Constructing Colorings for Diagrams *

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Introduction and Overview

An undirected graph $G = (V, E)$ is a (Hasse–) diagram if there is a poset $P = (V, <)$ and an orientation $\vec{E}$ of E such that $(x, y) \in \vec{E}$ iff $x < y$ in P and there is no z with $x < z < y$. We then write $G = D_P$. A pair $(x, y) \in \vec{E}$ is called a covering and denoted by $x \prec y$.

The chromatic number $\chi(G)$ is the least number of colors needed to color the vertices of G such that no two adjacent vertices obtain the same color. The interest in the chromatic number of diagrams is motivated by a lack of small explicit examples of triangle-free graphs with chromatic number > 3. In [NR87] Nešetřil and Rödl showed that calculating the chromatic number is NP-hard for diagrams. In the first part we give bounds for the chromatic number of an arbitrary diagram.

In the second part we investigate the chromatic number of diagrams under the restriction that P belongs to a special class of orders. The first results are: $\chi(D_P) \leq 3$ for semi-orders and series-parallel orders.

It is know that lattices (Bollobás [Bo77]) and 2-dimensional posets (Kříž and Nešetřil [KN91]) can have diagrams of arbitrarily large chromatic number. We construct a family I_k of orders with $\chi(D_{I_k}) = k$. An order I_k in this family is both, an interval orders and N-free. This leads us to the investigation of these two classes. In the case of N-free orders we can show that $\lceil \log_2(height(P) + 1) \rceil$ is an upper and lower bound for the chromatic number. For interval orders we provide an upper bound of $\left\lceil \frac{3}{\log_2 3} \log_2(height(P)) \right\rceil$.

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For all the constructions related to upper bounds there exist fast algorithms which give colorings in that bound. For the algorithms we will always assume that G is given together with a diagram orientation of a corresponding poset P. This assumption can not be removed since the decision problem whether a given graph is a diagram or not is NP-complete. This is another result of [NR87].

1 General Bounds

Some simple bounds

For an order $P = (V, <)$ the height of a vertex $v \in V$ is defined by

$$\mathit{height}\,(\,v\,) = \begin{cases} 1 & \text{iff } v \text{ is minimal} \\ \max\limits_{w<v} \mathit{height}\,(\,w\,) + 1 & \text{otherwise.} \end{cases} \tag{1}$$

The $\mathit{height}\,(\,P\,)$ is the maximal height of a vertex. The maximal difference of the height of two covering points is

$$\mathit{\Delta}\,(\,P\,) = \max_{v \prec w} \mathit{height}\,(\,w\,) - \mathit{height}\,(\,v\,) \tag{2}$$

Remark 1.1 $\chi\,(\,D_P\,) \leq \Delta\,(\,P\,) + 1$ *and such a coloring can be given in linear time.*

Proof: Assign the color $[\mathit{height}\,(\,v\,) \bmod (\mathit{\Delta}\,(\,P\,) + 1)]$ to a vertex v. By definition of $\mathit{\Delta}\,(\,P\,)$ two points v, w with $w \prec v$ never obtain the same color. □

Since $\mathit{\Delta}\,(\,P\,) < \mathit{height}\,(\,P\,)$ we also obtain $\chi\,(\,D_P\,) \leq \mathit{height}\,(\,P\,)$.

Let $L = v_1, \ldots, v_n$ be a list of the elements of V. The *list coloring* of $G = (V, E)$ with respect to L is obtained by the following rule.

for $v = v_1$ **to** v_n **do**
Color v_i with the first color which is not used for a v_j with $j < i$ and $\{v_j, v_i\} \in E$.

The list L induces an orientation $\vec{E}$ on E, which may be used to bound the number of colors needed in the list coloring by $(\max\limits_{v \in V} \mathit{indegree}_{\vec{E}}(v)) + 1$.

Let L be a linear extension of a poset P. Apply list coloring with respect to L to the diagram of P and note that $\mathit{indegree}\,(\,v\,) \leq \mathit{width}\,(\,P\,)$. Hence,

Remark 1.2 $\chi\,(\,D_P\,) \leq \mathit{width}(P) + 1$ *and such a coloring can be given in linear time.*

Combining Height and Width

For an order $P = (V, <)$ consider a partition of V into V^h and V^w. Since we may color both induced suborders independently it is then clear that,

$$\chi(D_P) \leq height\left(P\big|_{V^h}\right) + width\left(P\big|_{V^w}\right) + 1. \tag{3}$$

To make use of this observation we need a partition of V such that $height\left(P\big|_{V^h}\right) + width\left(P\big|_{V^w}\right)$ is small. This is the aim of the following theorem.

Theorem 1.3 *Let $P = (V, <)$ then there is a partition $V_1, \ldots, V_k$ of V such that each V_i is either a chain or an antichain and*

$$k < \left\lceil \sqrt{2|V|} \right\rceil \tag{4}$$

The following proof is due to U. Faigle.

