Order Dimension, Grids, and Products

Stefan Felsner[1], Torsten Mütze[2], Maximilian Wittmann[3]

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Interested in the dimension of the face lattices of d-dimensional grids, we are led to the study of fences and their products. This yields a characterization of generalized fences: F is a generalized fence if and only if $\dim(P \times F) \leqslant \dim(P) + 1$ for every poset P.

Whether $\dim(P \times Q) \geqslant \dim(P) + \dim(Q) - 2$ for all posets P and Q is a long-standing question in dimension theory. Having understood products where one factor is a fence we further investigate the dimension of products where the factors are crowns and other 3-dimensional posets. We reconsider the *covering property* defined by Reuter for Ferrers relations and show that in many cases the gap between $\dim(P \times Q)$ and $\dim(P) + \dim(Q)$ can be explained with the covering properties of one of the factors.

1 Introduction

Below in Subsection 1.1 we recall basic notions and concepts of posets and their dimension. Readers with little background on the topic may want to go there first. A thorough introduction to dimension of posets can be found in Trotter's monograph [Tro92].

Our primary interest was to study the dimension of the face poset of a d-dimensional grid. Faces of the d-grid are the elements of a d-dimensional product of paths. Hence, the inclusion poset of faces of the d-grid is a product of fences, the incidence (inclusion) orders of paths. This leads us to the study of products of fences and other posets. As results we have a dimension-theoretic characterization of generalized fences, i.e., of posets whose cover graph is a forest of paths. We also contribute some insights regarding one of the big open questions in order theory.

Question 1. *Do posets P and Q exist such that* $\dim(P \times Q) < \dim(P) + \dim(Q) - 2$ *?*

It is unclear when the dimension of products was first studied. It arises quite naturally from the definition given by Dushnik and Miller [DM41]. The dimension of products is a topic of a survey of Kelly and Trotter [KT82] but was already studied in unpublished work of Baker [Bak61]. The following upper and lower bounds on the dimension of a product belong to the folklore of dimension theory.

[1]Technische Universität Berlin, Germany. E-mail: `felsner@math.tu-berlin.de`
[2]Universität Kassel, Germany. E-mail: `tmuetze@mathematik.uni-kassel.de`
[3]Technische Universität Berlin, Germany. E-mail: `math@mwittmann.eu`

Fact 2. For arbitrary posets P and Q we have

$$\max(\dim(P), \dim(Q)) \leqslant \dim(P \times Q) \leqslant \dim(P) + \dim(Q).$$

Baker has shown that if P and Q are posets with **0** and **1**, i.e., with a least and a greatest element, then $\dim(P \times Q) = \dim(P) + \dim(Q)$ (a proof can be found in [Tro92, page 40]). Hence the upper bound can be tight. On the other hand, if A_n is an antichain, then $\dim(A_n) = 2$ and $A_n \times P$ is just the n-fold disjoint union of P. Hence, the dimension of $A_n \times P$ equals $\max\bigl(\dim(P), 2\bigr)$, and if $\dim(P) \geqslant 2$, then $\dim(A_n \times P) = \dim(P) = \dim(P) + \dim(A_n) - 2$.

So far nobody found an example with $\dim(P \times Q) < \dim(P) + \dim(Q) - 2$, however, the lower bound of Fact 2 is still the best we know. Several competing conjectures concerning the gap have been posed:

Conjecture 3 (Trotter [Tro85])**.** *For every m, n with $1 \leqslant m \leqslant n$, there exist posets P and Q with $\dim(P) = m$, $\dim(Q) = n$ and $\dim(P \times Q) = n$.*

Conjecture 4 (Kelly and Trotter [KT82])**.** $\dim(P \times Q) \geqslant \dim(P) + \dim(Q) - 2$ *for all posets P and Q.*

Reuter identified a situation where the lower bound of Fact 2 is not tight:

Theorem 5 (Reuter [Reu89])**.** $\dim(P \times P) \geqslant 4$ *for all P with* $\dim(P) = 3$.

A proof of the theorem is given in Subsection 2.4. We also provide some results which suggest that $\dim(P \times Q) \geqslant 4 = \dim(P) + \dim(Q) - 2$ for all P and Q with $\dim(P) = \dim(Q) = 3$. If true this would disprove Conjecture 3 and provide additional evidence for Conjecture 4.

Our interest in the dimension of the face poset of a d-dimensional grid was motivated by possible applications in discrete geometry. We found the question interesting also because it closely relates with the order dimension of polytopes, a research topic which has resulted in a collection of deep and interesting results.

- Schnyder [Sch89] proved that the face lattice of a 3-dimensional simplicial (triangulated) polytope has order dimension 4. Moreover, the dimension drops to 3 upon removal of a single face.
- Reuter [Reu90] proved that the order dimension of the face lattice of a d-polytope is at least $d + 1$ and observed that in fixed dimension $d \geqslant 4$ there are polytopes with arbitrarily large order dimension. This is due to the fact that in these dimensions there are neighbourly polytopes, i.e., polytopes whose faces of dimension 0 and 1 induce a complete graph.
- Brightwell and Trotter [BT93] generalized Schnyder's result, they prove that the face lattice of arbitrary 3-dimensional polytopes has order dimension 4, while the dimension drops to 3 upon removal of a single face. Simplified proofs of this result can be found in [Fel01, Fel03, FZ08].

The study of the dimension of the face poset of d-dimensional cubic grids in Section 2 leads us to studying the dimension of products.

1.1 Elements of Poset Dimension

We mostly consider posets $P = (X, \leqslant_P)$ with a reflexive order relation and corresponding irreflexive order relation $<_P$. When $x, y \in X$ and $x \nleqslant_P y$ and $y \nleqslant_P x$, then x and y are incomparable and we write $x \parallel y$ in P.

A poset P is a *linear order* if any two of its elements are comparable. If P and Q are partial orders on the same ground set, we say Q is an *extension* of P if $x \leqslant_P y$ implies $x \leqslant_Q y$. We call Q a *linear extension* of P if Q is a linear order which is an extension of P.

A family R of linear extensions of P is a *realizer* of P if for all $x, y \in X$, $x \leqslant_P y$ if and only if $x \leqslant_L y$ for every $L \in R$.

The definition of a realizer can be converted to the following:

Observation 6. A set R of linear extensions of P is a realizer if and only if for all $x, y \in P$ with $x \nleqslant_P y$ there is some extension $L \in R$ such that $y <_L x$.

The *dimension* of the poset P, denoted $\dim(P)$ is the least integer t so that P has a realizer $R = \{L_1, L_2, \ldots, L_t\}$ of cardinality t. A poset is *t-irreducible* if it has dimension t but the removal of any element leaves a subposet of dimension less than t, in this case the dimension of the subposet is $t - 1$.

The *product* of two posets $P = (X, \leqslant_P)$ and $Q = (Y, \leqslant_Q)$ is the poset $P \times Q = (X \times Y, \leqslant_{P\times Q})$ where $(x, y) \leqslant_{P\times Q} (x', y')$ if and only if $x \leqslant_P x'$ and $y \leqslant_Q y'$. The *dual* of a poset $P = (X, \leqslant_P)$ is the poset $P^{\mathsf{d}} = (X, \leqslant_{P^{\mathsf{d}}})$ where $x \leqslant_{P^{\mathsf{d}}} y$ if and only if $y \leqslant_P x$.

