On the Recognition of Four-Directional Orthogonal Ray Graphs

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Abstract. A 4-directional orthogonal ray graph (4-DORG) is the intersection graph of horizontal and vertical rays. If the rays are only pointing into the positive x and y directions, the intersection graph is a 2-DORG. For 3-DORGs horizontal rays are unrestricted but vertical rays only use the positive direction.
The recognition of 2-DORGs is known to be polynomial, they form a nice subclass of bipartite comparability graphs. The recognition problems for 3-DORGs and 4-DORGs, however, are open. Recently is has been shown that the recognition of unit grid intersection graphs, a superclass of 4-DORGs, is NP-complete.
Suppose G is given with a partition $\{L, R, U, D\}$ of its vertices and the question is whether G has a 4-DORG representation, where the four classes of the partition correspond to the four directions of rays. We show that this problem can be solved in polynomial time. For the proof we construct an auxiliary graph $\widetilde{G}$ with directed and undirected edges and show that G has a 4-DORG representation exactly when $\widetilde{G}$ has a transitive orientation respecting its directed edges. There is a gap in the proof of Theorem 1. We have not been able to fix it
With an independent approach, we show that if we are given a permutation π of the vertices of U but the partition $\{L, R\}$ of $V \setminus U$ is not given, then we can still efficiently check whether G has a 3-DORG representation. Here, π is the order of y-coordinates of endpoints of rays for U.

1 Introduction

Segment graphs, i.e., the intersection graphs of segments in the plane, have been the subject of wide spread research activities (see, e.g., [2, 12]). More tractable subclasses of segment graphs are obtained by restricting the number of directions for the segments to some fixed positive integer k [4, 11]. These graphs are called *k-directional segment graphs.* The easiest case is the case of two directions, which can be assumed to be parallel to the x- and y-axis. If intersections of parallel segments are forbidden, then 2-directional segment graphs are bipartite and the corresponding class of graphs is also known as *grid intersection graphs* (GIG), see [9]. The recognition of GIGs is NP-complete [10].

Since segment graphs are a fairly complex class, it is natural to study the subclass of *ray intersection graphs* [1]. Again, the number of directions can be restricted by an integer k, which yields the class of *k-directional ray intersection graphs*. Particularly interesting is the case where all rays are parallel to the x- or y-axis. The resulting class is the class of *orthogonal ray graphs*, the object of this paper. A *k-directional orthogonal ray graph*, for short a k-DORG ($k \in \{2, 3, 4\}$), is an orthogonal ray graph with rays in k directions. If $k = 2$ we assume that all rays point in the positive x- and the positive y-direction, if $k = 3$ we additionally allow the negative x-direction.

The class of 2-DORGs was introduced in [19]. There it is shown that the class of 2-DORGs coincides with the class of bipartite graphs whose complements are circular arc graphs, i.e., intersection graphs of arcs on a circle. This characterization implies the existence of a polynomial recognition algorithm (see [13]), as well as a characterization based on forbidden subgraphs [5]. Alternatively, 2-DORGs can also be characterized as the comparability graphs of ordered sets of height two and interval dimension two. This yields another polynomial recognition algorithm (see, e.g., [7]), and due to the classification of 3-interval irreducible posets ([6], [21, sec 3.7]) a complete description of minimally forbidden subgraphs. A very nice recent contribution on 2-DORGs is [20], the paper contains a clever solution to the jump number problem for the corresponding class of posets and shows a close connection between this problem and a hitting set problem for axis aligned rectangles in the plane.

Our contribution. The recognition problem for 4-DORGs was posed in [19]. We show that if a graph G is given with a partition $\{L, R, U, D\}$ of its vertices, then it can be efficiently checked whether G has a 4-DORG representation such that the vertices of L correspond to the rays pointing to the left, rays for R point to the right, rays for U point upwards, and rays for vertices of D point downwards. The result is obtained by constructing an auxiliary mixed graph $\varGamma_G$, i.e., a graph with some undirected and some directed edges, such that $\varGamma_G$ has a transitive orientation respecting its directed edges if and only if G admits a 4-DORG representation. The result trivially implies that for a graph G and a partition $\{L, R, U\}$ of its vertices it can be efficiently checked whether G has a 3-DORG representation respecting the partition. With an independent approach, we show that if we are given a permutation π of the vertices of U but the partition $\{L, R\}$ of $V \setminus U$ is unknown, then we can still efficiently check whether G has a 3-DORG representation. Here, π denotes the order of y-coordinates of endpoints of rays for U. While the results do not imply a complete polynomial recognition algorithm for 3-DORGs or 4-DORGs, we believe that the tools used in the proof can be adapted for other situations.

4-DORGs in VLSI design. In [18] 4-DORGs were introduced as a mathematical model for defective nano-crossbars in PLA (programmable logic arrays) design. A nano-crossbar is a rectangular circuit board with $m \times n$ orthogonally crossing wires. Fabrication defects may lead to disconnected wires. The bipartite intersection graph that models the surviving crossbar is an orthogonal ray graph.

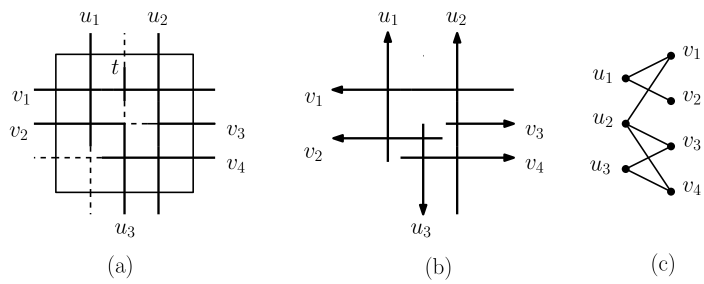


Fig. 1. a) A nano-wire crossbar with disconnected wire defects. b) The bipartite graph modeling the crossbar on the left. Note that vertex t is not present, since the corresponding wire is not connected to the crossbar boundary, hence with the remaining circuit

We briefly mention two problems for 4-DORGs that are tackled in [18]. One of them is that of finding, in a nano-crossbar with disconnected wire defects, a maximal surviving square (perfect) crossbar, which translates to finding a maximal k such that the balanced complete bipartite graph $K_{k,k}$ is a subgraph of the orthogonal ray graph modeling the crossbar. This *balanced biclique problem* is NP-complete for general bipartite graph but turns out to be polynomially solvable on 4-DORGs. The other problem, posed in [16], asks how difficult it is to find a subgraph that would model a given logic mapping and is shown to be NP-hard.

4-DORGs and UGIGs. A *unit grid intersection graph* (UGIG) is a GIG that admits an orthogonal segment representation with all segments of equal (unit) length. Every 4-DORG is a GIG. This can be seen by intersecting the ray representation with a rectangle R, that contains all intersections between the rays in the interior. To see that every 4-DORG is a UGIG, we first fix an appropriate length for the segments, e.g., the length d of the diagonal of R. If we only keep the initial part of length d from each ray we get a UGIG representation. Essentially this construction was already used in [18].

Unit grid intersection graphs were considered in [15]. There it is shown that UGIG contains P_6-free bipartite graphs, interval bigraphs and bipartite permutation graphs. Actually, these classes are already contained 2-DORG. Another contribution of [15] is to provide an example showing that the inclusion of UGIG in GIG is proper. In [17] it is shown that interval bigraphs belong to UGIG. Hardness of Hamiltonian cycle and graph isomorphism for inputs from UGIG has been shown in [22]. Very recently [14] it was shown that the recognition of UGIGs is NP-complete. With this last result we find 4-DORG nested between 2-DORG and UGIG with easy and hard recognition, respectively. This observation was central for our motivation to attack the recognition problem for 4-DORGs.

2 Preliminaries

We consider in this article simple undirected and directed graphs. For a graph G, we denote its vertex and edge set by $V(G)$ and $E(G)$, respectively. In an undirected graph G, the edge between vertices u and v is denoted by uv, and in this case u and v are said to be *adjacent* in G. The set $N(v) = \{u \in V : uv \in E\}$ is called the *neighborhood* of the vertex v of G. If the graph G is directed, we denote by $\langle uv \rangle$ the oriented arc from u to v. If G is the complete graph (i.e. a clique), we call an orientation λ of all (resp. of some) edges of G a (*partial*) *tournament* of G. If in addition λ is transitive, then we call it a (partial) transitive tournament. Given two matrices A and B of size $n \times n$ each, we call by $O(\mathrm{MM}(n))$ the time needed by the fastest known algorithm for multiplying A and B; currently this can be done in $O(n^{2.376})$ time [3].

Let G be a 4-DORG. Then, in a 4-DORG representation of G, every ray is completely determined by one point on the plane and the direction of the ray. We call this point the *endpoint* of this ray. Given a 4-DORG G along with a 4-DORG representation of it, we may not distinguish in the following between a vertex of G and the corresponding ray in the representation, whenever it is clear from the context. Furthermore, for any vertex u of G we will denote by u_x and u_y the x-coordinate and the y-coordinate of the endpoint of the ray of u in the representation, respectively.

3 4-Directional Orthogonal Ray Graphs

In this section we investigate some fundamental properties of 4-DORGs and their representations, which will then be used for our recognition algorithm. In particular, given a graph $G = (V, E)$ with a vertex partition $\{L, R, U, D\}$, we first construct in Section 3.1 two cliques G_1 and G_2 with specific orientation constraints on their edges. Then, in Section 3.2, we augment G_1 and G_2 into two larger cliques G_1^* and G_2^* with some further orientation constraints on their edges. Finally we combine in Section 3.3 the cliques G_1^* and G_2^* into an augmented clique G^* with a third kind of additional orientation constraints on its edges. Then, as we prove in Section 3.3, the initial graph G has a 4-DORG representation with respect to the vertex partition $\{L, R, U, D\}$ if and only if the clique G^* can be transitively oriented subject to some specific orientation constraints. The next observation on a 4-DORG representation is crucial for the rest of the section.

Observation 1 *Let $G = (V, E)$ be a graph that admits a 4-DORG representation, in which L (resp. R, U, D) is the set of leftwards (resp. rightwards, upwards, downwards) oriented rays. If $u \in U$ and $v \in R$ (resp. $v \in L$), then $uv \in E$ if and only if $u_x > v_x$ (resp. $u_x < v_x$) and $u_y < v_y$. Similarly, if $u \in D$ and $v \in R$ (resp. $v \in L$), then $uv \in E$ if and only if $u_x > v_x$ (resp. $u_x < v_x$) and $u_y > v_y$.*

For the remainder of the section, let $G = (V, E)$ be an arbitrary input graph with vertex partition $V = L \cup R \cup U \cup D$, such that $E \subseteq (L \cup R) \times (U \cup D)$.

3.1 The oriented cliques G_1 and G_2

In order to decide whether the input graph $G = (V, E)$ admits a 4-DORG representation, in which L (resp. R, U, D) is the set of leftwards (resp. rightwards, upwards, downwards) oriented rays, we build in this section two auxiliary cliques G_1 and G_2 with $|V|$ vertices each. We partition the vertices of G_1 (resp. G_2) into the sets L_x, R_x, U_x, D_x (resp. L_y, R_y, U_y, D_y). The intuition behind this notation for the vertices of G_1 and G_2 is that, if G has a 4-DORG representation with respect to the partition $\{L, R, U, D\}$, then each of these vertices of G_1 (resp. G_2) corresponds to the x-coordinate (resp. y-coordinate) of the endpoint of a ray of G in this representation.

In the following we define orientation of the edges of G_1 and G_2, cf. Lemma 1. The intuition behind these orientations is the following. If the input graph G is a 4-DORG, then it admits a 4-DORG representation such that, for every $u \in U \cup D$ and $v \in L \cup R$, we have that $u_x > v_x$ (resp. $u_y > v_y$) in this representation if and only if $\langle u_x v_x \rangle$ (resp. $\langle u_y v_y \rangle$) is an oriented edge of the clique G_1 (resp. G_2). That is, since all x-coordinates (resp. y-coordinates) of the endpoints of the rays in a 4-DORG representation can be linearly ordered, these orientations of the edges of G_1 (resp. G_2) build a transitive tournament.

In the next lemma we provide a characterization of 4-DORGs in terms of two particular transitive tournaments of the cliques G_1 and G_2. This characterization follows by Observation 1: the transitive tournament λ_1 (resp. λ_2) of G_1 (resp. of G_2) corresponds bijectively to the linear ordering of the x-coordinates (resp. the y-coordinates) of the endpoints of the rays in a 4-DORG representation of G.

Lemma 1. *The next two conditions are equivalent:*

1. *the graph $G = (V, E)$ has a 4-DORG representation with respect to the vertex partition $\{L, R, U, D\}$,*
2. *there exist two transitive tournaments λ_1 and λ_2 of G_1 and G_2, respectively, such that, whenever $u \in U \cup D$ and $v \in R \cup L$:*
   - *(a) if $uv \in E$, $u \in U$, and $v \in R$ (resp. $v \in L$) then $\langle u_x v_x \rangle \in \lambda_1$ (resp. $\langle v_x u_x \rangle \in \lambda_1$) and $\langle v_y u_y \rangle \in \lambda_2$,*
   - *(b) if $uv \in E$, $u \in D$, and $v \in R$ (resp. $v \in L$) then $\langle u_x v_x \rangle \in \lambda_1$ (resp. $\langle v_x u_x \rangle \in \lambda_1$) and $\langle u_y v_y \rangle \in \lambda_2$,*
   - *(c) if $uv \notin E$, $u \in U$, and $v \in R$ (resp. $v \in L$) then $\langle u_x v_x \rangle \notin \lambda_1$ (resp. $\langle v_x u_x \rangle \notin \lambda_1$) or $\langle v_y u_y \rangle \notin \lambda_2$,*
   - *(d) if $uv \notin E$, $u \in D$, and $v \in R$ (resp. $v \in L$) then $\langle u_x v_x \rangle \notin \lambda_1$ (resp. $\langle v_x u_x \rangle \notin \lambda_1$) or $\langle u_y v_y \rangle \notin \lambda_2$.*

Proof. ($\Rightarrow$) Assume that G has a 4-DORG representation with respect to the vertex partition $\{L, R, U, D\}$. We define the transitive tournament λ_1 of the clique G_1 as follows: for every two vertices u_x and v_x of G_1, $\langle u_x v_x \rangle \in \lambda_1$ if and only if $u_x > v_x$ in the 4-DORG representation of G. Similarly we define the transitive tournament λ_2 of the clique G_2, such that for every two vertices u_y and v_y of G_2, $\langle u_y v_y \rangle \in \lambda_2$ if and only if $u_y > v_y$ in the 4-DORG representation of G. Then the condition 2 of the lemma follows immediately by Observation 1.