Proof: We proceed by induction on $n = |P|$. We may assume that $height(P)$ and $width(P)$ are both greater than $\sqrt{2n}$ since otherwise we could easily cover P with $height(P)$ antichains or $width(P)$ chains.

So there is a chain C and an antichain A such that their union U is of size greater than $2 \cdot \sqrt{2n} - 1$.

By induction we may partition $V \setminus U$ into at most $\sqrt{2 \cdot (n - 2 \cdot \sqrt{2n} + 1)}$ chains and antichains. This value is always less than $\sqrt{2n} - 2$ and so we are done. $\square$

Observe that the bound given in Theorem 1.3 is sharp. A family of examples are the orders given by k parallel chains of size $1, \ldots, k$.

A decomposition theorem like ours has independently been obtained by Caro [Ca90] in the more general context of perfect graphs. Indeed the proof given here can also easily be adapted to perfect graphs.

Corollary 1.4 *If a poset P has n elements then $\chi(D_P) \leq \left\lceil \sqrt{2n} \right\rceil$ and a coloring using at most this number of colors can be computed in $O(n^3)$.*

Proof: Let V^w be the union of the chains and V^h the union of the antichains we obtain from Theorem 1.3. Now, the inequality is an immediate consequence of Formula (3).

The complexity of the corresponding coloration algorithm is dominated by the complexity of the computation of an appropriate partition. According to our proof above we have to cover the order with $height(P)$ antichains or $width(P)$ chains or we have to find maximum chains and antichains iteratively.

The later can happen at most $O(\sqrt{n})$ times. In each iteration we have to find a maximum chain and antichain. A maximum chain together with a partition into *height* (P) antichains can be found by labeling and scanning in time $O(n^2)$. A maximum antichain together with a partition into *width* (P) chains can be found by using matching in bipartite graphs in time $O(n^{2.5})$. So the running time is bounded by $O(n^3)$. □

A Bound that is Asymptotically Better

The bound on χ given by the previous construction in Corollary 1.4 is asymptotically improved by the following theorem and proof that are due to Tuza, [Tu91].

Theorem 1.5 *Triangle free graphs, and hence diagrams, admit colorings with $O(\frac{\sqrt{n}}{\log\log n})$ colors.*

Proof: First we combine two bounds on the size of independent sets in a triangle free graph. According to Ajtai et al. [AKS80], see also [Gr83], a triangle free graph (e.g. a diagram) of average degree d contains an independent set of size $\Omega(\frac{n \ln d}{d})$. On the other hand it contains an independent set of size d, namely the neighborhood of some vertex. A simple calculation using the threshold $d = \sqrt{n}\frac{\log n}{\log\log n}$ warrants an independent set of size $\Omega(\sqrt{n}\log\log n)$. This large independent set can be found efficiently.

Now to show the overall bound, take a large independent set, assign a new color to it and iterate with the points that are not yet colored. After $O(\frac{\sqrt{n}}{\log\log n})$ steps all vertices are colored. □

2 Bounds for Special Classes of Orders

In this part we investigate the chromatic number problem on diagrams corresponding to special classes of partial orders (we refer to [Mö89] for undefined terms). The hope was that – as with other NP-hard problems – the additional structural properties of these classes lead to better bounds or even to polynomial algorithms. It turned out that this is indeed true.

Some Classes with Bounded Chromatic Number

With $1 \oplus k$ we denote the disjoint union of a k-element chain with a single point.

Remark 2.1 *If D_P has no $1 \oplus k$ as induced subdiagram then $\Delta(P) < k$, and hence – with Remark 1.1 – we have $\chi(D_P) \leq k$.*

Semi-orders are those interval orders which do not have $1 \oplus 3$ as induced suborder, therefore:

Remark 2.2 *If P is a semi-order then $\chi(D_P) \leq 3$ and the chromatic number can be calculated in linear time.*

To get linear time we need to know if $\chi(D_P) = 1$ or $\chi(D_P) = 2$, checking this is straightforward.

Theorem 2.3 *If P is series-parallel then $\chi(D_P) \leq 3$ and $\chi(D_P)$ can be calculated in linear time.*

This is a direct consequence of

Lemma 2.4 *If P is series-parallel then there is a 3-coloring of D_P with colors a, b and c such that all minimal elements of P are colored a and all maximal elements are colored either a or b.*

Proof: We proceed by induction on the decomposition.

If P is a parallel composition the statement is trivial.

So let $P = P_1 \times P_2$ be a series composition with P_1 as lower part. By induction they have colorings as desired (see Fig. 1).

