesses w and w' and if $v < v'$, then $w > w'$.

In addition we can choose the outer face such that the vertex v_* and the initial parts of the edges $\{v_*, 1\}$ and $\{v_*, n - 1\}$ are incident to it.

The nesting property implies that we can label the bad edges as b_1, \dots, b_m for some $m \geq 2$, such that if $b_i = \{v_i, v_i + 1\}$, then $1 < v_1 < v_2 < \dots < v_m = n - 2$. Moreover, let w_i^L and w_i^R denote the leftmost (smallest index) and the rightmost (largest index) witness of the bad edge b_i , respectively. Then $1 \leq w_m^L \leq w_m^R < w_{m-1}^L \leq w_{m-1}^R < \dots < w_1^L \leq w_1^R$. Sidedness additionally implies $w_i^R < v_i$ for all $i = 1, \dots, m$. Figure 17 shows the situation for two bad edges b_i and b_{i+1} . Note that $v_i + 1 = v_{i+1}$ is possible.

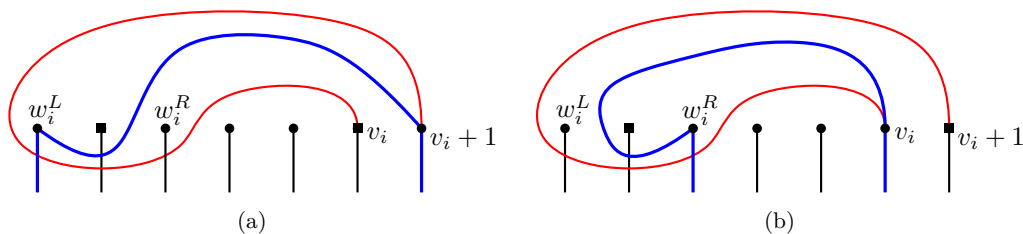


■ **Figure 17** Illustration of sidedness and nestedness for two bad edges. All vertical edges are incident to the star vertex v_* .

In a first step, we consider edges between one of the two vertices of a bad edge b_i and a witness of b_i . In particular, we show that $\{w_i^L, v_i + 1\}, \{w_i^R, v_i\}$ are not star-crossing.

► **Lemma 4.6.** For all $i = 1, \dots, m$, neither $\{w_i^L, v_i + 1\}$ nor $\{w_i^R, v_i\}$ is star-crossing.

Proof. For every fixed i , Lemma 2.2 shows that both edges are contained in the non-convex side of the triangle T_{b_i} . Assume $\{w_i^L, v_i + 1\}$ crosses a star edge $\{x, v_*\}$. Then x is a witness of b_i with $w_i^L < x$. However, the side of the triangle $\{w_i^L, v_i + 1, v_*\}$ that contains x is not convex due to the edge $\{x, v_*\}$. Additionally, the other side is not convex due to the edge $\{v_i, v_i + 1\}$; a contradiction. A similar argument holds for the edge $\{w_i^R, v_i\}$. The two situations are depicted in Figures 18(a) and 18(b). ◀



■ **Figure 18** The edges $\{w_i^L, v_i + 1\}$ and $\{w_i^R, v_i\}$ cannot be star-crossing because otherwise the blue triangle has no convex side, as witnessed by edges incident to the vertices marked with squares.

For $i = 1, \dots, m - 1$ let $L_i = \{x \mid w_{i+1}^R < x < w_i^L\}$ and $R_i = \{x \mid v_i + 1 \leq x \leq v_{i+1}\}$ denote the left and the right blocks of vertices between two consecutive bad edges b_i and b_{i+1} ;

see Figure 17. Note that R_i is non-empty since it always contains $v_i + 1$ and v_{i+1} but L_i might be empty.

In the following we seek edges from L_i to R_i that are not star-crossing. First, we show that the properties of a convex drawing imply that the edges between vertices of L_i and R_i stay in the region between the two bad edges b_i and b_{i+1} . This implies that the only star edges which can be crossed by those edges are $\{x, v_\star\}$ with $x \in L_i \cup R_i$.