($\Leftarrow$) Assume that the condition 2 of the lemma is satisfied. In order to construct 4-DORG representation with respect to the partition $\{L, R, U, D\}$, we define for every ray u of G its endpoint (u_x, u_y) in the representation, as follows. We define each of the sets $\{u_x : u \in V\}$ and $\{u_y : u \in V\}$ to be equal to $\{1, 2, \ldots, |V|\}$. Furthermore we linearly order the values of $\{u_x : u \in V\}$ such that $u_x > v_x$ if and only if $\langle u_x v_x \rangle \in \lambda_1$, and we linearly order the values of $\{u_y : u \in V\}$ such that $u_y > v_y$ if and only if $\langle u_y v_y \rangle \in \lambda_2$. Since every vertex of G is assigned also an orientation for its row (i.e. leftwards, rightwards, upwards, and downwards, according to the partition of V into the sets L, R, U, D), this completes the construction of the 4-DORG representation. Finally it follows by Observation 1 and the condition 2 of the lemma that this 4-DORG representation captures exactly the adjacencies of G, i.e. it is a 4-DORG representation for the graph G. $\square$

As we proved in Lemma 1, the input graph G admits a 4-DORG representation if and only if some edges of G_1, G_2 are forced to have specific orientations in the transitive tournaments λ_1, λ_2 (cf. conditions 2(a)-(b) of Lemma 1) and some pairs of edges of G_1, G_2 are not allowed to have a specific *pair* of orientations in λ_1, λ_2 (cf. conditions 2(c)-(d) of Lemma 1). Motivated by this, we introduce in the next two definitions the notions of *type-1-mandatory* orientations and of *forbidden pairs* of orientations, which will be crucial for our analysis in the remainder of Section 3.

Definition 1 (type-1-mandatory orientations). *Let $u \in U \cup D$ and $v \in L \cup R$, such that $uv \in E$. If $u \in U$ and $v \in R$ (resp. $v \in L$) then the orientations $\langle u_x v_x \rangle$ (resp. $\langle v_x u_x \rangle$) and $\langle v_y u_y \rangle$ of G_1 and G_2 are called* type-1-mandatory. *If $u \in D$ and $v \in R$ (resp. $v \in L$) then the orientations $\langle u_x v_x \rangle$ (resp. $\langle v_x u_x \rangle$) and $\langle u_y v_y \rangle$ of G_1 and G_2 are called* type-1-mandatory. *The set of all type-1-mandatory orientations of G_1 and G_2 is denoted by M_1.*

Note that, using the terminology of Definition 1, the edge orientations that are being forced in conditions 2(a)-(b) of Lemma 1 are exactly the type-1-mandatory orientations, i.e. the set M_1.

Definition 2 (forbidden pairs of orientations). *Let $u \in U \cup D$ and $v \in R \cup L$, such that $uv \notin E$. If $u \in U$ and $v \in R$ (resp. $v \in L$) then the pair $\{\langle u_x v_x \rangle, \langle v_y u_y \rangle\}$ (resp. the pair $\{\langle v_x u_x \rangle, \langle v_y u_y \rangle\}$) of orientations of G_1 and G_2 is called* forbidden. *If $u \in D$ and $v \in R$ (resp. $v \in L$) then the pair $\{\langle u_x v_x \rangle, \langle u_y v_y \rangle\}$ (resp. the pair $\{\langle v_x u_x \rangle, \langle u_y v_y \rangle\}$) of orientations of G_1 and G_2 is called* forbidden.

In the following, whenever the orientation of an edge uv of G_1 or G_2 is type-1-mandatory, we may say for simplicity of the presentation that the *edge* uv (instead of its *orientation*) is type-1-mandatory. Moreover, for simplicity of notation in the remainder of the paper, we introduce in the next definition the notion of *optional edges*.

Definition 3 (optional edges). *Let $\{\langle pq\rangle, \langle ab\rangle\}$ be a pair of forbidden orientations of G_1 and G_2. Then each of the (undirected)* edges *pq and ab is called* optional *edges.*

3.2 The augmented oriented cliques G_1^* and G_2^*

We iteratively augment the cliques G_1 and G_2 into the two larger cliques G_1^* and G_2^*, respectively, as follows. For every *optional* edge pq of G_1 (resp. of G_2), where $p \in U_x \cup D_x$ and $q \in L_x \cup R_x$ (resp. $p \in U_y \cup D_y$ and $q \in L_y \cup R_y$), we add two vertices $r_{p,q}$ and $r_{q,p}$ and we add all needed edges to make the resulting graph G_1^* (resp. G_2^*) a clique. Note that, if the initial graph G has n vertices and m non-edges (i.e. $\binom{n}{2} - m$ edges), then G_1^* and G_2^* are cliques with $n + 2m$ vertices each. Furthermore, since G_1 and G_2 are induced subgraphs of G_1^* and G_2^*, respectively, Definitions 1-3 carry over to G_1^* and G_2^*, i.e. the type-1-mandatory orientations, the forbidden pairs of orientations, and the optional edges are the same in the cliques G_1, G_2 and in the cliques G_1^*, G_2^*. We now introduce the notion of *type-2-mandatory* orientations of G_1^* and G_2^*.

Definition 4 (type-2-mandatory orientations). *For every optional edge pq of G_1^*, the orientations $\langle pr_{p,q}\rangle$ and $\langle qr_{q,p}\rangle$ of G_1^* are called* type-2-mandatory *orientations of G_1^*. For every optional edge pq of G_2^*, the orientations $\langle r_{p,q}p\rangle$ and $\langle r_{q,p}q\rangle$ of G_2^* are called* type-2-mandatory *orientations of G_2^*. The set of all type-2-mandatory orientations of G_1^* and G_2^* is denoted by M_2.*

Whenever the orientation of an edge uv of G_1^* or G_2^* is type-2-mandatory (cf. Definition 4), we may say for simplicity of the presentation that the *edge uv* (instead of its *orientation*) is type-2-mandatory. The type-2-mandatory orientations in G_1^* and in G_2^* (with respect to the optional edge pq) are illustrated in Figure 2(a) and 2(b), respectively. In Figure 2 only the graph induced by the vertices $\{p, q, r_{p,q}, r_{q,p}\}$ is shown, while the orientations of the type-2-mandatory edges $pr_{p,q}$ and $qr_{q,p}$ are drawn with double arrows for better visibility.

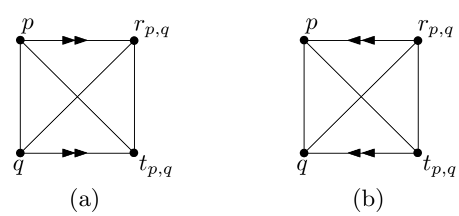


Fig. 2. An example for the construction of (a) the clique G_1^* from G_1 and of (b) the clique G_2^* from G_2, where pq is an optional edge and (a) $p \in U_x \cup D_x, q \in L_x \cup R_x$, (b) $p \in U_y \cup D_y$ and $q \in L_y \cup R_y$. In both (a) and (b), the type-2-mandatory orientations are drawn with double arrows.

In the next lemma we extend Lemma 1 by providing a characterization of 4-DORGs in terms of two transitive tournaments of the augmented cliques G_1^* and

G_2^*. The characterization of Lemma 2 will be then used in Section 3.3, in order to prove our main result of Section 3, namely the characterization of 4-DORGs in Theorem 1.

Lemma 2. *The next two conditions are equivalent:*

1. *the graph $G = (V, E)$ has a 4-DORG representation with respect to the vertex partition $\{L, R, U, D\}$,*
2. *there exist two transitive tournaments λ_1^* and λ_2^* of G_1^* and G_2^*, respectively, such that λ_1^* and λ_2^* satisfy the conditions 2(a)-(d) of Lemma 1, and in addition:*
   - *(a) $M_2 \subseteq \lambda_1^* \cup \lambda_2^*$,*
   - *(b) let pq be an optional edge of G_1^* and $pw \notin M_2$ be an incident edge of pq in G_1^*; then $\langle wr_{p,q}\rangle \in \lambda_1^*$ implies that $\langle wp\rangle \in \lambda_1^*$,*
   - *(c) let pq be an optional edge of G_2^* and $pw \notin M_2$ be an incident edge of pq in G_2^*; then $\langle r_{p,q}w\rangle \in \lambda_2^*$ implies that $\langle pw\rangle \in \lambda_2^*$,*
   - *(d) let pq be an optional edge of G_1^* (resp. G_2^*), where $p \in U_x \cup D_x$ (resp. $p \in U_y \cup D_y$); then we have:*
     - *(i) either $\langle pq\rangle, \langle r_{p,q}q\rangle, \langle r_{p,q}r_{q,p}\rangle \in \lambda_1^*$ (resp. $\in \lambda_2^*$) or $\langle qp\rangle$, $\langle qr_{p,q}\rangle$, $\langle r_{q,p}r_{p,q}\rangle \in \lambda_1^*$ (resp. $\in \lambda_2^*$),*
     - *(ii) for any incident optional edge pq' of G_1^* (resp. G_2^*), either $\langle pq\rangle, \langle r_{p,q'}q\rangle \in \lambda_1^*$ (resp. $\in \lambda_2^*$) or $\langle qp\rangle, \langle qr_{p,q'}\rangle \in \lambda_1^*$ (resp. $\in \lambda_2^*$),*
     - *(iii) for any incident optional edge $p'q$ of G_1^* (resp. G_2^*), either $\langle r_{p,q}q\rangle, \langle r_{p,q}r_{q,p'}\rangle \in \lambda_1^*$ (resp. $\in \lambda_2^*$) or $\langle qr_{p,q}\rangle, \langle r_{q,p'}r_{p,q}\rangle \in \lambda_1^*$ (resp. $\in \lambda_2^*$).*

Proof. ($\Rightarrow$) Assume that G has a 4-DORG representation with respect to the vertex partition $\{L, R, U, D\}$. Then the vertices of the clique G_1 (resp. of G_2) can be ordered by a linear ordering P_1 (resp. P_2), according to the x-coordinates (resp. the y-coordinates) of the endpoints of the rays in the 4-DORG representation. That is, $u >_{P_1} v$ (resp. $u >_{P_2} v$) if and only if $u_x > v_x$ (resp. $u_y > v_y$) in the 4-DORG representation of G. Now we iteratively extend the linear ordering P_1 (resp. P_2) of the vertices of G_1 (resp. G_2) to the linear ordering P_1^* (resp. P_2^*) of the vertices of G_1^* (resp. G_2^*) as follows. For every optional edge pq of G_1^*, we place the vertex $r_{p,q}$ immediately to the *left* of p in P_1^* (i.e. $r_{p,q} <_{P_1^*} p$ and no vertex lies between $r_{p,q}$ and p in P_1^*). Furthermore, for every optional edge pq of G_2^*, we place the vertex $r_{p,q}$ immediately to the *right* of p in P_2^* (i.e. $p <_{P_2^*} r_{p,q}$ and no vertex lies between p and $r_{p,q}$ in P_2^*).

We define the transitive tournament λ_1^* of the clique G_1^* as follows: for every two vertices u and v of G_1^*, $\langle uv\rangle \in \lambda_1^*$ if and only if $u >_{P_1^*} v$. Similarly, we define the transitive tournament λ_2^* of the clique G_2^* as follows: for every two vertices u and v of G_2^*, $\langle uv\rangle \in \lambda_2^*$ if and only if $u >_{P_2^*} v$. Since the linear ordering P_1^* (resp. P_2^*) is an extension of the linear ordering P_1 (resp. P_2), it follows now easily (similarly to the proof of Lemma 1) that the transitive tournaments λ_1^* and λ_2^* satisfy the conditions 2(a)-(d) of Lemma 1.

Proof of the condition 2(a) of the lemma. Let pq be an arbitrary optional edge of G_1^* (resp. of G_2^*). Then $r_{p,q} <_{P_1^*} p$ and $r_{q,p} <_{P_1^*} q$ (resp. $p <_{P_2^*} r_{p,q}$

and $q <_{P_2^*} r_{q,p}$) by the construction of the linear ordering P_1^* (resp. P_2^*) from the linear ordering P_1 (resp. P_2). Therefore, by the definition of the transitive tournament λ_1^* (resp. λ_2^*) from the linear ordering P_1^* (resp. P_2^*), it follows that $\langle pr_{p,q}\rangle, \langle qr_{q,p}\rangle \in \lambda_1^*$ (resp. $\langle r_{p,q}p\rangle, \langle r_{q,p}q\rangle \in \lambda_2^*$). That is, due to Definition 4, the orientations of the set M_2 are included in the union of λ_1^* and λ_2^*.