Given a poset $P = (X, \leqslant_P)$, let $\mathrm{inc}(P) = \{(x, y) \in X \times X : x \parallel y \text{ in } P\}$. A linear extension L of P with $y <_L x$ *reverses* the incomparable pair (x, y). A set R of linear extensions of P is a realizer if and only if for every $(x, y) \in \mathrm{inc}(P)$, some $L \in R$ reverses the pair (x, y). For this reason, it is convenient to have a criterion to determine whether there is a linear extension reversing a given subset $S \subset \mathrm{inc}(P)$. For an integer $k \geqslant 2$, a subset $S = \{(x_i, y_i) : 1 \leqslant i \leqslant k\} \subset \mathrm{inc}(P)$ is called an *alternating cycle* when $x_i \leqslant y_{i+1}$ in P, for all $i = 1, 2, \ldots, k$, where $y_{k+1} = y_1$. Note that a linear extension can not revert all the incomparable pairs of an alternating cycle. In fact, alternating cycles characterize reversible sets: A subset S of $\mathrm{inc}(P)$ can be reverted by a single linear extension of P if and only if S does not contain an alternating cycle.

A pair $(x, y) \in \mathrm{inc}(P)$ is a *critical pair* if $u <_P x$ implies $u <_P y$ and $y <_P v$ implies $x <_P v$, for all $u, v \in X$. The relevance of critical pairs is due to the following fact: A family of linear extensions is a realizer if and only if it reverses the set of critical pairs. Thus the dimension of P equals the minimum size of a family of linear extensions reversing all critical pairs.

2 Grids, Fences and Crowns

The d-grid can be defined as the product of paths. The notion of product applied here is the product of cell-complexes, i.e., if c is a k-cell in the d-dimensional grid G_d viewed as $G_a \times G_b$ with $a + b = d$, then there there is a k'-cell c' in G_a and a k''-cell c'' in G_b with $c = c' \times c''$ and $k = k' + k''$. From this it follows that the face order of G_d is the product of the face posets of d paths. The face poset of a path of length $k \geqslant 1$ is known as the *k-fence*, this is the poset F_k with elements $\{a_0, \ldots, a_k\} \cup \{b_1, \ldots, b_k\}$ and relations $a_i \leqslant b_j$ if $i \in \{j-1, j\}$. See Figure 1 and Figure 2 for examples.

Remark. *Face lattices* belong to convex polytopes, they include a unique minimal element (the empty set) and a unique maximal element (the full body). In contrast *face posets*, i.e., inclusion posets of faces, can be associated to general cell complexes and can have many minimal and maximal elements.

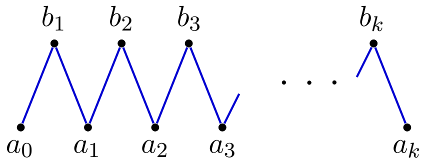

Figure 1: The k-fence F_k.

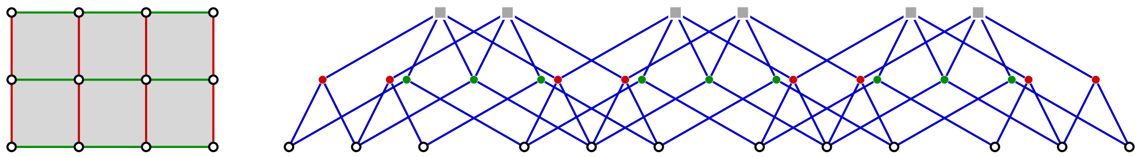

Figure 2: A 3×2 grid and its face poset $F_3 \times F_2$.

To understand the dimension of a product $F_{k_1} \times \cdots \times F_{k_d}$ of fences, we study the dimension of products $F \times P$ where F is a fence in Subsection 2.1. In Subsection 2.4 we study products with crowns and more generally products of 3-dimensional posets. This results in a new proof of Theorem 5.

2.1 Fence Posets

In this subsection we prove the following theorem.

Theorem 7.

$$\dim(F_{k_1} \times \ldots \times F_{k_d}) = d + 1.$$

We begin with a simple observation which covers the case $d = 1$ of the theorem.

Observation 8. The unique 2-realizer of the k-fence F_k is given by

$$L_{\text{left}} = a_0a_1b_1a_2b_2 \ldots a_kb_k \quad \text{and} \quad L_{\text{right}} = a_ka_{k-1}b_k \ldots a_1b_2a_0b_1.$$

The next proposition will give the upper bound for the theorem. Actually we bound the dimension of $\dim(F \times P)$ where F is a fence and P is arbitrary. We include the full proof even though later we will prove more general statements. We begin by introducing a useful concept.

Infixing sequences. For a sequence $S = x_1 \ldots x_s$ of elements of a set X and a sequence $T = y_1 \ldots y_t$ of elements of a set Y we define the *infixion of T in S* as a sequence in $X \times Y$ by

$$S(T) = (x_1, y_1) \ldots (x_1, y_t)(x_2, y_1) \ldots (x_2, y_t) \ldots (x_s, y_1) \ldots (x_s, y_t).$$

The building block of this operation is the infixion of T in an element x of X, this is $x(T) = (x, y_1) \ldots (x, y_t)$.

If realizers of products are constructed via infixions the condition of Observation 6 for being a realizer is typically easy to check. The proof of the following proposition gives an example.

Proposition 9. *Let P be an arbitrary finite poset, then* $\dim(P \times F_k) \leqslant \dim(P) + 1$.

Proof. Let d be the dimension of P and let $R = \{L_1, \ldots, L_d\}$ be a realizer of P. Define a set $R' = \{L'_1 \ldots, L'_d, L''_d\}$ of linear extensions of $P \times F_k$ by

$$\begin{aligned} L'_i &:= L_i(a_0 \ldots a_k b_1 \ldots b_k) \text{ for } 1 \leqslant i < d \\ L'_d &:= L_d(a_0)L_d(a_1b_1)\ldots L_d(a_kb_k) \\ L''_d &:= L_d(a_k)L_d(a_{k-1}b_k)\ldots L_d(a_0b_1). \end{aligned}$$

To certify that R' is a realizer we consider $(p, f), (\tilde{p}, \tilde{f}) \in P \times F_k$ with $(p, f) \nleqslant_{P\times F_k} (\tilde{p}, \tilde{f})$ and distinguish two cases:

- $p \nleqslant_P \tilde{p}$: Since R is a realizer of P there is some L_i such that $\tilde{p} <_i p$. If $i < d$ then $(\tilde{p}, \tilde{f}) <_{i'} (p, f)$. If $i = d$ and $f \nleqslant_{F_k} \tilde{f}$, then we have $\tilde{f}$ before f in one of the two linear extensions of the 2-realizer of F_k, whence $L_d(\tilde{f})$ precedes $L_d(f)$ in L'_d or L''_d. Finally, if $f \leqslant_{F_k} \tilde{f}$ then $L_d(f\tilde{f})$ is a subsequence of L'_d or L''_d which has $(\tilde{p}, \tilde{f})$ before (p, f).