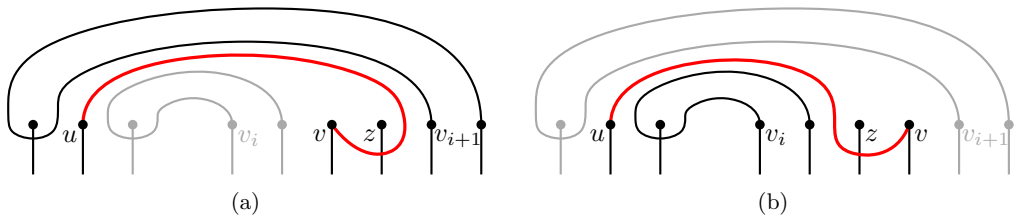
► **Lemma A.2.** *No edge $\{u, v\}$ with $u, v \in L_i \cup R_i$ crosses star edges $\{z, v_\star\}$ with $z \in V \setminus (L_i \cup R_i)$.*

Proof. The convex sides S_i and S_{i+1} of the triangles T_{b_i} and $T_{b_{i+1}}$, respectively, have a common intersection which is partitioned into three regions by the edges $\{w_{i+1}^R, v_\star\}$ and $\{w_i^L, v_\star\}$. Both vertices u, v are contained in the region that is bounded by both edges $\{w_{i+1}^R, v_\star\}$ and $\{w_i^L, v_\star\}$. Since the edge $\{u, v\}$ lies in S_i and S_{i+1} and crosses $\{w_{i+1}^R, v_\star\}$ and $\{w_i^L, v_\star\}$ at most once, it is contained in the same region. This shows that we cannot cross star edges $\{z, v_\star\}$ where z is outside of this region, i.e., $z \in V \setminus (L_i \cup R_i)$. ◀

Moreover, we show that an edge from a vertex in L_i to a vertex in R_i cannot cross star edges with vertices in R_i .

► **Lemma A.3.** *No edge $\{u, v\}$ with $u \in L_i$ and $v \in R_i$ crosses star edges $\{z, v_\star\}$ with $z \in R_i$.*

Proof. Assume towards a contradiction that the edge $\{u, v\}$ crosses a star edge $\{z, v_\star\}$ with $z \in R_i$. Let w_i and w_{i+1} be witnesses of the bad edges $b_i = \{v_i, v_i + 1\}$ and $b_{i+1} = \{v_{i+1}, v_{i+1} + 1\}$, respectively. We distinguish the two cases of $z > v$ and $z < v$, which are depicted in Figures 19(a) and 19(b). For $z > v$, we consider the subdrawing \mathcal{D}_1 induced by the 7 distinct vertices $w_{i+1}, u, v, z, v_{i+1}, v_{i+1} + 1, v_\star$, where $v_i + 1 = v$ is possible. In \mathcal{D}_1 the vertices u and v are consecutive in the rotation of v_\star , thus, $\{u, v\}$ and b_{i+1} are both bad edges. However, the respective cyclic order of the vertices around v_\star violates property (iv) of Lemma 4.5; a contradiction. For $z < v$, we consider the subdrawing \mathcal{D}_2 induced by $u, w_i, v_i, v_i + 1, z, v, v_\star$, where $v = v_{i+1}$ is possible. Again, since u and v are consecutive in the cyclic order around v_\star , the edge $\{u, v\}$ is a bad edge with witness z in \mathcal{D}_2 and the order of vertices corresponding to the two bad edges violates property (iv) of Lemma 4.5. ◀

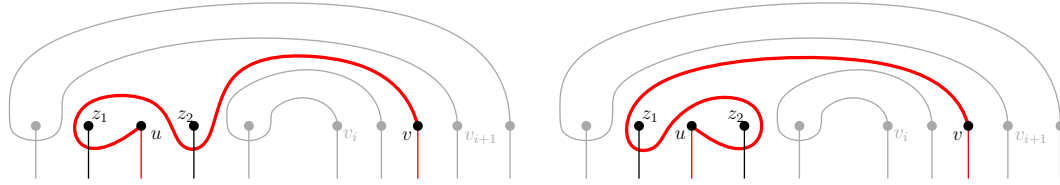


■ **Figure 19** Illustration of the two forbidden configurations to prove Lemma A.3. The red edges cannot cross the star edge $\{z, v_\star\}$ as depicted. The vertices and incident edges that are drawn gray are deleted to achieve a subdrawing contradicting the convexity.