Proof of the conditions 2(b) and 2(c) of the lemma. Let pq be an arbitrary optional edge of G_1^* (resp. of G_2^*) and let $pw \notin M_2$ be an incident edge of pq in G_1^* (resp. G_2^*). Then, similarly to the previous paragraph, the relative position of w with p in P_1^* (resp. in P_2^*) is the same as the relative position of w with $r_{p,q}$ in P_1^* (resp. in P_2^*). That is, $p >_{P_1^*} w$ if and only if $r_{p,q} >_{P_1^*} w$ (resp. $p >_{P_2^*} w$ if and only if $r_{p,q} >_{P_2^*} w$). Therefore, by the definition of the transitive tournament λ_1^* (resp. λ_2^*), it follows that $\langle pw\rangle \in \lambda_1^*$ (resp. $\in \lambda_2^*$) if and only if $\langle r_{p,q}w\rangle \in \lambda_1^*$ (resp. $\in \lambda_2^*$). This implies the conditions 2(b) and 2(c) of the lemma.

Proof of the condition 2(d) of the lemma. Let pq be an optional edge of G_1^*, where $p \in U_x \cup D_x$ (the proof for the case where pq is an optional edge of G_2^*, where $p \in U_y \cup D_y$, is exactly the same, so we omit it here for simplicity of the presentation). The relative position of p with q in P_1^* is the same as the relative position of $r_{p,q}$ with q in P_1^* (by the construction of the linear ordering P_1^* from P_1). Similarly, the relative position of p with q in P_1^* is the same as the relative position of $r_{p,q}$ with $r_{q,p}$ in P_1^*. That is, either $p >_{P_1^*} q$ and $r_{p,q} >_{P_1^*} q$ and $r_{p,q} >_{P_1^*} r_{q,p}$, or $p <_{P_1^*} q$ and $r_{p,q} <_{P_1^*} q$ and $r_{p,q} <_{P_1^*} r_{q,p}$. Therefore, by the definition of the transitive tournament λ_1^* from the linear ordering P_1^*, it follows that either $\langle pq\rangle, \langle r_{p,q}q\rangle, \langle r_{p,q}r_{q,p}\rangle \in \lambda_1^*$ or $\langle qp\rangle, \langle qr_{p,q}\rangle, \langle r_{q,p}r_{p,q}\rangle \in \lambda_1^*$. This proves the condition 2(d)(i) of the lemma.

Consider now an arbitrary optional edge pq' of G_1^* that is incident to the optional edge pq. Then, similarly to the above, the relative position of p with q in P_1^* is the same as the relative position of $r_{p,q'}$ with q in P_1^*. Therefore, either $p >_{P_1^*} q$ and $r_{p,q'} >_{P_1^*} q$, or $p <_{P_1^*} q$ and $r_{p,q'} <_{P_1^*} q$. Thus, either $\langle pq\rangle, \langle r_{p,q'}q\rangle \in \lambda_1^*$ or $\langle qp\rangle, \langle qr_{p,q'}\rangle \in \lambda_1^*$. This proves the condition 2(d)(ii) of the lemma.

Finally, consider an arbitrary optional edge $p'q$ of G_1^* that is incident to the optional edge pq. Then, by the construction of the linear ordering P_1^*, the relative position of p with q in P_1^* is the same as the relative position of each of $\{p, r_{p,q}\}$ with each of $\{q, r_{q,p'}\}$ in P_1^*. Therefore, either $r_{p,q} >_{P_1^*} q$ and $r_{p,q} >_{P_1^*} r_{q,p'}$, or $r_{p,q} <_{P_1^*} q$ and $r_{p,q} <_{P_1^*} r_{q,p'}$. Thus, either $\langle r_{p,q}q\rangle, \langle r_{p,q}r_{q,p'}\rangle \in \lambda_1^*$ or $\langle qr_{p,q}\rangle, \langle r_{q,p'}r_{p,q}\rangle \in \lambda_1^*$. This proves the condition 2(d)(iii) of the lemma.

For an illustration of the condition 2(d) of the lemma, see Figure 3.

($\Leftarrow$) Assume that there exist two transitive tournaments λ_1^* and λ_2^* of G_1^* and G_2^*, respectively, which satisfy the whole condition 2 of the lemma. Then, in particular, λ_1^* and λ_2^* satisfy the conditions 2(a)-(d) of Lemma 1. Now we define λ_1 (resp. λ_2) to be the restriction of λ_1^* (resp. λ_2^*) to the clique G_1 (resp. G_2). Then λ_1 and λ_2 are transitive tournaments of G_1 and G_2, respectively, and they satisfy the conditions 2(a)-(d) of Lemma 1. Thus G has a 4-DORG representation with respect to the vertex partition $\{L, R, U, D\}$ by Lemma 1. $\square$

An example of the orientations of condition 2(d) in Lemma 2 (for the case of G_1^*) is shown in Figure 3. For simplicity of the presentation, in Figure 3 only the the type-2-mandatory edges, the optional edges, and their associated edges are shown (the formal definition of the *associated edges* to an optional edge pq is given in the next definition).

Definition 5 (associated edges). *Let pq be an* optional *edge of G_1 (resp. G_2), where $p \in U_x \cup D_x$ and $q \in L_x \cup R_x$ (resp. $p \in U_y \cup D_y$ and $q \in L_y \cup R_y$). The* associated *edges to the optional edge pq are the edges defined in condition 2(d) of Lemma 2, i.e. the edges $r_{p,q}q$, $r_{p,q}r_{q,p}$, $r_{p,q'}q$, and $r_{p,q}r_{q,p'}$.*

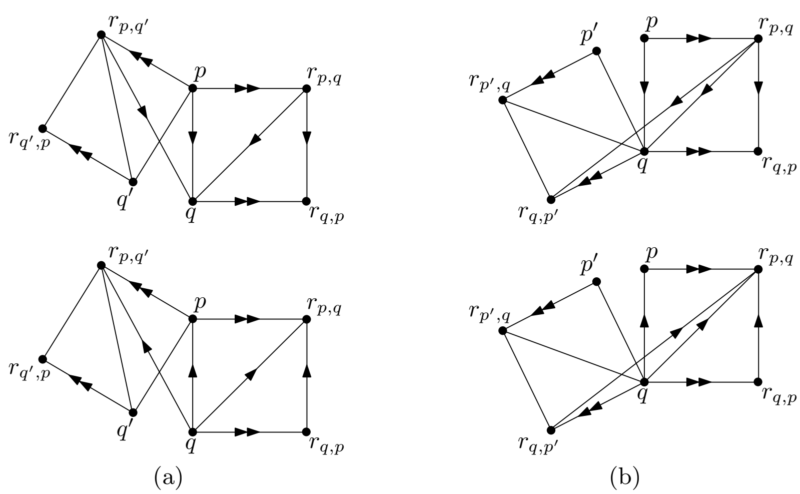


Fig. 3. An example of the orientations of the clique G_1^* in the transitive tournament λ_1^*, where $p \in U_x \cup D_x$ (cf. condition 2(d) in Lemma 2): (a) both possible orientations where the optional edges pq and pq' are incident and (b) both possible orientations where the optional edges pq and $p'q$ are incident. In both (a) and (b), the orientations of the type-2-mandatory edges are drawn with double arrows. The case for G_2 is the same, except that the orientation of the type-2-mandatory edges is the opposite.

The next corollary follows easily from the proof of Lemma 2.

Corollary 1. *Lemma 2 is true also if we remove condition 2(c).*

Proof. The first direction ($\Rightarrow$) of the proof is identical to the proof of Lemma 2. For the second direction ($\Leftarrow$) of the proof, similarly to the proof of Lemma 2, we define the restriction λ_1 of λ_1^* (resp. the restriction λ_2 of λ_2^*) to the clique G_1 (resp. G_2). Then λ_1 and λ_2 are transitive tournaments of G_1 and G_2, respectively, and they satisfy the conditions 2(a)-(d) of Lemma 1. Thus G has a 4-DORG representation with respect to the vertex partition $\{L, R, U, D\}$ by Lemma 1.

□

3.3 The coupling of G_1^* and G_2^* into the oriented clique G^*

Now we iteratively construct the clique G^* from the cliques G_1^* and G_2^*, as follows. Initially G^* is the union of G_1^* and G_2^*, together with all needed edges such that G^* is a clique. Then, for every pair $\{\langle pq \rangle, \langle ab \rangle\}$ of *forbidden orientations* of G_1^* and G_2^* (where $pq \in E(G_1)$ and $ab \in E(G_2)$, cf. Definition 2), we merge in G^* the vertices $r_{b,a}$ and $r_{p,q}$, i.e. we have $r_{b,a} = r_{p,q}$ in G^*. Recall that each of the cliques G_1^* and G_2^* has $n + 2m$ vertices. Therefore, since G_1^* and G_2^* have m pairs $\{\langle pq \rangle, \langle ab \rangle\}$ of forbidden orientations, the resulting clique G^* has $2n + 3m$ vertices. Furthermore, since G_1^* and G_2^* are induced subgraphs of G^*, Definitions 1-5 carry over to G^*, i.e. the type-1-mandatory and type-2-mandatory orientations, the forbidden pairs of orientations, the optional edges, and their associated edges are the same in the cliques G_1^*, G_2^* and in the clique G^*. We now introduce the notion of *type-3-mandatory* orientations of G^*.

Definition 6 (type-3-mandatory orientations). *For every pair $\{\langle pq \rangle, \langle ab \rangle\}$ of forbidden orientations of G_1^* and G_2^*, the orientation $\langle r_{q,p} r_{a,b} \rangle$ is called a* type-3-mandatory *orientation of G^*. The set of all type-3-mandatory orientations of G^* is denoted by M_3.*

Whenever the orientation of an edge uv of G^* is type-3-mandatory (cf. Definition 6), we may say for simplicity of the presentation that the *edge uv* (instead of its *orientation*) is type-3-mandatory. An example for the construction of G^* from G_1^* and G_2^* is illustrated in Figure 4, where it is shown how two optional edges $pq \in E(G_1^*)$ and $ab \in E(G_2^*)$ are joined together in G^*, where $\{\langle pq \rangle, \langle ab \rangle\}$ is a pair of forbidden orientations of G_1^* and G_2^*. For simplicity of the presentation, only the optional edges pq and ab, the type-2-mandatory edges $pr_{p,q}$, $qr_{q,p}$, $ar_{a,b}$, $br_{b,a}$, and two of the associated edges (namely $r_{p,q}r_{q,p}$ and $r_{a,b}r_{b,a}$) are shown in Figure 4. Furthermore, the type-2-mandatory orientations $\langle pr_{p,q} \rangle$, $\langle qr_{q,p} \rangle$, $\langle r_{a,b}a \rangle$, and $\langle r_{b,a}b \rangle$, as well as the type-3-mandatory orientation $\langle r_{q,p} r_{a,b} \rangle$, are drawn with double arrows in Figure 4 for better visibility.

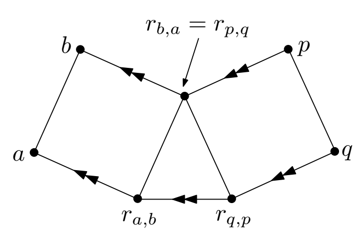


Fig. 4. An example of joining in G^* the pair of optional edges $\{pq, ab\}$, where $pq \in E(G_1)$ and $ab \in E(G_2)$.

In the next theorem we extend Lemmas 1 and 2 by providing a characterization of 4-DORGs in terms of one transitive tournament λ^* of the clique G^*.

The main novelty of the characterization of Theorem 1 in comparison to Lemmas 1 and 2 is that Theorem 1 does not rely any more on the conditions 2(c)-(d) of Lemma 1, which concern the *forbidden pairs* of orientations. This characterization will be used in Section 5, in order to provide our main result of the paper, namely the recognition of 4-DORGs with respect to the vertex partition $\{L, R, U, D\}$.

Theorem 1. *The next two conditions are equivalent:*

1. *the graph $G = (V, E)$ with n vertices has a 4-DORG representation with respect to the vertex partition $\{L, R, U, D\}$,*
2. *there exists a transitive tournament λ^* of G^*, such that λ^* satisfies the conditions 2(b)-(d) of Lemma 2 and $M_1 \cup M_2 \cup M_3 \subseteq \lambda^*$.*

Furthermore, given such a transitive tournament λ^* of G^*, a 4-DORG representation of G can be computed in $O(n^2)$ time.

Proof. ($\Rightarrow$) Assume that G has a 4-DORG representation with respect to the vertex partition $\{L, R, U, D\}$. Then Lemma 2 implies that there exist two transitive tournaments λ_1^* and λ_2^* of the cliques G_1^* and G_2^*, respectively, which satisfy the conditions 2(a)-(d) of Lemma 1 (and thus also the conditions 2(a)-(b) of Lemma 1) and the conditions 2(a)-(d) of Lemma 2 we have no proof for this statement – it may be false.