- $p \leqslant_P \tilde{p}$ and $f \nleqslant_{F_k} \tilde{f}$: We have $\tilde{f}$ before f in one of the two linear extensions of the 2-realizer of F_k, whence $L_k(\tilde{f})$ precedes $L_d(f)$ in L'_d or L''_d, hence, either $(\tilde{p}, \tilde{f}) <_{d'} (p, f)$ or $(\tilde{p}, \tilde{f}) <_{d''} (p, f)$.

Since R' is a $(d + 1)$-realizer of $P \times F_k$ we conclude that $\dim(P \times F_k) \leqslant \dim(P) + 1$. □

To see that the bound given in the proposition is tight let P be any d-dimensional poset with **0** and **1**. Due to Baker's theorem the dimension of P with the 2-chain **2** equals $\dim(P)+\dim(\mathbf{2}) = d + 1$. The poset $P \times \mathbf{2}$ is a subposet of $P \times F_k$, whence $\dim(P \times F_k) \geqslant d + 1$.

Products of fences have no **0**, hence, to show the lower bound for the theorem we need another argument. The following theorem, originally due to Reuter, shows that there are posets P such that $\dim(P \times F_1) = \dim(P) + 1$. The dual of F_1 is the poset V on three elements $\{0, 1, 2\}$ with the two relations $0 \leqslant 1, 0 \leqslant 2$ and $1 \parallel 2$.

Theorem 10 (Reuter [Reu90])**.** *Let P be a poset with a* **0***. Then,* $\dim(P \times V) = \dim(P) + 1$.

If P is the face lattice of a polytope with the **1** removed, then $P \times V$ corresponds to the face lattice of a bipyramid with base P without the maximal element.

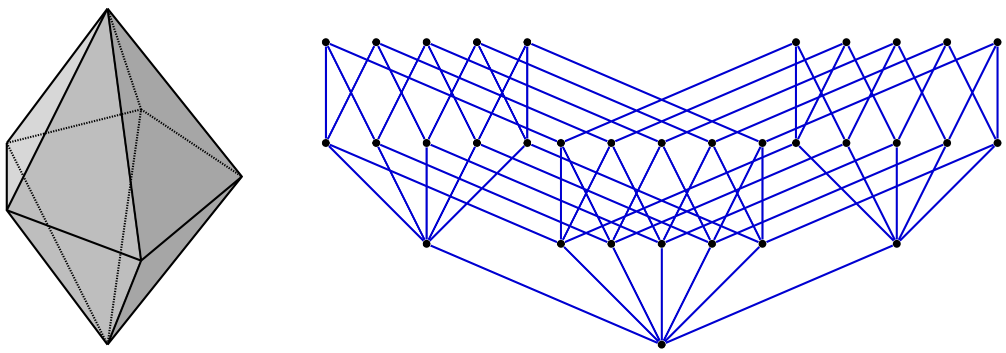

Figure 3: The face lattice of a bipyramid over a pentagon without **1**.

In his proof of the theorem Reuter [Reu90] argues on the basis of Ferrers dimension. Since this generalization of order dimension is not generally well known we have adapted the proof.

Proof. Given Proposition 9 we only have to show the lower bound. Let P be a poset with a $\mathbf{0}$ and $\dim(P) = d$. Aiming for a contradiction we assume that there is a d-realizer $L_1, \ldots, L_d$ of $P \times V$. Observe that for $x, y \in P$ we have $x \leqslant_P y$ if and only if $(x, 0) <_{P\times V} (y, 1)$. For $i \in \{0, 1, 2\}$ we define a subset $\check{I}_i$ of $\mathrm{inc}(P \times V)$ by

$$\check{I}_i = \{((x, 0), (y, i)) \in \mathrm{inc}(P \times V)\}$$

and for $j \in \{1, \ldots, d\}$ we let $\check{I}_i^j$ be the set of pairs in $\check{I}_i$ which are reversed in L_j, i.e.,

$$\check{I}_i^j = \big\{((x, 0), (y, i)) \in \mathrm{inc}(P \times V) \bigm| (y, i) <_j (x, 0)\big\}.$$

Note that the definition of a realizer implies $\bigcup_j \check{I}_i^j = \check{I}_i$. With I_i we denote the projection of $\check{I}_i$ to P, i.e.,

$$I_i = \big\{(x, y) \bigm| x, y \in P, ((x, 0), (y, i)) \in \mathrm{inc}(P \times V)\big\},$$

and with I_i^j we denote the projection of $\check{I}_i^j$ to P, i.e.,

$$I_i^j = \big\{(x, y) \bigm| (y, i) <_j (x, 0)\big\}.$$

The set I_0 equals $\mathrm{inc}(P)$ while I_1 and I_2 contain $\mathrm{inc}(P)$ together with all pairs (x, y) with $y <_P x$, still for $i = 0, 1, 2$ we have $\bigcup_j I_i^j = I_i$.

Note that a subset of $\check{I}_i$ is an alternating cycle in $P \times V$ exactly if the projection onto pairs of P contains an alternating cycle. Hence, all the sets I_0^j, I_1^j and I_2^j are reversible subsets of I_0, I_1 and I_2 respectively. For $(x, y) \in I_1^j$ we have $(y, 0) <_j (y, 1) <_j (x, 0)$, hence $I_1^j \subseteq I_0^j$. A symmetric argument yields $I_2^j \subseteq I_0^j$ whence $I_1^j \cup I_2^j \subseteq I_0^j$.

For $j \in \{1, \ldots, d\}$ we define

$$B^j = \{x \in P \mid (x, 2) <_j (\mathbf{0}, 1)\}.$$

From $(\mathbf{0}, 1) \parallel (\mathbf{0}, 2)$ in $P \times V$ it follows that there is some k such that $(\mathbf{0}, 1) <_k (\mathbf{0}, 2)$, together with $(\mathbf{0}, 2) \leqslant (x, 2)$ for all $x \in P$ this implies $B^k = \varnothing$.

For $j \neq k$ let $\widehat{I}^j = I_1^j \cup \big((P \times B^j) \cap I_1^k\big)$. We will show that $\mathrm{inc}(P) \subseteq \bigcup_{i \neq k} \widehat{I}^j$ and that each $\widehat{I}^j$ is a reversible set. This shows that P has a $(d-1)$-realizer whence $\dim(P) \leqslant d - 1$, a contradiction.

To show that $\widehat{I}^j$ is reversible we assume an alternating cycle $\{(a_i, b_i) : 1 \leqslant i \leqslant \ell\}$ with all pairs in $\widehat{I}^j$. Since I_1^j and I_1^k are reversible as witnessed by L_j and L_k the cycle contains consecutive pairs $(a_i, b_i) \in I_1^j$ and $(a_{i+1}, b_{i+1}) \in (P \times B^j) \cap I_1^k$. Being consecutive in the alternating cycle requires $a_i \leqslant_P b_{i+1}$, however, $b_{i+1} \in B^j$ implies $(b_{i+1}, 2) <_j (\mathbf{0}, 1)$ and $(a_i, b_i) \in I_1^j$ implies $(b_i, 1) <_j (a_i, 0)$. This yields

$$(b_{i+1}, 0) <_j (b_{i+1}, 2) <_j (\mathbf{0}, 1) \leqslant_j (b_i, 1) <_j (a_i, 0),$$

so $a_i \nleqslant_P b_{i+1}$. This is a contradiction.