The previous lemmas show that if $\{u, v\}$ with $u \in L_i$ and $v \in R_i$ crosses a star edge $\{z, v_\star\}$, then $z \in L_i$. We now analyze crossings in L_i and show that the edge $\{u, v\}$ cannot cross two star edges $\{z_1, v_\star\}$ and $\{z_2, v_\star\}$ with $z_1 < u < z_2$ and $z_1, z_2 \in L_i$; see Figure 20.

► **Lemma A.4.** *No edge $\{u, v\}$ with $u \in L_i$ and $v \in R_i$ crosses two star edges $\{z_1, v_\star\}$ and $\{z_2, v_\star\}$ with $z_1 < u < z_2$ and $z_1, z_2 \in L_i$.*

Proof. Assume towards a contradiction that $\{u, v\}$ crosses both edges $\{z_1, v_\star\}$ and $\{z_2, v_\star\}$. Then z_1 and z_2 are on different sides of the triangle spanned by $\{u, v, v_\star\}$. Since both star edges $\{z_1, v_\star\}$ and $\{z_2, v_\star\}$ cross the triangle, there is no convex side; a contradiction. \blacktriangleleft



■ **Figure 20** Illustration for the proof of Lemma A.4. The red triangle $\{u, v, v_\star\}$ has no convex side. Witnesses for the non-convexity are the edges $\{z_1, v_\star\}$ and $\{z_2, v_\star\}$.

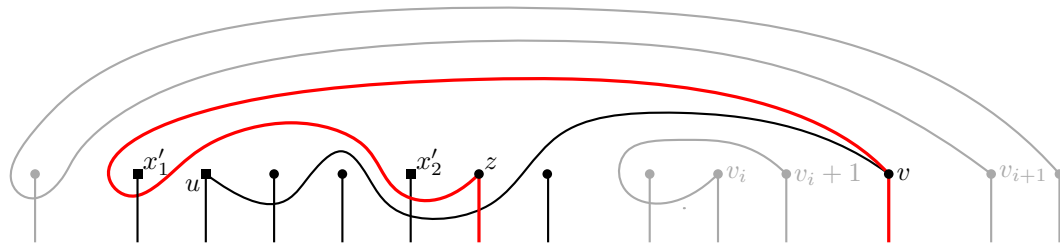
Even though edges from L_i to R_i cannot cross star edges incident to vertices in L_i with smaller and larger indices at the same time, crossings on one side cannot be avoided. We focus on edges which only cross star edges with larger indices than the end-vertex in L_i . As the following lemma shows, those edges help us to find edges from L_i to R_i that are not star-crossing.

► **Lemma A.5.** *Let $u \in L_i$ and $v \in R_i$ and let z be the largest index in L_i with $z > u$ such that the edge $\{u, v\}$ crosses the star edge $\{z, v_\star\}$. Then the following two statements hold:*

- (a) *The edge $\{z, v\}$ is not star-crossing.*
- (b) *The edge $\{z + 1, v\}$ does not cross any star edge $\{x', v_\star\}$ with $x' \in L_i$ and $x' \leq z$.*

Proof. To show (a), assume towards a contradiction that the edge $\{z, v\}$ crosses a star edge $\{x', v_\star\}$. From Lemmas A.2 and A.3 we know that $x' \in L_i$. Moreover, because a simple drawing of K_4 has at most one crossing and the edges $\{u, v\}$ and $\{z, v_\star\}$ already cross, the edge $\{z, v\}$ has no crossing with the triangle $T_{\{u, v\}}$. In particular, $\{z, v\}$ cannot cross any star edge in the convex side of $T_{\{u, v\}}$, i.e., the side not containing z . The choice of z implies that all edges $\{x, v_\star\}$ with $x \in L_i$ and $x > z$ do not cross $\{u, v\}$. Hence $\{v, z\}$ can only cross star edges $\{x', v_\star\}$ with $x' \in L_i$ and $x' < z$.

However, the triangle $T_{\{z, v\}}$ has no convex side. The side containing the vertex x' is not convex since the edge $\{x', v_\star\}$ crosses $\{z, v\}$. The other side contains the vertex u and is not convex since the edge $\{u, v\}$ crosses $\{z, v_\star\}$; a contradiction. Figure 21 shows two examples with $x' = x'_1 < u$ and $x' = x'_2 > u$.