We now define the *partial* tournament $\widehat{\lambda}_1^*$ (resp. $\widehat{\lambda}_2^*$) as the restriction of the tournament λ_1^* (resp. λ_2^*) to the some specific oriented edges of G_1^* (resp. of G_2^*), as follows. First, we include in $\widehat{\lambda}_1^*$ (resp. $\widehat{\lambda}_2^*$) (i) the *type-1-mandatory* and *type-2-mandatory* orientations (cf. Definitions 1 and 4), i.e. the oriented edges of $M_1 \cup M_2$ in G_1^* (resp. in G_2^*) and (ii) the orientations of the *optional* edges and their *associated* edges (cf. Definitions 3 and 5). Furthermore, consider an optional edge pq of G_1^* and an incident edge $pw \notin M_2$ in G_1^*. Then, if *both* edges $wr_{p,q}$ and wp are oriented as $\langle wr_{p,q}\rangle$ and $\langle wp\rangle$ in the tournament λ_1^* of G_1^* (cf. condition 2(b) of Lemma 2), we augment the partial tournament $\widehat{\lambda}_1^*$ by the orientations $\langle wr_{p,q}\rangle$ and $\langle wp\rangle$. Finally, consider an optional edge pq of G_2^* and an incident edge $pw \notin M_2$ in G_2^*. Then, if *both* edges $wr_{p,q}$ and wp are oriented as $\langle r_{p,q}w\rangle$ and $\langle pw\rangle$ in the tournament λ_2^* of G_2^* (cf. condition 2(c) of Lemma 2), we augment the partial tournament $\widehat{\lambda}_2^*$ by the orientations $\langle r_{p,q}w\rangle$ and $\langle pw\rangle$. This completes the definition of the partial tournaments $\widehat{\lambda}_1^*$ and $\widehat{\lambda}_2^*$. Note that each of $\widehat{\lambda}_1^*$ and $\widehat{\lambda}_2^*$ satisfies the conditions 2(b)-(d) of Lemma 2.

We define now the *partial* tournament $\widehat{\lambda}^*$ of G^* as $\widehat{\lambda}^* = \widehat{\lambda}_1^* \cup \widehat{\lambda}_2^* \cup M_3$. By the construction of $\widehat{\lambda}^*$, it follows that $M_1 \cup M_2 \cup M_3 \subseteq \widehat{\lambda}^*$ and that $\widehat{\lambda}^*$ satisfies the conditions 2(b)-(d) of Lemma 2.

We now prove that the partial tournament $\widehat{\lambda}^*$ of G^* is *acyclic*. Suppose otherwise that $\widehat{\lambda}^*$ contains an induced directed cycle C. Note that, since $\widehat{\lambda}_1^*$ (resp. $\widehat{\lambda}_2^*$) is a restriction of the transitive tournament λ_1^* (resp. λ_2^*), it follows that $\widehat{\lambda}_1^*$ (resp. $\widehat{\lambda}_2^*$) is an acyclic partial tournament of G_1^* (resp. of G_2^*). Therefore, since $\widehat{\lambda}_1^* \subseteq \widehat{\lambda}^*$ and $\widehat{\lambda}_2^* \subseteq \widehat{\lambda}^*$, the directed cycle C of $\widehat{\lambda}^*$ must contain edges from both $\widehat{\lambda}_1^*$ and $\widehat{\lambda}_2^*$. Let $z_1 \in V(G_1^*) \setminus V(G_2^*)$ and $z_2 \in V(G_2^*) \setminus V(G_1^*)$ be two vertices of C. Then, in particular, $\widehat{\lambda}^*$ contains a directed path Q from z_2 to z_1. Now, by the construction of $\widehat{\lambda}^*$ it follows that there must exist a pair $\{\langle pq \rangle, \langle ab \rangle\}$ of forbidden orientations of G_1^* and G_2^* (cf. Definition 2), where $pq \in E(G_1^*)$ and $ab \in E(G_2^*)$, such that Q passes from the vertices of G_2^* to the vertices of G_1^* through the vertex $r_{b,a} = r_{p,q}$, cf. Figure 4.

Observe now (by the construction of the partial tournament $\widehat{\lambda}_1^*$ of G_1^*) that the only possible outgoing edges from the vertex $r_{p,q}$ in $\widehat{\lambda}_1^*$ are some of the associated edges to the optional edge pq, cf. Definition 5. As an example see the oriented edges $\langle r_{p,q}q \rangle$, $\langle r_{p,q}r_{q,p} \rangle$, and $\langle r_{p,q}r_{q,p'} \rangle$ in the upper parts of Figures 3(a) and 3(b), respectively. Similarly, the only possible incoming edges to the vertex $r_{b,a} = r_{p,q}$ in $\widehat{\lambda}_2^*$ are some of the associated edges to the optional edge ab.

Since $\widehat{\lambda}_1^*$ (resp. $\widehat{\lambda}_2^*$) satisfies the condition 2(d) of Lemma 2, the associated edges of pq (resp. of ab) that are adjacent to vertex $r_{b,a} = r_{p,q}$ are either *all* incoming to $r_{p,q}$ or *all* outgoing from $r_{p,q}$ in $\widehat{\lambda}_1^*$ (resp. in $\widehat{\lambda}_2^*$). Since by assumption the directed path Q passes from the vertices of G_2^* to the vertices of G_1^* through $r_{b,a} = r_{p,q}$, it follows that these edges in $\widehat{\lambda}_1^*$ (resp. in $\widehat{\lambda}_2^*$) are all outgoing from (resp. incoming to) $r_{b,a} = r_{p,q}$. Therefore, in particular, the edge $r_{p,q}r_{q,p}$ is oriented as $\langle r_{p,q}r_{q,p} \rangle$ in $\widehat{\lambda}_1^*$ and the edge $r_{a,b}r_{b,a}$ is oriented as $\langle r_{a,b}r_{b,a} \rangle$ in $\widehat{\lambda}_2^*$. Thus the optional edges pq and ab are oriented as $\langle pq \rangle$ in $\widehat{\lambda}_1^*$ and $\langle ab \rangle$ in $\widehat{\lambda}_2^*$ (cf. condition 2(d)(i) of Lemma 2). This comes in contradiction to the fact that $\{\langle pq \rangle, \langle ab \rangle\}$ is a pair of forbidden orientations of G_1^* and G_2^*, cf. Definition 2 (recall that, due to Lemma 2, the conditions 2(c)-(d) of Lemma 1 are satisfied by the tournaments λ_1^* and λ_2^*). Therefore the partial tournament $\widehat{\lambda}^*$ of G^* is acyclic, and thus $\widehat{\lambda}^*$ can be extended to a *transitive* tournament λ^* of G^* such that $\widehat{\lambda}^* \subseteq \lambda^*$. Furthermore, since $\widehat{\lambda}^*$ satisfies the conditions 2(b)-(d) of Lemma 2, it follows that also λ^* satisfies these conditions. Finally $M_1 \cup M_2 \cup M_3 \subseteq \widehat{\lambda}^*$, since $M_1 \cup M_2 \cup M_3 \subseteq \lambda^*$.

($\Leftarrow$) Assume that there exists a transitive tournament λ^* of G^*, such that λ^* satisfies the conditions 2(b)-(d) of Lemma 2 and $M_1 \cup M_2 \cup M_3 \subseteq \lambda^*$. We define the transitive tournament λ_1^* (resp. λ_2^*) of G_1^* (resp. G_2^*) as the restriction of λ^* on the edges of the clique G_1^* (resp. G_2^*). Then, since λ^* satisfies the conditions 2(b)-(d) of Lemma 2, it follows that also λ_1^* and λ_2^* satisfy the conditions 2(b)-(d) of Lemma 2. Furthermore, since $M_2 \subseteq \lambda^*$ by assumption, it follows that $M_2 \subseteq \lambda_1^* \cup \lambda_2^*$, and thus λ_1^* and λ_2^* satisfy also the condition 2(a) of Lemma 2.

Due to Lemma 2, in order to prove that the graph $G = (V, E)$ has a 4-DORG representation with respect to the vertex partition $\{L, R, U, D\}$, it remains to show that λ_1^* and λ_2^* satisfy also the conditions 2(a)-(d) of Lemma 1. Recall that the edge orientations that are being forced in conditions 2(a) and 2(b) of Lemma 1 are exactly the type-1-mandatory orientations, i.e. the set M_1 (cf. Definition 1). That is, these two conditions are equivalent to the fact that $M_1 \subseteq \lambda_1^* \cup \lambda_2^*$, which is true since $M_1 \subseteq \lambda^*$ by assumption.

Now recall that the conditions 2(c) and 2(d) of Lemma 1 state that λ_1^* and λ_2^* do not contain any *forbidden pair* $\{\langle pq\rangle, \langle ab\rangle\}$ of orientations, where pq and ab are optional edges (cf. Definitions 2 and 3). For the sake of contradiction, consider a pair $\{\langle pq\rangle, \langle ab\rangle\}$ of forbidden orientations and assume that $\langle pq\rangle \in \lambda_1^*$ and $\langle ab\rangle \in \lambda_2^*$. Then, since λ_1^* and λ_2^* satisfy the condition 2(d)(i) of Lemma 2, it follows that $\langle r_{p,q}r_{q,p}\rangle \in \lambda_1^*$ and $\langle r_{a,b}r_{b,a}\rangle \in \lambda_2^*$, and thus $\langle r_{a,b}r_{b,a}\rangle, \langle r_{p,q}r_{q,p}\rangle \in \lambda^*$. Therefore, since λ^* is transitive by assumption and $r_{b,a} = r_{p,q}$ by the construction of G^* (cf. Figure 4), it follows that $\langle r_{a,b}r_{q,p}\rangle \in \lambda^*$. This is a contradiction, since $\langle r_{q,p}r_{a,b}\rangle \in M_3$ (cf. Definition 6) and $M_3 \subseteq \lambda^*$ by assumption. Therefore it is not true that both $\langle pq\rangle \in \lambda_1^*$ and $\langle ab\rangle \in \lambda_2^*$, and thus λ_1^* and λ_2^* satisfy the conditions 2(c) and 2(d) of Lemma 1.

Summarizing, λ_1^* and λ_2^* satisfy the conditions 2(a)-(d) of Lemma 1 and the conditions 2(a)-(d) of Lemma 2. Therefore Lemma 2 implies that the graph $G = (V, E)$ has a 4-DORG representation with respect to the vertex partition $\{L, R, U, D\}$.

Time analysis. First recall that, if the input graph G has n vertices and m non-edges (i.e. $\binom{n}{2} - m = O(n^2)$ edges), each of the cliques G_1, G_2, G_1^*, G_2^* has $O(n)$ vertices and $O(n^2)$ edges. Assume now that the condition 2 of the theorem is satisfied, i.e. there exists a transitive tournament λ^* of G^* such that λ^* satisfies the conditions 2(b)-(d) of Lemma 2 and $M_1 \cup M_2 \cup M_3 \subseteq \lambda^*$. First we construct the transitive tournament λ_1^* (resp. λ_2^*) of G_1^* (resp. G_2^*) as the restriction of λ^* on the edges of the clique G_1^* (resp. G_2^*). Second we construct the transitive tournament λ_1 (resp. λ_2) of G_1^* (resp. G_2^*) as the restriction of λ_1^* (resp. λ_2^*) on the edges of the clique G_1 (resp. G_2). The construction of λ_1 and λ_2 can be clearly done in $O(n^2)$ time, since the size of G^* is $O(n^2)$. Furthermore, due to the above proof of Theorem 1 and the proof of Lemma 2, it follows that the transitive tournaments λ_1 and λ_2 satisfy the conditions 2(a)-(d) of Lemma 1. Therefore, due to the proof of Lemma 1 (cf. the ($\Leftarrow$)-part of the proof), we can compute the endpoint for every ray in a 4-DORG representation of G by just sorting their coordinates. This can be done in $O(n \log n)$ time. Therefore, given the transitive tournament λ^* of G^*, a 4-DORG representation for G can be constructed in $O(n^2)$ time. $\square$

Similarly to Corollary 1, the next corollary follows easily from the proof of Theorem 1.

Corollary 2. *Theorem 1 is true also if λ^* does not have to satisfy the conditions 2(b) and 2(c) of Lemma 2.*

4 *S*-orientations of graphs

In this section we introduce a new way of augmenting an *arbitrary* graph G by adding a new vertex and some new edges to G. This type of augmentation process is done with respect to a particular edge $e_i = x_iy_i$ of the graph G, and is called the *deactivation* of e_i in G. By doing so, we establish structural properties that are needed in our 4-DORG recognition algorithm. In particular, if we apply iteratively the deactivation operation for a set of specific edges of the clique G^* of Section 3.3, we are able to reduce the recognition problem for 4-DORGs into the problem of extending a partial orientation of a graph into a transitive orientation (cf. Section 3). In order to do so, we first introduce in this section the crucial notion of an *S-orientation* of a graph G (cf. Definition 8), which extends the classical notion of a transitive orientation. For the remainder of this section, G denotes an arbitrary graph, and not the input graph discussed in Section 3.