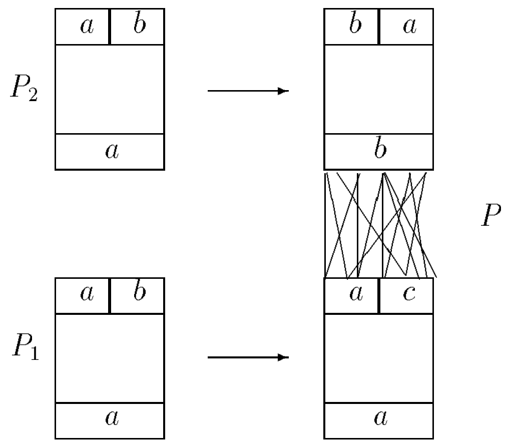

Figure 1: A Coloring for a series-composed order

Apply the transposition $\pi_1 = (b, c)$ to the coloring of P_1 and the transposition $\pi_2 = (a, b)$ to the coloring of P_2. After that the colorings of P_1 and P_2 fit together to a coloring of P, since the minimal elements of P_2 are now all colored with b and the maximal elements of P_1 are only colored with colors in $\{a, c\}$.

This coloring obviously has the desired properties. $\square$

Diagrams with High Chromatic Number

We give an explicit construction for diagrams with arbitrary chromatic number.

For nonnegative integers k let I_k be the interval order defined by the open intervals with endpoints in $\left\{1, \ldots, 2^k\right\}$. It has $height\,(\, I_k\,) = 2^k - 1$.

Two vertices v and w in I_k fulfill $v \prec w$ iff the right endpoint of the interval of v equals the left endpoint of w. The diagram of I_k has been studied under the name 'shift-graph'. We include the well-known proof of the next lemma (see e.g. [HE72]), since we will need similar methods when discussing the case of general interval orders.

Lemma 2.5

$$\chi\,(\, D_{I_k}\,) = k$$

Proof: Suppose we have a proper coloring of D_{I_k} with colors $\{1, \ldots, c\}$. With each point i associate the set $C(i)$ of colors used for the intervals having their right endpoint at i. Note that for $1 \leq i < j \leq 2^k$ we have $C(j) \not\subseteq C(i)$. Therefore all the 2^k subsets $C(i)$ of $\{1, \ldots, c\}$ are distinct. Hence $2^c \geq 2^k$ and $c \geq k$.

A coloring of D_{I_k} using k colors can recursively be obtained by the following construction. The intervals with both endpoints in $\left\{1, \ldots, 2^{k-1}\right\}$ are an I_{k-1}. A second I_{k-1} consists of those intervals with endpoints in $\left\{2^{k-1}+1, \ldots, 2^k\right\}$. Color the vertices of each copy with the same set of $k-1$ colors. A new color is used for the remaining vertices of I_k, i.e. for the antichain of intervals containing $2^{k-1} + \frac{1}{2}$. Since intervals of the two copies of I_{k-1} are separated by the interval $(2^{k-1}, 2^{k-1}+1)$ we have thus constructed a proper coloring. $\square$

Theorem 2.6 $\chi\,(\, D_P\,)$ *can be arbitrarily large for N-free orders and interval orders.*

Proof: I_k is an interval order. We have to show that it is also N-free.

Let u, v, w, x be an N, say $u \prec v, w \prec v, w \prec x$. Then we know that v and x have the same left endpoint and u and w have the same right endpoint. But then $u \prec x$, too. This contradicts the assumption that u, v, w, x induce an N. $\square$

N-free Orders

Remark 2.7 *For every $k \in I\!N$ there is an N-free order P such that*

$$\lceil \log_2(\text{height}\,(\,P\,) + 1)\,\rceil \leq \chi\,(\,D_P\,) = k\ . \tag{5}$$

The remark is a direct consequence of the previous theorem. We now give an upper bound for the chromatic number of diagrams of N-free orders which is logarithmic in *height* (P) too. The family $\{I_k\}$ of Lemma 2.5 is shown to consist of extremal N-free orders with respect to the chromatic number of the diagram. The technique used relies on the fact, that an N-free order can always be subdivided by an antichain.

Theorem 2.8 *If P is N-free then $\chi\,(\,D_P\,) \leq \lceil \log_2(\text{height}\,(\,P\,) + 1)\,\rceil$ and such a coloring can be calculated in linear time.*

Proof: The proof is by induction on *height* (P).