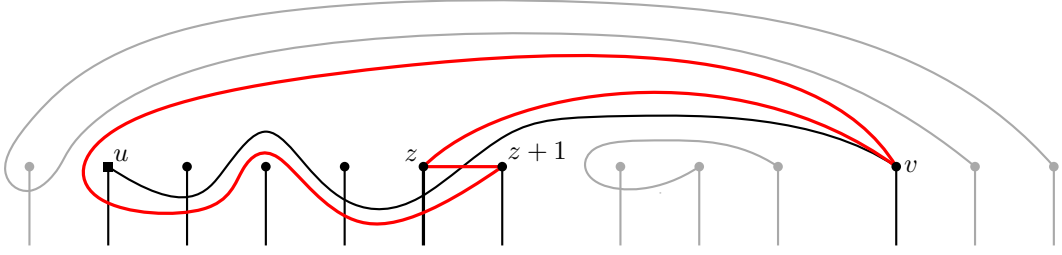
■ **Figure 21** Illustration for the proof of Lemma A.5(a). The red triangle $T_{\{z, v\}}$ has no convex side. Witnesses for the non-convexity are the edges $\{u, v\}$ and $\{x'_i, v_\star\}$.

To show (b), assume towards a contradiction that $\{z + 1, v\}$ crosses a star edge $\{x', v_\star\}$ with $x' \in L_i$ and $x' \leq z$. Since $\{z + 1, v\}$ cannot cross its adjacent edges $\{u, v\}$ and $\{v, v_\star\}$, the edge $\{z + 1, v\}$ crosses first $\{z, v_\star\}$ and consequently $\{u, v_\star\}$ in order to cross a star edge

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$\{x', v_\star\}$ with $x' \in L_i$. Note that this also holds if $z + 1 = w_i^L$. Now consider the triangle $T = \{z, z + 1, v\}$. The star edge $\{u, v_\star\}$ crosses exactly one of the edges of T , which is $\{z + 1, v\}$. The good edge $\{z, z + 1\}$ and $\{z, v\}$ are not star crossing. The latter follows by part (a). Thus u and v_\star lie in different sides of T .

However none of the two sides is convex. The edge $\{u, v\}$ crosses $\{z, v_\star\}$ but not $\{z + 1, v_\star\}$. Hence it crosses the good edge $\{z, z + 1\}$ which shows that the side of T containing u is not convex. The side of T containing v_\star is not convex because $\{z, v_\star\}$ crosses $\{z + 1, v\}$. This is a contradiction to the convexity. An illustration is given in Figure 22. ◀



■ **Figure 22** Illustration for the proof of Lemma A.5(b). The red triangle has no convex side. Witnesses for the non-convexity are the edges $\{z, v_\star\}$ and $\{u, v\}$.

We need one more definition before we can construct the plane Hamiltonian cycle. Consider a fixed vertex $r \in R_i$ and all edges $\{u, r\}$ with $u \in L_i$ that cross a star edge $\{z, v_\star\}$ with $z > u$. We are interested in the rightmost such vertex z , if it exists. More formally, we define

$$l(r) = \max(\{z \in L_i \mid \exists u \in L_i, u < z : \{u, r\} \text{ crosses } \{z, v_\star\}\} \cup \{w_{i+1}^R\}).$$

Note that in the case where $l(r)$ is set to w_{i+1}^R , there is no edge $\{u, r\}$ that crosses a star edge to the right of u . This especially implies that the edge $\{w_{i+1}^R + 1, r\}$ is not star-crossing. For most cases, we get two non-star-crossing edges for each vertex $r \in R_i$.

► **Lemma 4.7.** *For all $r \in R_i$, if $l(r) > w_{i+1}^R$ the edge $\{l(r), r\}$ is not star-crossing. If $l(r) + 1 < w_i^L$ the edge $\{l(r) + 1, r\}$ is not star-crossing.*