Definition 7. *Let $G = (V, E)$ be a graph and let (x_i, y_i), $1 \leq i \leq k$, be k* ordered *pairs of vertices of G, where $x_i y_i \in E$. Let V_{out}, V_{in} be two* disjoint *vertex subsets of G, where $\{x_i : 1 \leq i \leq k\} \subseteq V_{out} \cup V_{in}$. For every $i = 1, 2, \ldots, k$:*

- *the* special neighborhood *of x_i is a vertex subset $S(x_i) \subseteq \left(N(x_i) \cap \left(\bigcap_{x_j = x_i} N(y_j)\right)\right) \setminus \{x_j : 1 \leq j \leq k\}$,*
- *the* forced neighborhood orientation *of x_i is:*
  - *the set $F(x_i) = \{\langle x_i z\rangle : z \in S(x_i)\}$ of oriented edges of G, if $x_i \in V_{out}$,*
  - *the set $F(x_i) = \{\langle z x_i\rangle : z \in S(x_i)\}$ of oriented edges of G, if $x_i \in V_{in}$.*

Definition 8. *Let $G = (V, E)$ be a graph. For every $i = 1, 2, \ldots, k$ let $S(x_i)$ be a special neighborhood in G. Let T be a transitive orientation of G. Then T is an S-*orientation *of G on the special neighborhoods $S(x_i)$, $1 \leq i \leq k$, if for every $i = 1, 2, \ldots, k$:*

1. *$F(x_i) \subseteq T$ and*
2. *for every $z \in S(x_i)$, $\langle x_i y_i\rangle \in T$ if and only if $\langle z y_i\rangle \in T$.*

*Furthermore, if G admits such a transitive orientation T then G is S-*orientable *on these special neighborhoods.*

Definition 9. *Let $G = (V, E)$ be a graph. For every $i = 1, 2, \ldots, k$ let $S(x_i)$ be a special neighborhood in G. Let T be an S-orientation of G on the sets $S(x_i)$, $1 \leq i \leq k$. Then T is* consistent *if, for every $i = 1, 2, \ldots, k$, it satisfies the following conditions, whenever $zw \in E$, where $z \in S(x_i)$ and $w \in (N(x_i) \cap N(y_i)) \setminus S(x_i)$:*

- *if $x_i \in V_{out}$, then $\langle wz\rangle \in T$ implies that $\langle w x_i\rangle \in T$,*
- *if $x_i \in V_{in}$, then $\langle zw\rangle \in T$ implies that $\langle x_i w\rangle \in T$.*

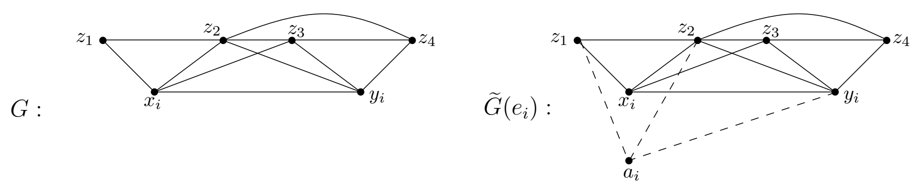
In the next definition we introduce the notion of *deactivating* an edge $e_i = x_i y_i$ of a graph G, where $S(x_i)$ is a special neighborhood in G. In order to deactivate edge e_i of G, we augment appropriately the graph G, obtaining a new graph $\widetilde{G}(e_i)$ that has one new vertex.

Definition 10. *Let $G = (V, E)$ be a graph and let $S(x_i)$ be a special neighborhood in G. The graph $\widetilde{G}(e_i)$ obtained by* deactivating *the edge $e_i = x_i y_i$ (with respect to S_i) is defined as follows:*

1. *$V(\widetilde{G}(e_i)) = V \cup \{a_i\}$,*

2. $E(\widetilde{G}(e_i)) = E \cup \{za_i : z \in N(x_i) \setminus S(x_i)\}$.

Note that, after deactivating the edge e_i, the edge a_iy_i is always an edge of $\widetilde{G}(e_i)$, since $y_i \notin S(x_i)$ by Definition 7. An example of the deactivation operation of Definition 10 is illustrated in Figure 5. For better visibility, the edges of $\widetilde{G}(e_i) \setminus E(G)$ are drawn dashed.



(a) (b)

Fig. 5. (a) A graph G and (b) the graph $\widetilde{G}(e_i)$ obtained after the deactivation of the edge $e_i = x_iy_i$ with respect to the edge neighborhood set $S(x_i) = \{z_3\}$. The edges of $\widetilde{G}(e_i) \setminus E(G)$ are drawn dashed.

Lemma 3. *Let $G = (V, E)$ be a graph and $S(x_i)$, $1 \leq i \leq k$, be a set of k special neighborhoods in G. Let $e_k = x_ky_k$ and let M_0 be an arbitrary set of edge orientations of G. If $\widetilde{G}(e_k)$ has an S-orientation $\widetilde{T}$ on $S(x_1), S(x_2), \ldots, S(x_{k-1})$ such that $M_0 \cup F(x_k) \subseteq \widetilde{T}$, then G has an S-orientation T on $S(x_1), S(x_2), \ldots, S(x_k)$ such that $M_0 \subseteq \widetilde{T}$.*

Proof. We will show that $\widetilde{T}$ is also an S-orientation of $\widetilde{G}(e_k)$ on S_k. First consider a vertex $z \in S(x_k)$, and thus $a_kz \notin E(\widetilde{G}(e_k))$ by Definition 10. Then, since $\widetilde{T}$ is transitive and $x_ka_k \notin E(\widetilde{G}(e_k))$ by Definition 10, it follows that $\langle x_ky_k \rangle \in \widetilde{T}$ if and only if $\langle a_ky_k \rangle \in \widetilde{T}$. Furthermore, since $a_kz \notin E(\widetilde{G}(e_k))$, it follows that $\langle a_ky_k \rangle \in \widetilde{T}$ if and only if $\langle zy_k \rangle \in \widetilde{T}$. Therefore, for every $z \in S(x_k)$, $\langle x_ky_k \rangle \in \widetilde{T}$ if and only if $\langle zy_k \rangle \in \widetilde{T}$. Thus, since $F(x_k) \subseteq \widetilde{T}$ by assumption, it follows by Definition 8 that $\widetilde{T}$ is also an S-orientation of $\widetilde{G}(e_k)$ on S_k. Thus, since $M_0 \subseteq \widetilde{T}$ and $\widetilde{T}$ is by assumption an S-orientation of $\widetilde{G}(e_k)$ on $S_1, S_2, \ldots, S_{k-1}$, it follows that the restriction T of $\widetilde{T}$ on G is an S-orientation of G on $S(x_1), S(x_2), \ldots, S(x_k)$ such that $M_0 \subseteq T$. $\square$

Lemma 4. *Let $G = (V, E)$ be a graph and $S(x_i)$, $1 \leq i \leq k$, be a set of k special neighborhoods in G. Let $e_k = x_ky_k$ and let M_0 be an arbitrary set of edge orientations of G. If G has a consistent S-orientation T on $S(x_1), S(x_2), \ldots, S(x_k)$ such that $M_0 \subseteq T$, then $\widetilde{G}(e_k)$ has a consistent S-orientation $\widetilde{T}$ on $S(x_1), S(x_2), \ldots, S(x_{k-1})$ such that $M_0 \cup F(x_k) \subseteq \widetilde{T}$.*

Proof. Assume that G has a consistent S-orientation T on $S(x_1), S(x_2), \ldots, S(x_k)$ such that $M_0 \subseteq T$. Then, in particular, $F(x_k) \subseteq T$ and

T is an S-orientation on $S(x_1), S(x_2), \ldots, S(x_{k-1})$ by Definition 8. We construct an orientation $\widetilde{T}$ of $\widetilde{G}(e_k)$ such that $T \subseteq \widetilde{T}$ (and thus also $M_0 \subseteq \widetilde{T}$), as follows. For every edge $a_k z \in E(\widetilde{G}(e_k))$, where $z \in N(x_k) \setminus S(x_k)$, we orient $\langle a_k z \rangle \in \widetilde{T}$ if and only if $\langle x_k z \rangle \in T$. It follows by the definition of $\widetilde{T}$ that $F(x_k) \subseteq T \subseteq \widetilde{T}$. Furthermore $\widetilde{T}$ is an S-orientation on $S(x_1), S(x_2), \ldots, S(x_{k-1})$, cf. Definition 8. In Figure 6 the orientation $\widetilde{T}$ is illustrated on a small example.

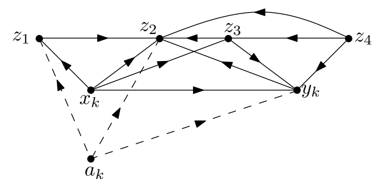


Fig. 6. An example for the orientation $\widetilde{T}$ of the graph $\widetilde{G}(e_i)$, $i = k$, of Figure 5, where $e_k = x_k y_k$.

We first prove that $\widetilde{T}$ remains a *consistent* S-orientation on the sets $S(x_1), S(x_2), \ldots, S(x_{k-1})$, cf. Definition 9. Consider the vertex pair (x_j, y_j), where $1 \leq j \leq k$. Furthermore consider two vertices z, w, where $zw \in E$, $z \in S(x_j)$, and $w \in (N(x_j) \cap N(y_j)) \setminus S(x_j)$. If $a_k \notin \{x_j, y_j, z, w\}$, then the four vertices $\{x_j, y_j, z, w\}$ satisfy the conditions of Definition 9, since the S-orientation T is consistent by assumption. Otherwise assume that $a_k \in \{x_j, y_j, z, w\}$, and thus $a_k = w$. Note that the vertices $\{x_j, y_j, z, a_k\}$ induce a clique, and thus $x_k \notin \{x_j, y_j, z, a_k\}$, since $x_k a_k \notin E(\widetilde{G}(e_k))$ by the construction of $\widetilde{G}(e_k)$. Furthermore note that $x_k \notin S(x_j)$ by Definition 7. Since each of $\{x_j, y_j, z\}$ is adjacent to a_k in $\widetilde{G}(e_k)$, it follows by the construction of $\widetilde{G}(e_k)$ that each of $\{x_j, y_j, z\}$ is also adjacent to x_k in $\widetilde{G}(e_k)$. Therefore the vertices $\{x_j, y_j, z, x_k\}$ satisfy the conditions of Definition 9. Now, the construction of the orientation $\widetilde{T}$ implies that $\langle a_k z \rangle \in \widetilde{T}$ if and only if $\langle x_k z \rangle \in T$; similarly $\langle a_k x_j \rangle \in \widetilde{T}$ if and only if $\langle x_k x_j \rangle \in T$. Therefore, since the vertices $\{x_j, y_j, z, x_k\}$ satisfy the conditions of Definition 9, it follows that the vertices $\{x_j, y_j, z, a_k\}$ satisfy also the conditions of Definition 9. Thus T is a consistent S-orientation on the sets $S(x_1), S(x_2), \ldots, S(x_{k-1})$ by Definition 9.

In order to complete the proof of the lemma, we now show that the orientation $\widetilde{T}$ is transitive. To do so, it suffices to prove that $\widetilde{T}$ is transitive on any induced subgraph of $\widetilde{G}(e_k)$ on three vertices and at least two edges. Since the the orientation T of G is transitive by assumption and a_k is the only new vertex in $\widetilde{G}(e_k)$, it suffices to consider only triples $\{z, w, a_k\}$ of vertices in $\widetilde{G}(e_k)$.

Suppose first that the vertices $\{z, w, a_k\}$ induce a triangle in $\widetilde{G}(e_k)$. Then $x_k \notin \{z, w\}$, since $x_k a_k \notin E(\widetilde{G}(e_k))$ by construction of $\widetilde{G}(e_k)$. Furthermore, since z and w are adjacent to a_k $\widetilde{G}(e_k)$, it follows that z and w are also adjacent to x_k. Moreover, the construction of $\widetilde{T}$ implies that $\langle a_k z \rangle \in \widetilde{T}$ if and only if

$\langle x_k z\rangle \in T$; similarly $\langle a_k w\rangle \in \widetilde{T}$ if and only if $\langle x_k w\rangle \in T$. That is, the triangle induced by $\{z, w, a_k\}$ is transitive in $\widetilde{T}$ if and only if the triangle induced by $\{z, w, x_k\}$ is transitive in T. Therefore, since T is transitive by assumption, the triangle induced by $\{z, w, a_k\}$ is transitive in $\widetilde{T}$.

Suppose now that the induced subgraph of $\widetilde{G}(e_k)$ on the vertices $\{z, w, a_k\}$ has two edges. Let first $za_k, a_k w \in E(\widetilde{G}(e_k))$ and $zw \notin E(\widetilde{G}(e_k))$. Then $x_k \notin \{z, w\}$ by the construction of $\widetilde{G}(e_k)$. Furthermore, by the construction of the orientation $\widetilde{T}$ it follows that $\langle a_k z\rangle \in \widetilde{T}$ if and only if $\langle x_k z\rangle \in T$; moreover $\langle a_k w\rangle \in \widetilde{T}$ if and only if $\langle x_k w\rangle \in T$. Therefore, the induced subgraph of $\widetilde{G}(e_k)$ on the vertices $\{z, w, a_k\}$ is transitively oriented in $\widetilde{T}$ if and only if the induced subgraph of G on the vertices $\{z, w, x_k\}$ is transitively oriented in T; this is true, since the orientation T is transitive by assumption.

Let now $zw, wa_k \in E(\widetilde{G}(e_k))$ and $za_k \notin E(\widetilde{G}(e_k))$. Then $x_k \neq w$, since $x_k a_k \notin E(\widetilde{G}(e_k))$, and thus $\langle a_k w\rangle \in \widetilde{T}$ if and only if $\langle x_k w\rangle \in T$ by the construction of the orientation $\widetilde{T}$. Thus, if $x_k = z$, it follows immediately that the induced subgraph of $\widetilde{G}(e_k)$ on $\{w, z, a_k\}$ is transitively oriented in $\widetilde{T}$. Suppose that $x_k \neq z$. We distinguish now the cases where $z \notin S(x_k)$ and $z \in S(x_k)$.

Case 1. $z \notin S(x_k)$. Then the construction of $\widetilde{G}(e_k)$ implies that vertex z is not adjacent to x_k, since z is not adjacent to a_k. Therefore, since the orientation T is transitive by assumption, $\langle x_k w\rangle \in T$ if and only if $\langle zw\rangle \in T$. Therefore, since $\langle a_k w\rangle \in \widetilde{T}$ if and only if $\langle x_k w\rangle \in T$, it follows that $\langle a_k w\rangle \in \widetilde{T}$ if and only if $\langle zw\rangle \in T$. Therefore, since $T \subseteq \widetilde{T}$ and T is transitive by assumption, it follows that the induced subgraph of $\widetilde{G}(e_k)$ on $\{w, z, a_k\}$ is transitively oriented in $\widetilde{T}$.