The elements v of P with $\textit{height}\,(\,v\,) = \left\lceil \frac{1}{2}\textit{height}\,(\,P\,)\right\rceil = k$ form an antichain A. It need not be maximal, but elements which are parallel to all the elements in A have smaller height. Let

$$A' = \left\{ v \in V \,\middle|\, \textit{height}\,(\,v\,) < k \text{ and } \forall w \succ v\ ,\ \textit{height}\,(\,w\,) > k \right\}. \tag{6}$$

Note that $A^* = A \cup A'$ is a maximal antichain. Now partition $V \setminus A^*$ into two sets

$$V^- = \left\{ v \,\middle|\, v < a \text{ for some } a \in A^* \right\} \qquad \text{and} \qquad V^+ = \left\{ v \,\middle|\, a < v \text{ for some } a \in A^* \right\} \tag{7}$$

It is then clear that the suborders induced by V^+ and V^-, P^- and P^+ say, both have height less than k. The N-freeness of P warrants that no element of P^- is covered by an element of P^+. So we may color both suborders independently and assign a new color to A^*. This gives a coloring as desired.

The color of an element only depends on its height and the height of its immediate successors. With $l(v) = \min\limits_{v \prec w}\{\textit{height}\,(\,w\,) - 1\}$, the color of v can be found by some bit operations on *height* (v) and $l(v)$. $\Box$

Interval Orders

A similar idea as for N-free orders can also be used for interval orders. We could e.g. divide an interval order into two parts, but then we would need two antichains to separate them in general. To get the constant small (≈ 1.8) we use a partition into three parts and get the following.

Theorem 2.9 *If P is an interval order then*

$$\chi(D_P) \leq \left\lceil \frac{3}{\log_2 3} \log_2(\text{height}(P)) \right\rceil \tag{8}$$

and such a coloring can be determined in linear time.

Proof: We proceed by induction on the height of P. We may assume that *height* $(P) \geq 3$. The following arguments are illustrated in Fig. 2.

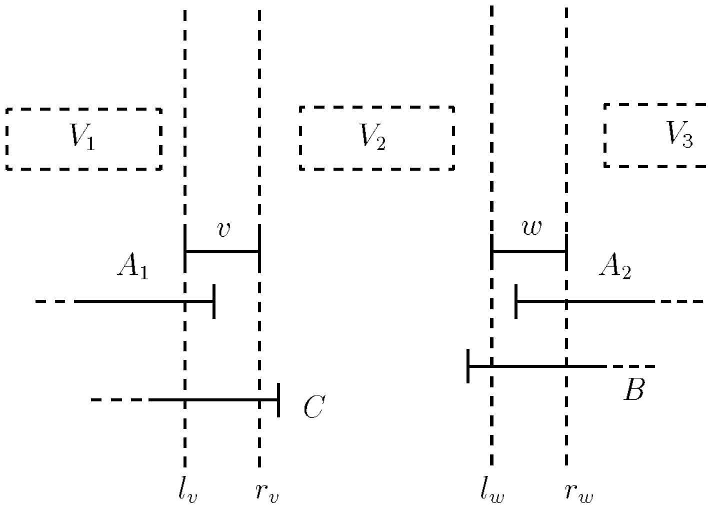

Figure 2: Splitting an interval order

Take an interval representation of P. For $v \in V$ let l_v and r_v be the left respectively right endpoint of the interval in this representation. We say that v dominates w if the interval of w is contained in the interval of v. Note that there is always a maximum chain of P such that no element of that chain dominates any other element.

In such a chain we choose elements v and w at height $\frac{1}{3}$ and $\frac{2}{3}$. Define seven sets of elements

$$\begin{aligned} V_1 &= \left\{x \,\middle|\, r_x < l_v\right\}, & A_1 &= \left\{x \,\middle|\, l_v \leq r_x \leq r_v\right\}, \\ V_2 &= \left\{x \,\middle|\, r_v < l_x < r_x < l_w\right\}, & A_2 &= \left\{x \,\middle|\, l_w \leq l_x \leq r_w\right\}, \\ V_3 &= \left\{x \,\middle|\, r_w < l_x\right\}, & B &= \left\{x \,\middle|\, l_x < l_w \leq r_x\right\}, \\ & & C &= \left\{x \,\middle|\, l_x \leq r_v < r_x < l_w\right\}. \end{aligned}$$

No cover relations holds between elements of different V_i , $i = 1, 2, 3$, since the interval of either v or w lies between them. So they may be colored independently by induction.

Each of A_1, A_2, B and C is an antichain. Use three new colors to color them. One for B, one for C and one for A_1 and A_2 together, this can be done as long as V_2 is not empty.

The total amount of colors needed then is 3 times the maximal recursion depth, which is $\log_3 height(P) = \frac{\log_2 height(P)}{\log_2 3}$.

To obtain the linear time complexity observe that the color of an element is determined by the first recursion level in which it falls into one of the sets A_1, A_2, B, C. But this can be calculated iteratively if we

1. determine a non-dominating maximum chain in scanning P from bottom to top

2. determine for each element the lowest and highest element of that chain which is parallel

$\square$

Recently, Felsner and Trotter [FT91] have obtained a much stronger result concerning the chromatic number of the diagram of interval orders: If P is an interval order then $\chi(D_P) \leq \lceil \log_2(height(P)) \rceil + 2$.

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