It remains to show that $\mathrm{inc}(P) \subseteq \bigcup_{j \neq k} \widehat{I}^j$. From $\bigcup_j B^j = P$ and $B^k = \varnothing$ we obtain $\bigcup_{j \neq k} B^j = P$. This implies $\bigcup_{j \neq k} (P \times B^j) \cap I_1^k = I_1^k$ whence

$$\bigcup_{j \neq k} \widehat{I}^j = \bigcup_{j \neq k} \big(I_1^j \cup ((P \times B^j) \cap I_1^k)\big) = \big(\bigcup_{j \neq k} I_1^j\big) \cup I_1^k = \bigcup_j I_1^j = I_1.$$

Since $\mathrm{inc}(P) \subseteq I_1$ this completes the proof. □

From the theorem we obtain $\dim(F_1^d) = \dim(V^d) \geqslant d + 1$. Since F_1^d is a subposet of every product of fences we have completed the proof of Theorem 7.

2.2 Geometric Applications

In this short subsection we collect some geometric consequences of Theorem 10 and open questions.

Observing that F_1^d with a **1** attached is the face lattice of the d-cube we obtain.

Corollary 11 (Reuter [Reu90]). *If H_d is the face lattice of the d-hypercube, then* $\dim(H_d) = d + 1$.

This is a special case of the dimension of the face poset of a grid.

Corollary 12. *The dimension of the face poset of a d-grid with side-lengths $k_1, \ldots, k_d$, $k_i \geqslant 1$ is equal to $d + 1$.*

In two dimensions it follows from Schnyder's theorem and its generalizations that the dimension of the face poset of finite patches of other plane tesselations is also 3, Figure 4 shows some examples.

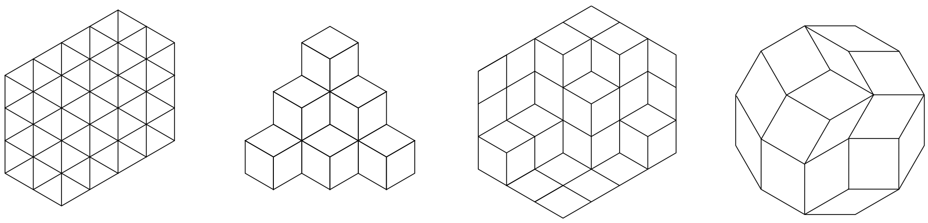

Figure 4: Patches of plane tesselations.

In $\mathbb{R}^d$ it is not clear that the dimension of the face posets of such examples remains $d + 1$, even in cases where all the full-dimensional cells are cubes.

2.3 Generalized Fences

A *generalized fence* is a poset P whose cover graph is a forest of paths. The name is motivated by the fact that every generalized fence can be obtained from some fence in two steps, taking a suborder and subdividing the edges of the diagram. Note that disjoint unions of chains belong to the class of generalized fences.

Our aim in this subsection is to prove the following characterization of generalized fences in terms of products and dimension.

Theorem 13. *A poset F is a generalized fence if and only if* $\dim(P \times F) \leqslant \dim(P) + 1$ *for every poset P.*

The proof that the product with a generalized fence is increasing the dimension by at most one is similar to the proof of Proposition 9.

To begin with, we observe that the dimension of generalized fences is at most 2, this is the generalization of Observation 8. To construct a realizer, fix a left to right order of the components and draw a diagram respecting this order. An example is shown in Figure 5. Based on this drawing we can define two linear extensions L_{left} and L_{right}. These linear extensions are constructed with the generic algorithm for linear extensions which iteratively picks a minimal element of the poset for the linear extension and removes it from the poset.

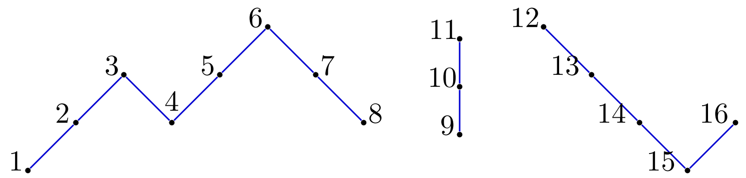

Figure 5: A generalized fence F with three components.

For L_{left} the minimal element chosen by the algorithm is always the leftmost and for L_{right} it is the rightmost. In the case of the generalized fence of Figure 5 we obtain:

$$L_{\text{left}} = 1, 2, 4, 3, 5, 8, 7, 6, 9, 10, 11, 15, 14, 13, 12, 16$$
$$L_{\text{right}} = 15, 16, 14, 13, 12, 9, 10, 11, 8, 7, 4, 5, 6, 1, 2, 3.$$

Let P be a k-dimensional poset with realizer $L_1, \ldots, L_k$ and let F be a generalized fence. We construct a realizer of $P \times F$ by mimicking the proof of Proposition 9. We let $L = L_k$ and use the technique of infixing subsequences of elements of F into linear extensions of P. Choose an arbitrary linear extension M of F and define $L'_i = L_i(M)$ for $i = 1, \ldots, k-1$. The linear extensions L'_k and L''_k are based on L, i.e., on L_k, and the linear extensions L_{left} and L_{right} of F respectively. To construct L'_k we first break L_{left} into pieces of consecutive elements, where each piece consists of all elements of a maximal down-slope. Pieces are one by one infixed into L in the order given by L_{left}. The construction of L''_k is alike but on the basis of L_{right} and the pieces correspond to maximal up-slopes.

We exemplify the construction with the example from Figure 5, the 2-realizer constructed above and with $M = 1, 4, 8, 9, 15, 2, 5, 7, 10, 14, 16, 3, 6, 11, 13, 12$ is:

$$L'_i = L_i(1, 4, 8, 9, 15, 2, 5, 7, 10, 14, 16, 3, 6, 11, 13, 12) \text{ for } 1 \leqslant i < k$$
$$L'_k = L(1)L(2)L(4, 3)L(5)L(8, 7, 6)L(9, 10, 11)L(15, 14, 13, 12)L(16)$$
$$L''_k = L(15, 16)L(14)L(13)L(12)L(9, 10, 11)L(8)L(7)L(4, 5, 6)L(1, 2, 3).$$

To certify that this is a realizer, we refer to Observation 6. Consider $(p, f), (\tilde{p}, \tilde{f}) \in P \times F$ with $(p, f) \nleqslant_{P\times F} (\tilde{p}, \tilde{f})$. The argument that $(\tilde{p}, \tilde{f})$ precedes (p, f) in one of the linear extensions is exactly as in the proof of Proposition 9.

- $p \nleqslant_P \tilde{p}$: There is some L_i such that $\tilde{p} <_i p$. If $i < k$ then $(\tilde{p}, \tilde{f}) <_{i'} (p, f)$. If $i = k$ and $f \nleqslant_F \tilde{f}$, then we have $\tilde{f}$ before f in L_{left} or L_{right} , whence $L_k(\tilde{f})$ precedes $L_k(f)$ in L'_k or L''_k. Finally, if $f \leqslant_F \tilde{f}$ then $L_k(f\tilde{f})$ is a subsequence of L'_k or L''_k which has $(\tilde{p}, \tilde{f})$ before (p, f).