Case 2. $z \in S(x_k)$. Then z is adjacent to both x_k and y_k in G, cf. Definition 7. Furthermore, since w is adjacent to a_k in $\widetilde{G}(e_k)$ by assumption, it follows by the construction of $\widetilde{G}(e_k)$ that $w \notin S(x_k)$. The remainder of the proof, which is done by contradiction, is based on the fact that the S-orientation $\widetilde{T}$ is *consistent*, cf. Definition 9.

Assume that the induced subgraph of $\widetilde{G}(e_k)$ on $\{w, z, a_k\}$ is not transitively oriented in $\widetilde{T}$, i.e. either $\langle a_k w\rangle, \langle wz\rangle \in \widetilde{T}$ or $\langle zw\rangle, \langle wa_k\rangle \in \widetilde{T}$. In the following we distinguish the cases where $x_k \in V_{out}$ and $x_k \in V_{in}$.

Case 2a. $x_k \in V_{out}$. Then $\langle x_k z\rangle \in T$. Suppose first that $\langle zw\rangle, \langle wa_k\rangle \in \widetilde{T}$. Then $\langle wx_k\rangle \in T$ by the construction of $\widetilde{G}$. That is, the vertices $\{x_k, z, w\}$ of G induce a directed cycle in T, which is a contradiction to the transitivity of T.

Suppose now that $\langle a_k w\rangle, \langle wz\rangle \in \widetilde{T}$. Then $\langle x_k w\rangle \in T$ by the construction of $\widetilde{T}$. Let $\langle y_k x_k\rangle \in T$. Then, since $\langle x_k w\rangle \in T$ and T is transitive by assumption, it follows in particular that $y_k w \in E(G)$. Otherwise let $\langle x_k y_k\rangle \in T$. Then, since T is an S-orientation by assumption (cf. Definition 8), it follows that $\langle zy_k\rangle \in T$. Therefore, since we assumed that $\langle wz\rangle \in T \subseteq \widetilde{T}$, it follows in particular by the transitivity of T that $y_k w \in E(G)$. That is, $y_k w \in E(G)$ in both cases where $\langle y_k x_k\rangle \in T$ and $\langle x_k y_k\rangle \in T$. Summarizing, $z \in S(x_k)$, $w \in (N(x_k) \cap N(y_k)) \setminus S(x_k)$, and $zw \in E(\widetilde{G}(e_k))$, while $\langle wz\rangle \in T \subseteq \widetilde{T}$ and $\langle x_k w\rangle \in T$. This is a contradiction to the first condition of Definition 9, since the orientation T is consistent.

Case 2b. $x_k \in V_{in}$. Then $\langle zx_k \rangle \in T$. Suppose first that $\langle a_k w \rangle, \langle wz \rangle \in \widetilde{T}$. Then $\langle x_k w \rangle \in T$ by the construction of $\widetilde{G}$. That is, the vertices $\{x_k, z, w\}$ of G induce a directed cycle in T, which is a contradiction to the transitivity of T.

Suppose now that $\langle zw \rangle, \langle wa_k \rangle \in \widetilde{T}$. Then $\langle wx_k \rangle \in T$ by the construction of $\widetilde{T}$. Let $\langle x_k y_k \rangle \in T$. Then, since $\langle wx_k \rangle \in T$ and T is transitive by assumption, it follows in particular that $y_k w \in E(G)$. Otherwise let $\langle y_k x_k \rangle \in T$. Then, since T is an S-orientation by assumption (cf. Definition 8), it follows that $\langle y_k z \rangle \in T$. Therefore, since we assumed that $\langle zw \rangle \in T \subseteq \widetilde{T}$, it follows in particular by the transitivity of T that $y_k w \in E(G)$. That is, $y_k w \in E(G)$ in both cases where $\langle x_k y_k \rangle \in T$ and $\langle y_k x_k \rangle \in T$. Summarizing, $z \in S(x_k)$, $w \in (N(x_k) \cap N(y_k)) \setminus S(x_k)$, and $zw \in E(\widetilde{G}(e_k))$, while $\langle zw \rangle \in T \subseteq \widetilde{T}$ and $\langle wx_k \rangle \in T$. This is a contradiction to the second condition of Definition 9, since the orientation T is consistent.

Therefore the orientation $\widetilde{T}$ is transitive. This completes the proof of the lemma. □

After deactivating the edge e_k of G, obtaining the graph $\widetilde{G}(e_k)$, we can continue by sequentially deactivating the edges $e_{k-1}, e_{k-2}, \ldots, e_1$, obtaining eventually the graph $\widetilde{G}$. The next theorem follows by iterative application of Lemmas 3 and 4.

Theorem 2. *Let $G = (V, E)$ be a graph and $S(x_i)$, $1 \leq i \leq k$, be a set of k special neighborhoods in G. Let M_0 be an arbitrary set of edge orientations of G, and let $\widetilde{G}$ be the graph obtained after deactivating all edges $e_i = x_i y_i$, where $1 \leq i \leq k$.*

- *If G has a consistent S-orientation T on $S(x_1), S(x_2), \ldots, S(x_k)$ such that $M_0 \subseteq T$, then $\widetilde{G}$ has a transitive orientation $\widetilde{T}$ such that $M_0 \cup F(x_i) \subseteq \widetilde{T}$ for every $i = 1, 2, \ldots, k$.*
- *If $\widetilde{G}$ has a transitive orientation $\widetilde{T}$ such that $M_0 \cup F(x_i) \subseteq \widetilde{T}$ for every $i = 1, 2, \ldots, k$, then G has an S-orientation T on $S(x_1), S(x_2), \ldots, S(x_k)$ such that $M_0 \subseteq T$.*

5 Efficient Recognition of 4-DORGs

In this section we complete our analysis in Sections 3 and 4 and we present our 4-DORG recognition algorithm. Let $G = (V, E)$ be an arbitrary input graph that is given along with a vertex partition $V = L \cup R \cup U \cup D$, such that $E \subseteq (L \cup R) \times (U \cup D)$. First we construct from G the cliques G_1, G_2 (cf. Section 3.1), the augmented cliques G_1^*, G_2^* (cf. Section 3.2), and finally we combine G_1^* and G_2^* in order to produce the clique G^* (cf. Section 3). Then, for a specific choice of k ordered pairs (x_i, y_i) of vertices, $1 \leq i \leq k$, and for particular sets $S(x_i)$ and neighborhood orientations $F(x_i)$, $1 \leq i \leq k$ (cf. Definitions 7 and 8), we iteratively deactivate the edges x_iy_i, $1 \leq i \leq k$ (cf. Section 4), constructing thus the graph $\widetilde{G}^*$. Then, for a specific partial orientation of the graph $\widetilde{G}^*$, we prove in Theorems 3 and 4 that $\widetilde{G}^*$ has a transitive orientation that extends this partial orientation if and only if the input graph G has a 4-DORG representation with respect to the vertex partition $\{L, R, U, D\}$.

Assume that the input graph G has n vertices and m non-edges (i.e. $\binom{n}{2} - m$ edges). Recall by Definition 2 (in Section 3.1) that each of the m non-edges of G corresponds to exactly one pair $\{\langle pq\rangle, \langle ab\rangle\}$ of forbidden orientations of G_1 and G_2, where $pq \in E(G_1)$ and $ab \in E(G_2)$ are two (undirected) optional edges, cf. Definition 3. Denote by p_iq_i and a_ib_i the optional edges that participate in the ith of these pairs of forbidden orientations, such that $p_i \in U_x \cup D_x$, $q_i \in L_x \cup R_x$, $a_i \in U_y \cup D_y$, $b_i \in L_y \cup R_y$. Note that, by this notation for the optional edges p_iq_i and a_ib_i, the corresponding pair of forbidden orientations may be either $\{\langle p_iq_i\rangle, \langle a_ib_i\rangle\}$, or $\{\langle q_ip_i\rangle, \langle a_ib_i\rangle\}$, $\{\langle p_iq_i\rangle, \langle b_ia_i\rangle\}$, or $\{\langle q_ip_i\rangle, \langle b_ia_i\rangle\}$.

Now, for each pair $\{p_iq_i, a_ib_i\}$ of optional edges, where $1 \leq i \leq m$, we define four *ordered pairs* of vertices (cf. Definition 7), as follows: $(x_{2i-1}, y_{2i-1}) = (p_i, q_i)$, $(x_{2i}, y_{2i}) = (q_i, r_{p_i,q_i})$, $(x_{2m+2i-1}, y_{2m+2i-1}) = (a_i, b_i)$, and $(x_{2m+2i}, y_{2m+2i}) = (b_i, r_{a_i,b_i})$. Furthermore we define the disjoint vertex subsets V^*_{out} and V^*_{in} of the clique G^* (cf. Definition 7) as $V^*_{out} = \{x_i : 1 \leq i \leq 2m\}$ and $V^*_{in} = \{x_i : 2m+1 \leq i \leq 4m\}$. That is, V^*_{out} (resp. V^*_{in}) contains exactly the vertices x_i that correspond to endpoints of the optional edges in G_1 (resp. G_2).

Let now $\widetilde{G}^*$ is the graph constructed from G^* by iteratively deactivating the edges x_iy_i, where $1 \leq i \leq 4m$. Denote by M_1, M_2, M_3 the type-1-mandatory, the type-2-mandatory, and the type-3-mandatory orientations of G^*, respectively (cf. Definitions 1, 4, and 6). Since G^* is an induced subgraph of $\widetilde{G}^*$, Definitions 1, 4, and 6 carry over to $\widetilde{G}^*$, i.e. all three types of mandatory orientations are the same in the clique G^* and in the graph $\widetilde{G}^*$.

Moreover we define for every x_i, $1 \leq i \leq 4m$, its special neighborhood $S(x_i) = \{r_{x_j,y_j} : x_j = x_i\}$, cf. Definition 7. Note that $x_j \notin S(x_i)$ for every $i, j \in \{1, 2, \ldots, 4m\}$, cf. Definition 7. Note furthermore that, due to the above definition of the vertex sets V^*_{out}, V^*_{in}, and $S(x_i)$, $1 \leq i \leq 4m$, the forced neighborhood orientation $F(x_i)$ of vertex x_i is exactly the set of oriented type-2-mandatory edges of G^* (and thus also of $\widetilde{G}^*$) that are incident to x_i, cf. Definition 4. That is, $\cup_{i=1}^{4m} F(x_i) = M_2$, where M_2 is the set of all type-2-mandatory orientations in the graph $\widetilde{G}^*$ (cf. Definition 4).

Let $e_i = x_iy_i$, where $1 \leq i \leq 4m$. We iteratively deactivate the edges $e_{4m}, e_{4m-1}, \ldots, e_1$ in $\widetilde{G}^*$ (cf. Section 4), obtaining eventually the graph $\widetilde{G}^*$. In the next two theorems we provide necessary and sufficient conditions (in therms of the graph $\widetilde{G}^*$) such that the input graph G to have a 4-DORG representation with respect to a given vertex partition $\{L, R, U, D\}$.

Theorem 3. *If the graph $G = (V, E)$ has a 4-DORG representation with respect to the vertex partition $\{L, R, U, D\}$, then $\widetilde{G}^*$ has a transitive orientation $\widetilde{T}$ such that $M_1 \cup M_2 \cup M_3 \subseteq \widetilde{T}$.*

Proof. Let G have a 4-DORG representation with respect to the vertex partition $\{L, R, U, D\}$. Then Theorem 1 implies that there exists a transitive tournament (and thus a transitive orientation) λ^* of the clique G^*, such that λ^* satisfies the conditions 2(b)-(d) of Lemma 2 and $M_1 \cup M_2 \cup M_3 \subseteq \lambda^*$.

We will prove that λ^* is a consistent S-orientation of G^* on the sets $S(x_i)$, $1 \leq i \leq 4m$. To this end, we first prove that λ^* is an S-orientation, cf. Definition 8. Recall that $\cup_{i=1}^{4m} F(x_i) = M_2$ by the definition of the forced neighborhood orientation $F(x_i)$ of vertex x_i, $1 \leq i \leq 4m$. Therefore, since $M_2 \subseteq \lambda^*$ by assumption, it follows that the transitive orientation λ^* of G^* satisfies the condition 1 of Definition 8.

Regarding the condition 2 of Definition 8, consider first an ordered pair $(x_{2i-1}, y_{2i-1}) = (p_i, q_i)$, where $1 \leq i \leq m$. Then $S(x_{2i-1}) = \{r_{p_i,q_i}\} \cup \{r_{p_i,q_j} : p_i q_j \text{ is an optional edge in } G_1^*\}$. Therefore, since λ^* satisfies in particular the conditions 2(d)(i) and 2(d)(ii) of Lemma 2, it follows that either $\langle p_i q_i \rangle, \langle r_{p_i,q_i} q_i \rangle, \langle r_{p_i,q_j} q_i \rangle \in \lambda^*$ or $\langle q_i p_i \rangle, \langle q_i r_{p_i,q_i} \rangle, \langle q_i r_{p_i,q_j} \rangle \in \lambda^*$. That is, for every $i \in \{1, 2, \ldots, m\}$ and for every $z \in S(x_{2i-1})$, $\langle x_{2i-1}, y_{2i-1} \rangle = \langle p_i q_i \rangle \in \lambda^*$ if and only if $\langle z y_{2i-1} \rangle \in T$. Consider now an ordered pair $(x_{2i}, y_{2i}) = (q_i, r_{p_i,q_i})$, where $1 \leq i \leq m$. Then $S(x_{2i}) = \{r_{q_i,p_i}\} \cup \{r_{q_i,p_j} : p_j q_i \text{ is an optional edge in } G_1^*\}$. Therefore, since λ^* satisfies in particular the conditions 2(d)(i) and 2(d)(iii) of Lemma 2, it follows that either $\langle r_{p_i,q_i} q_i \rangle, \langle r_{p_i,q_i} r_{q_i,p_i} \rangle, \langle r_{p_i,q_i} r_{q_i,p_j} \rangle \in \lambda^*$ or $\langle q_i r_{p_i,q_i} \rangle, \langle r_{q_i,p_i} r_{p_i,q_i} \rangle, \langle r_{q_i,p_j} r_{p_i,q_i} \rangle \in \lambda^*$. That is, for every $i \in \{1, 2, \ldots, m\}$, $\langle x_{2i}, y_{2i} \rangle = \langle q_i, r_{p_i,q_i} \rangle \in \lambda^*$ if and only if $\langle z y_{2i} \rangle \in T$, for every $z \in S(x_{2i})$.