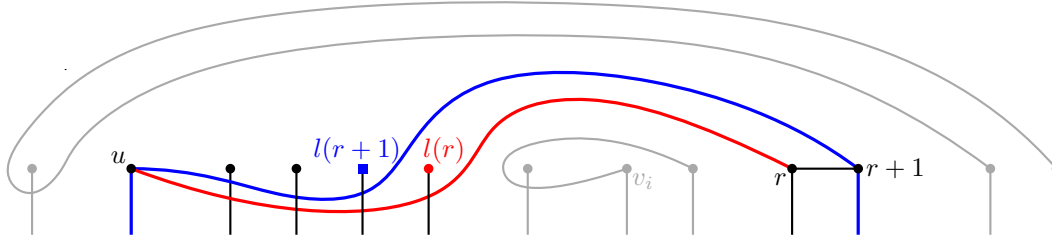
Proof. The edge $\{l(r), r\}$ is not star-crossing by Lemma A.5(a) if $l(r) \in L_i$, i.e., $l(r) > w_{i+1}^R$. Moreover, if $l(r) > w_{i+1}^R$ Lemma A.5(b) implies that $\{l(r) + 1, r\}$ does not cross any star edge $\{x, v_\star\}$ with $x \leq l(r)$. By the maximality of $l(r)$, the edge $\{l(r) + 1, r\}$ does not cross any star edge $\{x, v_\star\}$ with $x > l(r) + 1$ if $l(r) + 1 \in L_i$. The condition $l(r) + 1 \in L_i$ is fulfilled exactly if $l(r) + 1 < w_i^L$. In the case in which $l(r) = w_{i+1}^R$ only appears if there is no edge $\{u, r\}$ crossing a star edge right of u . Hence the edge $\{l(r) + 1, r\}$ is not star-crossing due to the definition of $l(r)$. ◀

Together with Lemma 4.6 this gives us our desired non-star-crossing edges from L_i to R_i . To make sure that these edges do not cross each other, we show that the $l(r)$ have decreasing indices with increasing r .

► **Lemma 4.8.** *For $r, r' \in R_i$ with $r < r'$, we have $l(r) \geq l(r')$.*

Proof. We consider the case $r' = r + 1$. For $r' > r + 1$ the claim then follows by transitivity. If $l(r + 1) = w_{i+1}^R$, the claim clearly holds. In the following, let $l(r + 1) > w_{i+1}^R$ and $u < l(r + 1)$ such that $\{u, r + 1\}$ crosses $\{l(r + 1), v_\star\}$. Consider the triangle $T_{\{u, r+1\}}$, which is drawn blue

in Figure 23. The side of $T_{\{u, r+1\}}$ containing $l(r+1)$ is not convex, which is witnessed by $\{l(r+1), v_\star\}$. By the convexity of the drawing, the side not containing $l(r+1)$ is convex. By Lemma A.3, the edge $\{u, r+1\}$ does not cross $\{r, v_\star\}$. Hence the vertex r and consequently the edge $\{u, r\}$ are contained in the unique convex side of $T_{\{u, r+1\}}$. This implies that $\{u, r\}$ crosses the star edge $\{l(r+1), v_\star\}$, which shows $l(r) \geq l(r+1)$. \blacktriangleleft



■ **Figure 23** Illustration of the proof of Lemma 4.8.

The monotonicity of the vertices $l(r)$ implies the following claim, which we need for the construction of the Hamiltonian cycle.

► **Observation A.6.** *Let $r \in R_i$.*

- *If $l(r) = w_{i+1}^R$ for $r < v_{i+1}$, then for all $r' \in R_i$ with $r' > r$ it is $l(r') = l(r)$.*
- *If $l(r) + 1 = w_i^L$ for $r > v_i + 1$, then for all $r' \in R_i$ with $r' < r$ it is $l(r') = l(r)$.*

The general idea to construct a plane Hamiltonian cycle that is not star-crossing is now as follows: Starting at v_1 (the leftmost vertex incident to a bad edge), in each step we add the next unvisited vertex to the left (smaller index) or to the right (larger index) using non-star-crossing edges. This yields a path through all vertices except v_\star , which is plane by Observation 2.3. By adding the two star edges incident to the end-vertices of this path, we then obtain the desired Hamiltonian cycle.

To determine which non-star-crossing edges are suitable for our path, we relate vertices from the right block $R = \bigcup R_i = \{v_1 + 1, \dots, n - 2\}$ with vertices from the left block $L = \{1, \dots, v_1\}$. As we have shown, for each $i = 1, \dots, m - 1$ and each vertex $r \in R_i$ there is a corresponding vertex $l(r) \in L_i \cup \{w_{i+1}^R\}$. If $w_{i+1}^R < l(r) < w_i^L - 1$ the two edges $\{l(r), r\}$ and $\{l(r) + 1, r\}$ are not star-crossing (cf. Lemma 4.7). In the two remaining cases, i.e., if $l(r) = w_{i+1}^R$ or $l(r) + 1 = w_i^L$, at least one of the two edges is not star-crossing with the additional property that the next, respectively previous, vertex r' in R_i has the same value $l(r') = l(r)$ (cf. Observation A.6). Moreover, for increasing index of $r \in R$ the values $l(r) \in L$ are decreasing (cf. Lemma 4.8).