- $p \leqslant_P \tilde{p}$ and $f \nleqslant_F \tilde{f}$: We have $\tilde{f}$ before f in L_{left} or L_{right}, whence $L_k(\tilde{f})$ precedes $L_k(f)$ in L'_k or L''_k, which implies that $(\tilde{p}, \tilde{f})$ comes before (p, f) in L'_k or L''_k.

This completes the proof of the following proposition which is othe first implication needed of the theorem.

Proposition 14. *Let F be a generalized fence and P be an arbitrary poset, then* $\dim(P \times F) \leqslant \dim(P) + 1$.

For the proof of the theorem it remains to show that for every P which is not a generalized fence we can find some Q such that $\dim(Q \times F) > \dim(Q) + 1$. Due to the monotonicity of dimension we only have to consider posets P which are minimal in the suborder relation with respect to not being generalized fences. All minimal examples are shown in Figure 6.

Lemma 15. *If Q is a poset which is not a generalized fence, then Q contains one of posets Z_3, Z_3^{d}, Y, Y^{d}, B_2, C_2, or C_k with $k \geqslant 3$ shown in Figure 6 as a subposet.*

Proof. The cover graph of every poset which is not a generalized fence contains a vertex of degree three or a cycle. If it contains a vertex of degree three it has Z_3 or Y or one of the duals Z_3^{d} or Y^{d} as a subposet. Now suppose that the poset contains a cycle. Consider the orientation of the edges of the cycle induced by the order relation. Let k be the number of sinks in this orientation and note that the number of sources on the cycle is also k. If $k = 1$ the cycle decomposes into two directed paths from source to sink. Since the cover graph has no transitive edges the length of each path is at least two, whence the poset contains an induced B_2. If $k \geqslant 2$ take the poset induced by the k sources and the k sinks and note that it is isomorphic to C_k. □

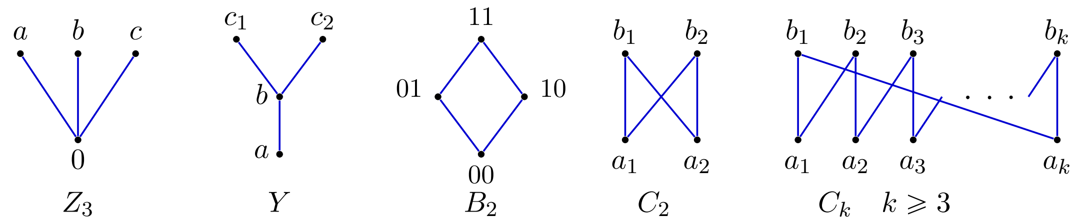

Figure 6: Obstructions for generalized fences: the posets Z_3, Y, B_2, C_2, and C_k with $k \geqslant 3$.

We go through the cases one by one, starting with the easy ones.

For the duals Z_3^{d} and Y^{d} of Z_3 and Y it is sufficient to note that $P^{\mathsf{d}} \times Q^{\mathsf{d}} = (P \times Q)^{\mathsf{d}}$ and that $\dim(P) = \dim(P^{\mathsf{d}})$ for all posets P and Q.

Case $Q = B_2$. The poset B_2 is the 2-dimensional Boolean lattice. The product $B_2 \times B_2$ is the Boolean lattice B_4, hence $\dim(B_2 \times B_2) = \dim(B_4) = 4 > 3 = \dim(B_2) + 1$.

Case $Q = C_k$, $k \geqslant 3$. Recall that $\dim(C_k) = 3$ for $k \geqslant 3$. Now consider the product of a chain L with C_k. We obtain $\dim(L \times C_k) \geqslant 3 > 2 = \dim(L) + 1$.

Case $Q = Z_3$. The dimension of Z_3 is obviously two. We will see that the dimension of $Z_3 \times Z_3$ is four. Together this yields $\dim(Z_3 \times Z_3) = 4 > \dim(Z_3) + 1 = 3$.

It is easily checked that $Z_3 \times Z_3$ is the incidence order of the complete bipartite graph $K_{3,3}$ with a $\mathbf{0}$ attached, hence $\dim(Z_3 \times Z_3) = \dim(K_{3,3})$. Schnyder [Sch89] proved the following remarkable characterization of planar graphs: *A graph G is planar if and only if the dimension of the incidence poset of G is at most three.* Since $K_{3,3}$ is nonplanar we conclude that $\dim(Z_3 \times Z_3) \geqslant 4$.

The history of the proof that the dimension of the incidence order of a non-planar graph is at least four is intricate. The result has frequently been attributed to Babai and Duffus (see e.g. [Tro92, page 128]), this, however, may be wrong. We are aware of a proof in Trotter's book [Tro92, page 129] and a proof in Schnyder's article [Sch89, Thm. 4.1]. Schnyder's proof is a 2-dimensional version of a more general theory of geometric realizations of abstract simplicial complexes which are defined via a collection of linear orders, we refer to Scarf [Sca73] and Ossona de Mendez [Oss99].

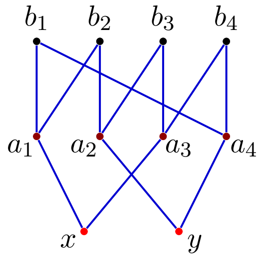

Figure 7: The 4-dimensional poset P_2.

For the two remaining cases we use the poset P_2 shown in Figure 7. Below in Lemma 16 we show that $\dim(P_2) = 4$.

Case $Q = C_2$. The dimension of C_2 is obviously two. The diagram of $C_2 \times C_2$ is shown in Figure 8 (left), there it is highlighted that P_2 is a subposet of $C_2 \times C_2$. Therefore we have $\dim(C_2 \times C_2) \geqslant 4 > 3 = \dim(C_2) + 1$.

Case $Q = Y$. The dimension of Y is obviously two. The diagram of $Y \times Y$ is shown in Figure 8 (right), there it is highlighted that P_2 is a subposet of $Y \times Y$. Therefore we have $\dim(Y \times Y) \geqslant 4 > 3 = \dim(C_2) + 1$.

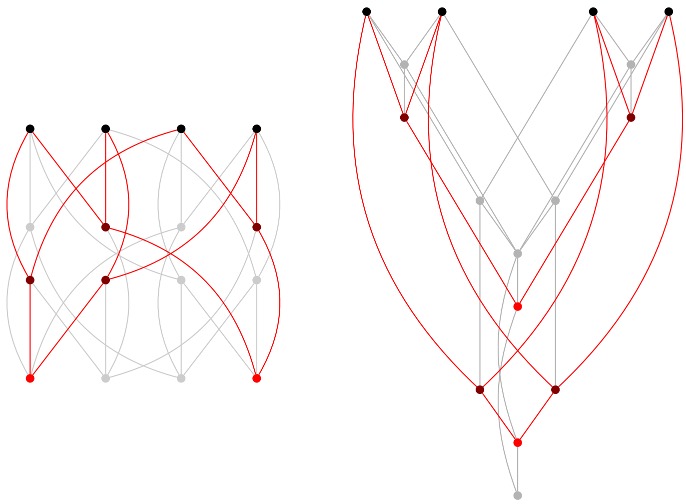

Figure 8: P_2 in $C_2 \times C_2$ and $Y \times Y$.