Summarizing, whenever $1 \leq i \leq 2m$, we have that $\langle x_i y_i \rangle \in \lambda^*$ if and only if $\langle z y_i \rangle \in \lambda^*$, for every $z \in S(x_i)$. By exactly the same arguments (just by considering edges of G_2^* instead of G_1^*) it follows by the conditions 2(d)(i)-(iii) of Lemma 2 that, whenever $2m+1 \leq i \leq 4m$, we have that $\langle x_i y_i \rangle \in \lambda^*$ if and only if $\langle z y_i \rangle \in \lambda^*$, for every $z \in S(x_i)$. Therefore the transitive orientation λ^* of G^* satisfies the condition 2 of Definition 8, and thus λ^* is an S-orientation of G^* on the sets $S(x_i)$, $1 \leq i \leq 4m$.

We now prove that λ^* is also consistent, cf. Definition 9. Recall first that, by the definition of the vertex subsets V_{out}^* and V_{in}^* of the graph $\widetilde{G}^*$, the vertices of $V_{out}^* \cup V_{in}^*$ are also a subset of the vertices of the clique G^*.

Consider first an ordered pair $(x_{2i-1}, y_{2i-1}) = (p_i, q_i)$, where $1 \leq i \leq m$. That is, $x_{2i-1} \in V_{out}^*$. Let $z \in S(x_{2i-1})$. Then, by the definition of $S(x_{2i-1})$, it follows that either $z = r_{p_i,q_i}$ or $z = r_{p_i,q_j}$, where $p_i q_j$ is an optional edge of G_1^* that is incident to $p_i q_i$. Furthermore let $w \notin S(x_{2i-1})$. If $x_{2i-1} w \in M_2$, then $w \in S(x_{2i-1})$ by the definition of $S(x_{2i-1})$, which is a contradiction. Therefore $x_{2i-1} w = p_i w \notin M_2$. Moreover, since G^* is a clique, it follows in particular that z is adjacent to w, as well as that w is adjacent to both x_{2i-1} and y_{2i-1}. We will prove that $\langle wz \rangle \in \lambda^*$ implies that $\langle w x_{2i-1} \rangle = \langle w p_i \rangle \in \lambda^*$. Suppose first that $z = r_{p_i,q_i}$. Then, since λ^* satisfies the condition 2(b) of Lemma 2, it follows that $\langle wz \rangle = \langle w r_{p_i,q_i} \rangle \in \lambda^*$ implies that $\langle w x_{2i-1} \rangle = \langle w p_i \rangle \in \lambda^*$. Suppose now that $z = r_{p_i,q_j}$, where $p_i q_j$ is an incident optional edge of G_1^*. Then the condition 2(b) of Lemma 2 implies that $\langle wz \rangle = \langle w r_{p_i,q_j} \rangle \in \lambda^*$ implies that $\langle w x_{2i-1} \rangle = \langle w p_i \rangle \in \lambda^*$.

Consider now an ordered pair $(x_{2i}, y_{2i}) = (q_i, r_{p_i,q_i})$, where $1 \leq i \leq m$. That is, $x_{2i} \in V^*_{out}$. Let $z \in S(x_{2i})$. Then, by the definition of $S(x_{2i})$, it follows that either $z = r_{q_i,p_i}$ or $z = r_{q_i,p_j}$, where $p_j q_i$ is an optional edge of G^*_1 that is incident to $p_i q_i$. Furthermore let $w \notin S(x_{2i})$. If $x_{2i}w \in M_2$, then $w \in S(x_{2i})$ by the definition of $S(x_{2i})$, which is a contradiction. Therefore $x_{2i}w = q_i w \notin M_2$. Moreover, since G^* is a clique, it follows in particular that z is adjacent to w, as well as that w is adjacent to both x_{2i} and y_{2i}. We will prove that $\langle wz \rangle \in \lambda^*$ implies that $\langle wx_{2i-1} \rangle = \langle wp_i \rangle \in \lambda^*$. Suppose first that $z = r_{q_i,p_i}$. Then, since λ^* satisfies the condition 2(b) of Lemma 2, it follows that $\langle wz \rangle = \langle wr_{q_i,p_i} \rangle \in \lambda^*$ implies that $\langle wx_{2i} \rangle = \langle wq_i \rangle \in \lambda^*$. Suppose now that $z = r_{q_i,p_j}$, where $p_j q_i$ is an incident optional edge of G^*_1. Then the condition 2(b) of Lemma 2 implies that $\langle wz \rangle = \left\langle wr_{q_i,p_j} \right\rangle \in \lambda^*$ implies that $\langle wx_{2i} \rangle = \langle wq_i \rangle \in \lambda^*$.

Summarizing, since $x_i \in V^*_{out}$ if and only if $1 \leq i \leq 2m$, we proved that the transitive ordering λ^* satisfies the first condition of Definition 9. By considering the vertices $x_i \in V^*_{in}$ (i.e., where $2m+1 \leq i \leq 4m$), it follows by exactly the symmetric arguments that λ^* satisfies also the second condition of Definition 9, and thus λ^* is a consistent S-orientation on the sets $S(x_i)$, $1 \leq i \leq 4m$.

Recall that $\widetilde{G}^*$ is the graph constructed from G^* by iteratively deactivating the edges $x_i y_i$, where $1 \leq i \leq 4m$ (cf. Section 4). Now, since λ^* is a consistent S-orientation on the sets $S(x_i)$, $1 \leq i \leq 4m$, such that $M_1 \cup M_2 \cup M_3 \subseteq \lambda^*$, the first part of Theorem 2 implies that $\widetilde{G}^*$ has a transitive orientation $\widetilde{T}$ such that $M_1 \cup M_2 \cup M_3 \cup F(x_i) \subseteq \widetilde{T}$ for every $i = 1, 2, \ldots, 4m$. Therefore, since $\cup_{i=1}^{4m} F(x_i) = M_2$, it follows that $M_1 \cup M_2 \cup M_3 \subseteq \widetilde{T}$. This completes the proof of the theorem. ☐

Theorem 4. *If $\widetilde{G}^*$ has a transitive orientation $\widetilde{T}$ such that $M_1 \cup M_2 \cup M_3 \subseteq \widetilde{T}$, then G has a 4-DORG representation with respect to the vertex partition $\{L, R, U, D\}$.*

Proof. Let $\widetilde{G}^*$ have a transitive orientation $\widetilde{T}$ such that $M_1 \cup M_2 \cup M_3 \subseteq \widetilde{T}$. Then also $\cup_{i=1}^{4m} F(x_i) \subseteq \widetilde{T}$, since $\cup_{i=1}^{4m} F(x_i) = M_2$. Therefore the second part of Theorem 2 implies that the graph G^* (i.e. the graph before deactivating the edges $x_i y_i$, $1 \leq i \leq 4m$) has an S-orientation T on the sets $S(x_i)$, $1 \leq i \leq 4m$, such that $M_1 \cup M_2 \cup M_3 \subseteq T$. Since T is a transitive orientation of G^* and G^* is a clique, T is a transitive tournament of G^*.

We will prove that T satisfies the condition 2(d) of Lemma 2. Consider first an optional edge $p_i q_i$ of G^*_1, $1 \leq i \leq m$, where $p_i \in U_x \cup D_x$ and $q_i \in L_x \cup R_x$. Then, due to the definition of the ordered vertex pairs $(x_1, y_1), \ldots, (x_{4m}, y_{4m})$, it follows that $x_{2i-1} = p_i$ and $y_{2i-1} = q_i$. Furthermore $S(x_{2i-1}) = \{r_{p_i,q_i}\} \cup \{r_{p_i,q_j} : p_i q_j \text{ is an optional edge in } G^*_1\}$. Since T is an S-orientation on the sets $S(x_i)$, $1 \leq i \leq 4m$, it follows by the condition 2 of Definition 8 that for every $z \in S(x_{2i-1})$ we have $\langle x_{2i-1} y_{2i-1} \rangle = \langle p_i q_i \rangle \in T$ if and only if $\langle z y_{2i-1} \rangle = \langle z q_i \rangle \in T$. That is, for every optional edge $p_i q_j$ of G^*_1 that is incident to $p_i q_i$, either $\langle p_i q_i \rangle, \langle r_{p_i,q_i} q_i \rangle, \left\langle r_{p_i,q_j} q_i \right\rangle \in T$ or $\langle q_i p_i \rangle, \langle q_i r_{p_i,q_i} \rangle, \left\langle q_i r_{p_i,q_j} \right\rangle \in T$ (cf. the conditions 2(d)(i)-(ii) of Lemma 2).

Furthermore, due to the definition of the ordered vertex pairs $(x_1, y_1), \ldots, (x_{4m}, y_{4m})$, it follows that $x_{2i} = q_i$ and $y_{2i} = r_{p_i,q_i}$. Moreover $S(x_{2i}) = \{r_{q_i,p_i}\} \cup \{r_{q_i,p_j} : p_j q_i$ is an optional edge in $G_1^*\}$. Since T is an S-orientation on the sets $S(x_i)$, $1 \leq i \leq 4m$, the condition 2 of Definition 8 implies that for every $z \in S(x_{2i})$ we have $\langle x_{2i} y_{2i} \rangle = \langle q_i r_{p_i,q_i} \rangle \in T$ if and only if $\langle z y_{2i} \rangle = \langle z r_{p_i,q_i} \rangle \in T$. That is, for every optional edge $p_j q_i$ of G_1^* that is incident to $p_i q_i$, either $\langle r_{p_i,q_i} q_i \rangle, \langle r_{p_i,q_i} r_{q_i,p_i} \rangle, \langle r_{p_i,q_i} r_{q_i,p_j} \rangle \in T$ or $\langle q_i r_{p_i,q_i} \rangle, \langle r_{q_i,p_i} r_{p_i,q_i} \rangle, \langle r_{q_i,p_j} r_{p_i,q_i} \rangle \in T$ (cf. the conditions 2(d)(i) and 2(d)(iii) of Lemma 2).

Summarizing, we proved that the transitive tournament T of G^* satisfies the condition 2(d) of Lemma 2 for all optional edges $p_i q_i$ of G_1^*, where $1 \leq i \leq m$. By considering the optional edges $a_i b_i$ of G_2^*, where $1 \leq i \leq m$, it follows by exactly the same arguments that T satisfies the condition 2(d) of Lemma 2 also for the optional edges of G_2^*. Therefore, since also $M_1 \cup M_2 \cup M_3 \subseteq T$, it follows by Corollary 2 (cf. also the statements of Theorem 1 and Lemma 2) that G has a 4-DORG representation with respect to the vertex partition $\{L, R, U, D\}$. □

We are now ready to present our 4-DORG recognition algorithm (Algorithm 1). The proof of correctness and the timing analysis are given in the next theorem.

Theorem 5. *Let $G = (V, E)$ be a graph with n vertices, given along with a vertex partition $V = L \cup R \cup U \cup D$, such that $E \subseteq (L \cup R) \times (U \cup D)$. Then Algorithm 1 constructs in $O(\mathrm{MM}(n^2))$ time a 4-DORG representation for G with respect to this vertex partition, or correctly announces that G does not have a 4-DORG representation.*

Proof. Let $\widetilde{G}^*$ be the graph that is constructed from G^* by iteratively deactivating the edges $x_i y_i$, $1 \leq i \leq 4m$, cf. line 12 of Algorithm 1. Then it follows directly by Theorems 3 and 4 that $\widetilde{G}^*$ has a transitive orientation $\widetilde{T}$ such that $M_1 \cup M_2 \cup M_3 \subseteq \widetilde{T}$ if and only if G has a 4-DORG representation with respect to the vertex partition $\{L, R, U, D\}$. Therefore, if $\widetilde{G}^*$ does not have such a transitive orientation $\widetilde{T}$, Algorithm 1 correctly concludes in line 14 that G does not have a 4-DORG representation with respect to the vertex partition $\{L, R, U, D\}$. Otherwise, if $\widetilde{G}^*$ has such a transitive orientation $\widetilde{T}$, Algorithm 1 computes and returns a 4-DORG representation of G with respect to this vertex partition, using the $O(n^2)$ algorithm of Theorem 1.