Using this notion, we explicitly describe the Hamiltonian cycle: We initialize $x := v_1$ and $r := v_1 + 1$. By repeating the following procedure, we iteratively extend the plane path visiting all vertices $\{x, \dots, r - 1\}$ to a plane path visiting all vertices of $\{x', \dots, r' - 1\}$ with $x' < x$ and $r' > r$.

We set $x' := l(r)$ and $r' := \min(\{\tilde{r} \in R \mid r < \tilde{r}, l(r) \neq l(\tilde{r})\} \cup \{n - 1\})$, i.e., r' is the first vertex to the right of r where $l(r') \neq l(r)$. Note that for $r \in R_i$ with $r < v_{i+1}$, $r' \in R_i$ and hence the edges connecting two consecutive vertices in the rotation of v_\star between r and r' are good edges. We traverse $x, x - 1, \dots, x' + 1$ via good edges, use the non-star-crossing edge $\{x' + 1, r\} = \{l(r) + 1, r\}$ to reach r , traverse $r, r + 1, \dots, r' - 1$ via good edges, and use the non-star-crossing edge $\{r' - 1, x'\} = \{r' - 1, l(r' - 1)\}$ to reach x' . Note the edge $\{x' + 1, r\} = \{l(r) + 1, r\}$ is not star-crossing for $x' + 1 = l(r) + 1 < w_i^L$ by Lemma 4.7. If

$x' + 1 = l(r) + 1 = w_i^L$, then by definition of r and Observation A.6 it is $r = v_i + 1$. Hence the edge $\{x' + 1, r\} = \{l(r) + 1, r\}$ is not star-crossing by Lemma 4.6. Similarly, the edge $\{r' - 1, x'\} = \{r' - 1, l(r' - 1)\}$ is not star-crossing. This follows for $l(r' - 1) > w_{i+1}^R$ from Lemma 4.7. In the remaining case, if $x' = l(r' - 1) = w_{i+1}^R$, then by definition of r' and Observation A.6 it is $r' - 1 = v_{i+1}$. Hence the edge $\{r' - 1, x'\} = \{r' - 1, l(r' - 1)\}$ is not star-crossing by Lemma 4.6.

We repeat this step with x' and r' in the roles of x and r , respectively, until we have included vertex $n - 2$ to the path. Then we traverse all remaining non-star vertices $x', x' - 1, \dots, 1, n - 1$ in this order via good edges, and close the Hamiltonian cycle via the two star edges $\{n - 1, v_\star\}$ and $\{v_\star, v_1\}$.

Running time: In a first preprocessing step, we compute all bad edges. This is possible in $\mathcal{O}(n^2)$ time: for each of the $n - 1$ edges $\{i, i + 1\}$ there are at most $n - 3$ potential witnesses to test. This also determines the number m of bad edges and all values v_i, w_i^L, w_i^R .

In a second preprocessing step, we compute the value $l(r)$ for every r . We claim that this can be done in $\mathcal{O}(n^2)$ time. To determine $l(r)$ for every $r \in R_i$, recall that $l(r + 1) \leq l(r)$ due to Lemma 4.8. Thus, we start with the smallest index $r \in R_i$ and with the largest index $l \in L_i$. We then iteratively check if $l = l(r)$ by testing whether $\{l, v_\star\}$ crosses some edge $\{l', r\}$ with $l' < l$. If the answer is yes, we have found $l = l(r)$ and increase r by one, otherwise $l \neq l(r)$ and we decrease l by one. In total, we consider a linear number of candidate pairs (l, r) in this process to determine all values $l(r)$. And for each such pair (l, r) , we check a linear number of potential crossings $\{l, v_\star\}$ with $\{l', r\}$. Altogether, this computes all values $l(r)$ in $\mathcal{O}(n^2)$ time.

Observe that after this preprocessing we can decide in constant time, which edge to add next to the path in each step of the algorithm. And since the final cycle has exactly n edges, building it takes $\mathcal{O}(n)$ time. Hence, the total running time is $\mathcal{O}(n^2)$, which completes the proof of Theorem 4.1.