Lemma 16. *The dimension of the poset P_2 shown in Figure 7 is four.*

Proof. Suppose that P_2 has a realizer L_1, L_2, L_3 of size three. Since P_2 has two minimal elements each has to be the first in one of the three linear extensions. By symmetry between x and y, we may assume that x is first in L_1 and y is first in L_2 and L_3. The critical pairs (y, a_1) and (y, a_3) can only be reversed in L_1, i.e., $a_1 <_1 y$ and $a_3 <_1 y$. Again due to symmetry we may assume that $a_2 <_1 a_4$.

The pair (x, a_2) has to be reversed, we may assume that this happens in L_2. We now know that the pairs (a_2, b_1) and (a_2, b_4) have not been reversed in L_1 because b_1 and b_2 are both larger than a_4 in P_2 and $a_2 <_1 a_4$. The two pairs are also not reversed in L_2 because b_1 and b_2 are both larger than x in P_2 and $a_2 <_2 x$. This implies $b_1 <_3 a_2$ and $b_4 <_3 a_2$. Together with the relations of P_2 we find that both of a_1, a_3 precede both of b_2, b_3 in L_3 and also in L_1.

Consequently, the two pairs (a_1, b_3) and (a_3, b_2) have to be reversed in L_2. However, this is impossible because they form an alternating cycle. Hence, there is no 3-realizer of P_2.

We have shown that $\dim(P_2) \geqslant 4$. The construction of a 4-realizer for P_2 is an easy exercise left to the reader. □

The poset P_2 is the first poset in a sequence of posets $P_n = (X_n, \leqslant_n)$, on $4n + 2$ elements $\{x, y\} \cup \{a_1, \ldots, a_{2n}\} \cup \{b_1, \ldots, b_{2n}\}$. The elements a_i and b_i of P_n induce a crown C_{2n} as in Figure 6. The element x is below all the odd indexed a_i and incomparable to the even indexed, while y is below all the even indexed a_i and incomparable to the odd indexed.

Proposition 17 (Wittmann [Wit23], Lemma 3.8.)**.** *The poset P_n is 4-dimensional and irreducible, i.e., removing any element results in a poset of dimension three.*

2.4 Crowns and Products

To show that there is a poset P with $\dim(P \times C_k) > \dim(P) + 1$ where C_k is a crown with $k \geqslant 3$ we have been using a chain, i.e., a 1-dimensional poset P. This choice was not by laziness. Indeed we next show that there are no other examples.

Theorem 18. *Let P be a finite poset and $k \geqslant 3$. Then,*

$$\dim(P \times C_k) \leqslant \max(3, \dim(P) + 1).$$

Proof. The proof of this theorem is similar to the proof given for fences in Proposition 9. We use a specific realizer of C_k and an arbitrary realizer $L_1, \ldots, L_d$ of P. The realizer of $P \times C_k$ is then constructed via infixion.

Note that unlike in the case of fences and generalized fences the minimum size realizer of C_k is not unique. We are going to use the following 3-realizer of C_k where M_2 and M_3 are the unique 2-realizer of the fence obtained by deleting b_1 from C_k while M_1 is only reverting the critical pairs involving b_1, i.e., the pairs (a_i, b_1) for $2 \leqslant i \leqslant k - 1$.

$$\begin{aligned} M_1 &= a_1 a_k b_1 a_2 \ldots a_{k-1} b_2 \ldots b_k, \\ M_2 &= a_1 a_2 b_2 a_3 b_3 a_4 b_4 \ldots a_k b_k b_1, \\ M_3 &= a_k a_{k-1} b_k a_{k-2} b_{k-1} \ldots a_2 b_3 a_1 b_2 b_1. \end{aligned} \tag{1}$$

Given a poset P with a realizer $L_1, \ldots, L_d$ we define a realizer $L'_1, \ldots, L'_{d-1}, L'_d, L''_d$ as follows.

$$\begin{aligned} L'_i \quad &:= L_i(a_1 \ldots a_k b_1 \ldots b_k) \text{ for } 1 \leqslant i \leqslant d-2, \\ L'_{d-1} &:= L_d(a_1 a_k b_1) L_{d-1}(a_2 \ldots a_{k-1} b_2 \ldots b_k), \\ L'_d \quad &:= L_d(a_1 a_2 b_2) L_d(a_3 b_3) \ldots L_{d-1}(a_k b_k b_1), \\ L''_d \quad &:= L_d(a_k a_{k-1} b_k) L_d(a_{k-2} b_{k-1}) \ldots L_{d-1}(a_1 b_2 b_1). \end{aligned}$$

Note that the last three linear extensions of this list contain subsequences where parts of the realizer of C_k have been injected into L_{d-1} and parts where the injection is into L_d. We come to this again in Section 3.

If $\dim(P) = 1$, then we use a 2-realizer consisting of two copies of the chain P and obtain a 3-realizer of $P \times C_k$ which is minimum because the 3-dimensional crown C_k is a subposet of the product.

It remains to show that $L'_1, \ldots, L'_{d-1}, L'_d, L''_d$ is a realizer of $P \times C_k$. Let $(p, c) \nleqslant_{P \times C_k} (\tilde{p}, \tilde{c})$ in $P \times C_k$ and consider the following two cases.

- $c \nleqslant_{C_k} \tilde{c}$: If $\tilde{c} \neq \{b_1, a_1, a_k\}$, then the pair is reversed in either L'_d or L''_d. If $\tilde{c} = b_1$, then the pair is reversed in L'_{d-1}. For $\tilde{c} = a_1$ and $c \neq a_2$, the pair is reversed in L'_d and for $c = a_2$ it is reversed in L'_{d-1}. An analogous reasoning applies for $\tilde{c} = a_k$ and $c = a_2$ or $c \neq a_2$ but with L''_d instead of L'_d.
- $p \nleqslant_P \tilde{p}$ and $c \leqslant_{C_k} \tilde{c}$: As $p \nleqslant_P \tilde{p}$, there is some $i \in \{1, \ldots, d\}$ such that $\tilde{p} <_i p$. If $i < n-1$, then the pair is reversed in L'_i. If $i = n-1$ and $c \notin \{a_1, a_k\}$, then the pair is reversed in $L_{d-1}(a_2 \ldots a_{k-1} b_2 \ldots b_k)$. If $c = a_1$ or $c = a_k$, then the part $L_{d-1}(a_1 b_2 b_1)$ of L''_d or the part $L_{d-1}(a_k b_k b_1)$ of L'_d respectively suffices. Lastly, if $i = n$ and $c \in \{a_1, a_k\}$, then the pair is reversed in one of the blocks $L_d(a_1 a_k b_1), L_d(a_1 a_2 b_2)$ or $L_d(a_k a_{k-1} b_k)$. For all $c \notin \{a_1, a_k\}$ with $c \leqslant \tilde{c}$ one of L'_d or L''_d contains $L_d(c\tilde{c})$ whence the pair is reversed. □

In Proposition 14 we have extended the result for fences (Proposition 9) to generalized fences, i.e., suborders of fences with subdivisions. Similarly, Theorem 18 can be extended to crowns with subdivisions.