The running time of Algorithm 1 is therefore dominated by the time needed to compute a transitive orientation $\widetilde{T}$ of the graph $\widetilde{G}^*$, given a partial orientation $M_1 \cup M_2 \cup M_3$ of $\widetilde{T}$. This can be done using one of the known transitive orientation algorithm; the fastest known algorithm requires matrix multiplication (currently achieved in $O(\mathrm{MM}) = O(n^{2.376})$ time for multiplying two $n \times n$ matrices [3]). Therefore, since the size of the graph $\widetilde{G}^*$ is $O(n^2)$, the transitive orientation $\widetilde{T}$ can be computed (if one exists) in time $O(\mathrm{MM}(n^2))$ (i.e. in the time needed for multiplying two $n^2 \times n^2$ matrices). □

Algorithm 1 Recognition of 4-DORGs

Input: An undirected graph $G = (V, E)$ with a vertex partition $V = L \cup R \cup U \cup D$

Output: A 4-DORG representation for G, or the announcement that G is not a 4-DORG graph

$n \leftarrow |V|$; $\quad m \leftarrow \binom{n}{2} - |E|$ {m is the number of non-edges in G}
Construct from G the clique G_1 with vertex set $L_x \cup R_x \cup U_x \cup D_x$ and the clique G_2 with vertex set $L_y \cup R_y \cup U_y \cup D_y$
Construct the set M_1 of type-1-mandatory orientations in G_1 and G_2
Construct the m forbidden pairs of orientations of G_1 and G_2
Construct from G_1, G_2 the augmented cliques G_1^*, G_2^* and the set M_2 of type-2-mandatory orientations
Construct from G_1^*, G_2^* the clique G^* and the set M_3 of type-3-mandatory orientations
for $i = 1$ to m **do**
  Let $p_i q_i \in E(G_1), a_i b_i \in E(G_2)$ be the optional edges in the ith pair of forbidden orientations, where $p_i \in U_x \cup D_x$, $q_i \in L_x \cup R_x$, $a_i \in U_y \cup D_y$, $b_i \in L_y \cup R_y$
  $(x_{2i-1}, y_{2i-1}) \leftarrow (p_i, q_i)$; $\quad (x_{2i}, y_{2i}) \leftarrow (q_i, r_{p_i,q_i})$
  $(x_{2m+2i-1}, y_{2m+2i-1}) \leftarrow (a_i, b_i)$; $\quad (x_{2m+2i}, y_{2m+2i}) \leftarrow (b_i, r_{a_i,b_i})$
  $S(x_i) \leftarrow \{r_{x_j,y_i} : x_j = x_i\}$
Construct the graph $\widetilde{G}^*$ by iteratively deactivating all edges $x_i y_i$, $1 \leq i \leq 4m$
if $\widetilde{G}^*$ has a transitive orientation $\widetilde{T}$ such that $M_1 \cup M_2 \cup M_3 \subseteq \widetilde{T}$ **then**
  return the 4-DORG representation of G computed by Theorem 1
else
  return "G is not a 4-DORG graph with respect to the partition $\{L, R, U, D\}$"

6 Recognizing three directional orthogonal ray graphs with partial representation restrictions

In this section we consider bipartite graphs $G(V, E)$, with vertex partitions $V = A \cup B$, having m, respectively n elements and an ordering $(v_1, v_2, \ldots, v_m)$ of the vertices of A.

The question we address is the following: Does G admit a representation as a three directional orthogonal ray graph, with A corresponding to rays oriented upwards, and B to rays oriented to the left or to the right, such that for any $v_i, v_j \in A$, the y-coordinate of v_i is larger than the y-coordinate of v_j if and only if $i < j$?

Our approach consists of using the adjacency relations in the graph, to recursively derive necessary conditions for the left to right ordering of the rays corresponding to the vertices in A. If at no point during the process a contradiction is reached, we will construct a 3-DORG representation, otherwise none exists.

Definition 11. *$A = A_1 \cup A_2 \cup \ldots \cup A_k$ be a partition of A. Then, $(A_1, A_2, \ldots A_k)$ is said to be a* consistent partition *for G, if the following holds: if G is a three directional orthogonal ray graph, then it admits a 3DORG representation such* *that for $i < j$, all the rays of the vertices in A_i are represented to the left of the rays of the vertices in A_j.*

Definition 12. *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$, $\mathcal{B} = \{B_1, B_2, \ldots, B_l\}$ be two partitions of A. If $\mathcal{B}$ is a refinement of $\mathcal{A}$, $\mathcal{B}$ is called* elementary *if and only if there exists no partition strictly contained between $\mathcal{A}$ and $\mathcal{B}$ with respect to refinement.*

Let n_1 denote the number of neighbors of v_1, and let $(u_1, u_2, \ldots u_{n_1})$ be any ordering of the elements of $N(v_1) \subseteq B$. Supposing G admits a three directional orthogonal ray representation. By symmetry, it can be assumed without that u_1 is oriented to the left.

Lemma 5. *If G has a 3DORG representation respecting the y-coordinate ordering $(v_1, v_2, \ldots, v_m)$ of A, then $(N(u_1), A \setminus N(u_1))$ is a consistent partition of A.*

Proof. Assume G is a three directional orthogonal ray graph and that there exist $v_i \in N(u_1), v_j \in A \setminus N(u_1)$ such that v_j lies to the left of v_i. Since it is a neighbor of v_1, the ray of u_1 is higher than the endpoint of v_j, and starts to the right of v_j due to adjacency to v_i, therefore it must cross the ray of v_j, contradiction.

For all neighbors $(u_1, u_2, \ldots u_{n_1})$ of v_1 we construct the following chain of ordered partition refinements $P_1, P_2, \ldots P_{n_1}$:

1) $P_1 = (N(u_1), A \setminus N(u_1))$.

2) For all $i \geq 1$, P_{i+1}, if it exists, is the consistent partition resulting from the intersection of P_i with either $(N(u_i), A \setminus N(u_i))$ or $(A \setminus N(u_i), N(u_i))$.

The following holds:

Proposition 1. *If G has a 3DORG representation that respects the y-coordinate ordering $(v_1, v_2, \ldots, v_m)$ of A and in which u_1 is oriented to the left, then $(P_1, P_2, \ldots P_{n_1})$ exists and is a consistent chain of elementary refinements.*

Proof. Consider a 3DORG representation of G as in the hypothesis. Then, by Lemma 5, P_1 is a consistent partition of G.

Similarly as in Lemma 5, it can be shown that at least one of $(N(u_1), A \setminus N(u_{i+1}))$ and $(A \setminus N(u_{i+1}), N(u_{i+1}))$ is a consistent partition for G (depending on whether u_{i+1} goes towards left or right). Let $P_i = (A_1, A_2, \ldots A_{k_i})$. By assumption, P_i is a consistent partition of G, i.e. if $j_1 < j_2$, all elements of A_{j_1} appear to the left of all elements of A_{j_2} for any such representation of G. Together with the previous remark, this implies there exists an $1 \leq l_i \leq k_i$ such that either $N(u_{i+1}) \subseteq A_1 \cup \ldots \cup A_{l_i}$ or $N(u_{i+1}) \subseteq A_{l_i} \cup \ldots \cup A_{k_i}$. Therefore, at most the set A_{l_i} is split in the refinement P_{i+1}, hence P_{i+1} is elementary. Finally, since P_{i+1} is the intersection of two consistent (ordered) partitions, it follows that $(P_i)_{1 \leq i \leq n_1}$ is a consistent chain of elementary refinements.

After processing all neighbors of v_1, we eliminate v_1 and its neighbors, obtaining an induced subgraph with a 3-DORG representation where v_2 is the highest vertical ray. We process the neighbors of v_2 in a similar fashion as in

Proposition 1, the starting partition being the last one obtained after processing v_1 (excluding v_1 itself). Repeating the procedure for the remaining vertices of A, we create a sequence of partitions which, if at no step a contradiction is reached with respect to their properties, will permit the construction of a 3-DORG representation with the desired constraints.

Formally, we construct chain $R_1, R_2, \ldots, R_m$ of partitions as follows:

1) $G_1 = G$ and $R_1 = P_{n_1}$, as in Proposition 1.

2) For $1 \leq i < m$, let G_{i+1} be defined as $G[V(G_i \setminus N[v_i]]$ (i.e., the induced subgraph on all vertices of G_i but $\{v_i, N(v_i)\}$). Starting with $R_i \setminus \{v_i\}$ as initial partition, one successively performs refinings as in Proposition 1 for each $u \in N(v_{i+1})$, with R_{i+1} being the final partition after all neighbors of v_{i+1} have been processed. If G has no 3-DORG representation, then at some step, for some vertex u, the refinement of the currently processed partition P is not elementary.

Theorem 6. *Let $G(V, E)$ be a bipartite graph , with vertex partitions $V = A \cup B$ and $(v_1, v_2, \ldots, v_m)$ an ordering of the vertices of A. Then, A admits a three directional orthogonal ray representation such that the ray of v_i is higher than that of v_j for any $i < j$ if and only if the above procedure yields a chain of elementary refinements for all $1 \leq i \leq m$.*

Proof. "$\Rightarrow$": Assume G has a three directional orthogonal ray representation. This induces a linear ordering R of the vertical rays by their x-coordinate. Applying Proposition 1 to $G_i, 1 \leq i \leq m$, it follows that R restricted to $V(G_i)$ is a refinement of any partition corresponding to a neighbor of v_i, therefore the sets R_i are well defined. Furthermore, since the chain of refinements is elementary, the conclusion follows.

"$\Leftarrow$" : We produce a 3DORG representation based on the chain of partitions $R_1, R_2, \ldots R_m$. We first assign to each $v_i \in A$ the y-coordinate $m - i$. Let now $A_1, A_2, \ldots A_{n_1}$ be the sets in the partition R_1. There must exist a $1 \leq k \leq n_1$ such that $v_1 \in A_k$. The vertex v_1 is now assigned the final x coordinate $|A_1| + |A_2| + \ldots + |A_k|$. For v_i, with $i > 1$, the x-coordinate is computed analogously with respect to R_i. However, to avoid collisions with rays of vertices $v_j, j < i$ we add 1 for each $v_j, j < i$ whose set in R_j was the left of the set of v_i in R_j. The y-coordinates of the horizontal rays are assigned as follows: for a vertex v_k in A, consider all neighbors for which v_k is their highest adjacency. The rays corresponding to these neighbors are now assigned distinct y-values between $m - k$ and $m - k + 1$. For the x-coordinates, consider a vertex $u \in B$ and let v_k be its highest neighbor. Then, $N(u)$ contains either the leftmost, or the rightmost set in R_k (since u was processed in the construction of R_k). In the first case, we orient u to the left, and its x-coordinate is assigned to that of the rightmost of its neighbors in G, after adding 0.5. If $N(u)$ contains the rightmost set of R_k, u will be oriented to the right, and its x-coordinate is assigned to that of its leftmost neighbor in G, after subtracting 0.5. This construction is highlighted in the example in Figure 6 below.

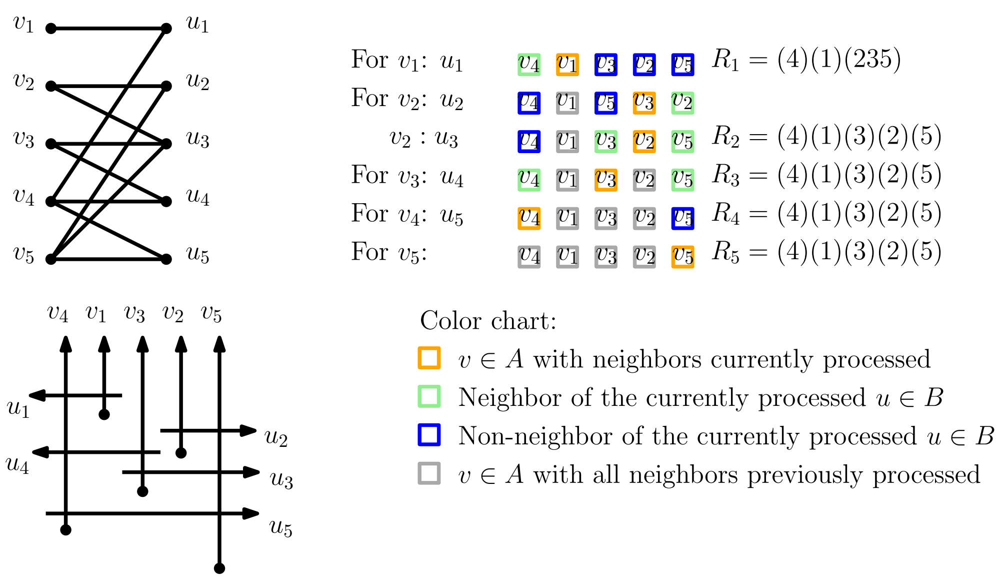


Fig. 7. Constructing a 3-DORG representation: Top left - the bipartite graph G with the given vertex order. Top-right - the consistent chain of partitions. Bottom right - The 3-DORG representation of G as read from the partition chain

The method used to prove the theorem can be viewed as a particular *partition refinement technique*. Such techniques have various applications in string sorting, automaton minimization, and graph algorithms. An overview is given in [8].

We now show that Theorem 6 yields an efficient algorithm for deciding whether G is a 3-DORG with the given vertex ordering.

Theorem 7. *It can be decided in $O(|V|^2)$ operations whether a bipartite graph G admits a three directional orthogonal ray representation with a given horizontal and vertical partition and a prescribed ordering of the y-coordinates of the vertical rays.*

Proof. According to Theorem 6 it is enough to construct the chain $R_1, R_2, \dots R_m$ of partitions. Note that each step of partition refinement corresponds to processing a vertex $u \in B$, for some G_k, $1 \leq k \leq m$. If all steps can be carried out consistently, we obtain a three directional orthogonal ray representation, otherwise none exists. The refinement for u is elementary if and only if at most one set in the current partition contains both neighbors and non-neighbors of u. Furthermore, no set containing a neighbor of u may lie between two sets containing non-neighbors of u and viceversa. Checking for both these properties requires a single sweep through the current partition, that is, $O(m)$ steps. There are n vertices in B to be processed, yielding a total number of $O(nm) \leq O(|V|^2)$ operations.

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