We believe that the upper bound given in the proposition is always tight, i.e., that there is no poset P with $\dim(P \times C_k) = \dim(P)$.

We now know that $3 \leqslant \dim(C_k \times C_\ell) \leqslant 4$, for all $k, \ell \geqslant 3$. The following proposition shows that the upper bound is tight.

Proposition 19. *$\dim(C_k \times C_\ell) = 4$, for $k, \ell \geqslant 3$.*

Proof. While it is not used in our proof, it is interesting to note that $C_k \times C_\ell$ is the incidence poset of vertices, edges and faces of a toroidal $k \times \ell$ grid.

The upper bound immediately follows from Theorem 18. For the lower bound we will show that subposet of C_k formed between the top and the bottom layer is of dimension 4.

Let L be a linear extension of $C_k \times C_\ell$ which belongs to a minimum realizer. Due to symmetry we may assume that (b_1, b_1) is the smallest element from the top layer in L, i.e., (b_1, b_1) is the first element of $\{b_1, \ldots, b_k\} \times \{b_1, \ldots, b_\ell\}$ in L. The set of elements of the bottom layer which are smaller than (b_1, b_1) is

$$D = \{(a_1, a_1), (a_1, a_\ell), (a_k, a_1), (a_k, a_\ell)\}.$$

The poset induced by $\{(b_1, b_2), (b_2, b_1), (b_1, b_\ell), (b_k, b_1)\} \cup D$ forms a 4-crown (see Figure 9). Because (b_1, b_1) is first among all (b_i, b_j) in L, none of the critical pairs of this 4-crown is reversed in L. Hence, besides L a realizer of $C_k \times C_\ell$ requires at least three linear extensions. □

In the introduction we have promised a proof of the fact that $\dim(P \times P) \geqslant 4$ for all P with $\dim(P) = 3$ (Theorem 5). For the proof it is enough to verify the inequality for all 3-irreducible posets. The complete list of 3-irreducible posets has been obtained by Kelly [Kel77] and Trotter and Moore [TM76], and can be found in Trotter's book [Tro92, pp. 62–66]. In this list all the posets except for the chevron and the crowns contain a copy of Z_3 or Z_3^{d}. The right side shows the 3-irreducible poset C from the list with an induced Z_3^{d}. From Subsection 2.3

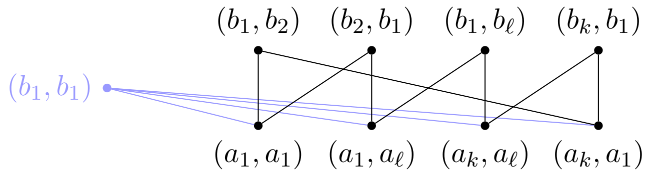

Figure 9: 4-crown in $C_k \times C_\ell$ blocked by (b_1, b_1).

we know that $\dim(Z_3 \times Z_3) = 4$, so $\dim(P \times P) \geqslant 4$ for these posets. We have also shown $\dim(C_2 \times C_2) = 4$. The dimension of the product $C_k \times C_k$ with C_k a crown with $k \geqslant 3$ has been investigated in Proposition 19. The chevron itself contains an induced copy of C_2, see Figure 10, hence, we are done with this case too and the proof of Theorem 5 is complete.

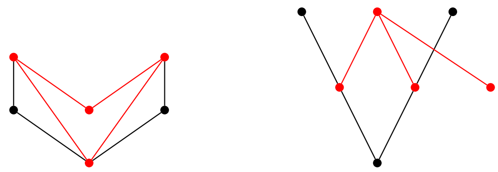

Figure 10: A C_2 subposet of the chevron and a Z_3^{d} subposet of the 3-irreducible poset C.

Products of 3-dimensional posets. We are convinced that $\dim(P \times Q) \geqslant 4$ for all P and Q with $\dim(P) \geqslant 3$ and $\dim(Q) \geqslant 3$. For the proof it would be sufficient to show the inequality for every pair of 3-irreducible posets. We have already observed that except for crowns and the chevron all 3-irreducible posets contain a copy of Z_3 or Z_3^{d}. Moreover, the chevron contains a C_2, hence, the result would follow from the following seven inequalities:

1) $\dim(Z_3 \times Z_3) \geqslant 4$, 2) $\dim(C_2 \times C_2) \geqslant 4$, 3) $\dim(Z_3 \times Z_3^{\mathrm{d}}) \geqslant 4$, 4) $\dim(Z_3 \times C_2) \geqslant 4$,
5) $\dim(C_k \times Z_3) \geqslant 4$, 6) $\dim(C_k \times C_2) \geqslant 4$, 7) $\dim(C_k \times C_\ell) \geqslant 4$.

Inequalities 1) and 2) have been shown as part of the proof of Proposition 14. Inequalities 3) and 4) can easily be checked by computer or with some patience by hand. Inequality 7) was the subject of Proposition 19. Inequalities 5) and 6), however, have been shown to be wrong in Theorem 18. In fact, the theorem shows that to obtain the result that the product of two 3-dimensional posets is at least 4-dimensional, the dimension of each of the products $C_k \times P$ with 3-irreducible P has to be considered.

Using computers (see [Wit]) we have determined the dimension of products of pairs of 3-irreducible posets. In the cases of infinite families the computations were restricted to the initial cases $n = 0, 1, 2$. The results of the computation do not depend on n with the only exception $\dim(F_0 \times F_0) = 5 < 6$. The values are shown in the following table:

3 The Covering Property

In [Reu89] Reuter defines a *covering property* for Ferrers relations which implies upper bounds for the dimension of products. It turns out that the upper bounds of Proposition 9, Proposition 14, and Theorem 18 can be described in these terms. In this section we describe the covering property and discuss its potential and its limitations.

	A_n	B	C	CX_1	CX_2	CX_3	D	E	EX_1	EX_2	FX_1	FX_2	H_n	I_n	F_n	G_n	J_n
A_n	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
B	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
C	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
CX_1	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
CX_2	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
CX_3	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
D	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
E	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
EX_1	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
EX_2	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
FX_1	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
FX_2	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
H_n	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
I_n	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
F_n	4	5	5	5	5	5	5	5	5	5	5	5	5	5	6	6	6
G_n	4	5	5	5	5	5	5	5	5	5	5	5	5	5	6	6	6
J_n	4	5	5	5	5	5	5	5	5	5	5	5	5	5	6	6	6

Table 1: The dimension of products of 3-irreducible posets. For drawings of the diagrams of the posets see [Tro92, pp. 62–66] and note that in the drawing of F_n the edge (c, d) is missing.

To introduce the idea let us look at the proof of Theorem 18 again. We injected consecutive pieces of the linear extensions M_1, M_2 and M_3 of C_k into two linear extensions L_{d-1} and L_d of P to make three linear extensions of the product. The following indicates the blocks used for injection (parentheses) and the linear extension into which the block was injected (color).

$$\begin{aligned} M_1 &= (a_1 a_k b_1)(a_2 \dots a_{k-1} b_2 \dots b_k), \\ M_2 &= (a_1 a_2 b_2)(a_3 b_3) \dots (a_{k-1} b_{k-1})(a_k b_k b_1), \\ M_3 &= (a_k a_{k-1} b_k)(a_{k-2} b_{k-1}) \dots (a_2 b_3)(a_1 b_2 b_1). \end{aligned}$$

We had to prove that a pair pair $(p, x) \nleqslant (q, y)$ is reversed. For the proof we distinguished two cases.

- $x \nleqslant y$: Here we use that there is some M_i such that y precedes x in M_i and the two elements are in distinct blocks of M_i. The L' obtained by injecting the blocks of M_i reverts the pair, i.e., (q, y) precedes (p, x) in L'.

- $p \nleqslant q$ and $x \leqslant y$: As $p \nleqslant q$, the realizer of P contains some L with $q <_L p$. If in one of the M_i there is a block B containing x and y such that the construction involves $L(B)$, i.e., the injection of a block B into L, then (q, y) precedes (p, x) in $L(B)$ and the pair is reversed.

We now turn the conditions established in this abstraction of the proof of Theorem 18 into a definition.

Definition 20. Let $Q = (X, \leqslant)$ be a poset and let $R = M_1, \dots, M_s$ be a realizer of Q. The s-realizer R has the (r, s)-*covering property* if it is possible to break the linear extensions of R into consecutive blocks and color these blocks with colors from $\{1, 2, \dots, r\}$ such that

1. If $x \nleqslant y$ then there is some M_i such that y precedes x in M_i and the two elements are in distinct blocks of M_i.

2. If $x \leqslant y$ then for each color j there is some M_i with a block of color j which contains x and y.

This definition can be viewed as the specialization of a definition given by Reuter [Reu89] in the context of Ferrers relations to the case of partial orders.

Theorem 21. *Let P, Q be posets. If Q has the (r, s)-covering property and* $\dim(P) \geqslant r$ *then*

$$\dim(P \times Q) \leqslant \dim(P) + s - r.$$

Proof. Let the (r, s)-covering property of Q be certified by the realizer $M_1, \dots, M_s$ where for each i the linear extension M_i is a sequence $B_{i,1} \dots B_{i,k_i}$ of blocks and $[i, j] \in \{1, \dots, r\}$ is the color of block $B_{i,j}$. Let $\dim(P) = d \geqslant r$ and let $L_1, \dots, L_d$ be a realizer of P.

We next define linear extensions of $P \times Q$. Let $a = d - r$ For $i \in \{1, \dots, a\}$ define

$$L'_i = L_i(M_1),$$

i.e, we inject M_1 in L_i. For $i \in \{1, \dots, s\}$ let

$$L''_i = L_{a+[i,1]}(B_{i,1}), L_{a+[i,2]}(B_{i,2}), \dots, L_{a+[i,k_i]}(B_{i,k_i}).$$

The claim is that $L'_1, \dots, L_{d-r}, L''_1 \dots, L''_s$ is a realizer of $P \times Q$.

We have to prove that a pair pair $(p, q) \nleqslant (p', q')$ is reversed.

- $q \nleqslant q'$: Condition 2 of the (r, s)-covering property asserts that there are i, j_1, j_2 with $j_1 < j_2$ such that $y \in B_{i,j_1}$ and $x \in B_{i,j_2}$. Hence (p', q') precedes (p, q) in L''_i.
- $p \nleqslant p'$ and $q \leqslant q'$: As $p \nleqslant p'$, the realizer of P contains some L_i reverting the pair (p, p'). If $i \leqslant a = d-r$, the the pairs are reversed in $L'_i = L_i(M_1)$. If $i = a+c$, then condition 1 of the (r, s)-covering property asserts that these is some block of color c containing q and q'. If this block is $B_{k,\ell}$, then L''_k contains the subsequences $L_{a+[k,\ell]}(B_{k,\ell})$ which reverts the pair. □

3.1 Examples of Coverings

We know of no example where a choice of $s > \dim(P)$ would yield an interesting result. Therefore we simplify notation and say that a poset P has the r-covering property, if it has the $\big(r, \dim(P)\big)$-covering property. We list known examples:

- Fences and generalized fences have the 1-covering property.
- Crowns C_k with $k \geqslant 3$ are 2-covering.
- Antichains and more generally disjoint sums of chains are 2-covering.
- Standard example are 2-covering.

Reuter [Reu89] mentioned that under certain conditions, which he did not describe in detail, the 2-covering property is maintained under the bipyramid construction discussed in 2.1. The following proposition provides a precise statement. A realizer $R = L_1, \dots, L_k$ of P has the *hereditary 2-covering property* if for each $i \in \{1, \dots, k\}$, the first block of L_i has color two and all other blocks have color one.

Proposition 22 (Wittmann [Wit23], Proposition 4.13). *If P is a poset with a k-realizer which has the hereditary 2-covering property, then $((P + \mathbf{0}) \times V) - (\mathbf{0}, \mathbf{0})$ has a $(k+1)$-realizer which has the hereditary 2-covering property.*

It is easy to verify that the sum of two chains, the crown C_k with $k \geqslant 3$ and the standard example S_n admit minimum realizers with the hereditary 2-covering property. Hence after adding a $\mathbf{0}$ we can use them to construct infinite series of posets with the 2-covering property.

It would be interesting to identify further posets with the (r, s)-covering property with $r - s \geqslant 2$. A single example with $r - s \geqslant 3$ would disprove Conjecture 4. Our computations (see Table 1) suggest that if P and Q both are 3-dimensional and contain a 3-irreducible subposet different from a crown C_k with $k \geqslant 3$, then $\dim(P \times Q) \geqslant 5$, hence, such posets can not be 2-covering.

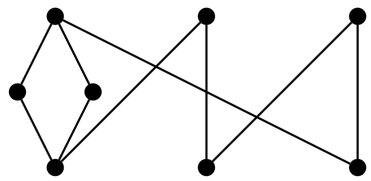

Figure 11: Connected poset P with $\dim(P) = 3$, $\dim(P^2) = 4$ and $\dim(P \times B_2) = 4$.

Let P be the poset shown in Figure 11 and note that $\dim(P) = 3$. The Boolean Lattice B_2 is a subposet of P, therefore $\dim(P \times B_2) \geqslant 4 > \dim(P) + 2 - 2$. This shows that P is not 2-covering. On the other hand $\dim(P \times P) = 4 = \dim(P) + \dim(P) - 2$. The example thus shows that the defect in the dimension of a product with respect to the sum of the dimensions of the factors is not always explained by the covering property.

4 Conclusion

We have studied problems related to the dimension of products of posets. On the way, we have obtained a characterization of generalized fences as the only posets satisfying the inequality $\dim(P \times F) \leqslant \dim(P) + 1$ for every poset P. We also discussed the *r-covering* property in the context of order dimension. The covering property was originally introduced by Reuter for Ferrers dimension. As a side product of this research, Wittman developed a computational framework for dimension problems based on SAT. This allows to compute the dimension of posets with more than 100 elements, whereas all previous algorithms would fail with such sizes. The algorithm is available in a public repository [Wit].

We are not alone with our interest in the dimension of products of posets. George Bergman has a draft [Ber23] with a collection of problems. Here is an interesting question from his collection:

- If P is a poset, and C a chain of more than one element, must $\dim(P \times C) = \dim(P \times \mathbf{2})$?

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