Coloring Circle Arrangements: New 4-Chromatic Planar Graphs*

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Felsner, Hurtado, Noy and Streinu (2000) conjectured that arrangement graphs of simple great-circle arrangements have chromatic number at most 3. Motivated by this conjecture, we study the colorability of arrangement graphs for different classes of arrangements of (pseudo-)circles.

In this paper the conjecture is verified for $\triangle$-*saturated* pseudocircle arrangements, i.e., for arrangements where one color class of the 2-coloring of faces consists of triangles only, as well as for further classes of (pseudo-)circle arrangements. These results are complemented by a construction which maps $\triangle$-saturated arrangements with a pentagonal face to arrangements with 4-chromatic 4-regular arrangement graphs. This *corona* construction has similarities with the *crowning* construction introduced by Koester (1985). Based on exhaustive experiments with small arrangements we propose three strengthenings of the original conjecture.

We also investigate fractional colorings. It is shown that the arrangement graph of every arrangement $\mathscr{A}$ of pairwise intersecting pseudocircles is "close" to being

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3-colorable. More precisely, the fractional chromatic number $\chi_f(\mathscr{A})$ of the arrangement graph is bounded from above by $\chi_f(\mathscr{A}) \leq 3+O(\frac{1}{n})$, where n is the number of pseudocircles of $\mathscr{A}$. Furthermore, we construct an infinite family of 4-edge-critical 4-regular planar graphs which are fractionally 3-colorable. This disproves a conjecture of Gimbel, Kündgen, Li, and Thomassen (2019).

1 Introduction

An *arrangement of pseudocircles* is a family of simple closed curves on the sphere or in the plane such that each pair of curves intersects at most twice. Similarly, an *arrangement of pseudolines* is a family of x-monotone curves such that every pair of curves intersects exactly once. An arrangement is *simple* if no three pseudolines/pseudocircles intersect in a common point and *intersecting* if every pair of pseudolines/pseudocircles intersects. Given an arrangement of pseudolines/pseudocircles, the *arrangement graph* is the planar graph obtained by placing vertices at the intersection points of the arrangement and thereby subdividing the pseudolines/pseudocircles into edges.

A *proper coloring* of a graph assigns a color to each vertex such that no two adjacent vertices have the same color. The *chromatic number* χ of a graph is the smallest number of colors needed for a proper coloring of the graph. For an arrangement $\mathscr{A}$, we denote the chromatic number of (the arrangement graph of) $\mathscr{A}$ by $\chi(\mathscr{A})$.

The famous Four Color theorem and also Brook's theorem imply the 4-colorability of planar graphs with maximum degree 4; hence also every arrangement graph is properly 4-colorable. This motivates the following question: which arrangement graphs have chromatic number 4 and which can be properly colored with fewer than four colors?

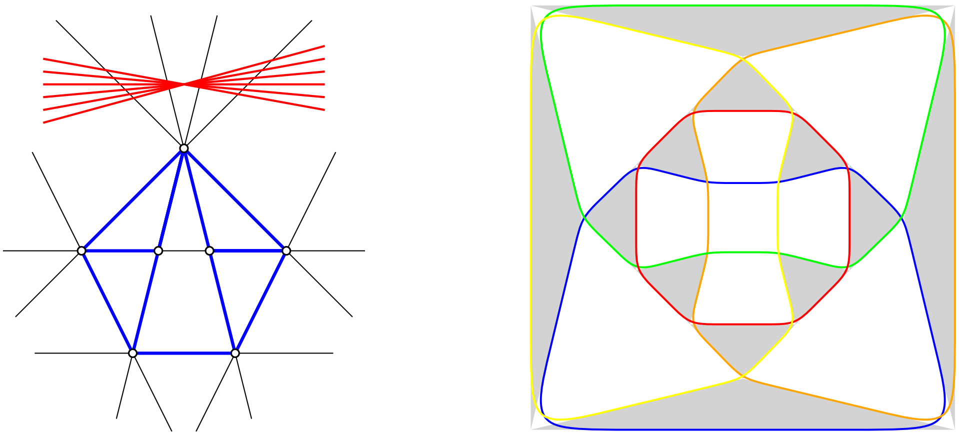
There exist arbitrarily large non-simple line arrangements that require 4 colors. For example, the construction depicted in Figure 1(a) contains the Moser spindle as subgraph which has chromatic number 4. Hence the construction cannot be properly 3-colored. Using an inverse central (gnomonic) projection which maps lines to great-circles, one gets a non-simple arrangement $\mathscr{A}$ of great-circles with $\chi(\mathscr{A}) = 4$ for any such line arrangement. Therefore, we restrict our attention to simple arrangements in the following.

Koester [Koe85] presented a simple arrangement $\mathscr{A}$ of 7 circles with $\chi(\mathscr{A}) = 4$ in which all but one pair of circles intersect; see Figure 8(b) in Section 3.1. Moreover, there also exist simple intersecting arrangements that require 4 colors. We invite the reader to verify this property for the example depicted in Figure 1(b).

In 2000, Felsner, Hurtado, Noy and Streinu [FHNS00] (cf. [FHNS06]) studied arrangement graphs of pseudoline and pseudocircle arrangements. They obtained results regarding connectivity, Hamiltonicity, and colorability of those graphs. In that work, they also stated the following conjecture:

Conjecture 1 (Felsner et al. [FHNS00, FHNS06])**.** *The arrangement graph of every simple arrangement of great-circles on the sphere is 3-colorable.*

While this conjecture is fairly well known (cf. [Ope09, Kal18, Wag02] and [Wag10, Chapter 17.7]) there has been little progress in the last 20 years. Aichholzer, Aurenhammer, and Krasser verified the conjecture for up to 11 great-circles [Kra03, Chapter 4.6.4]. They did not explicitly mention "non-realizable" arrangements, i.e., arrangements of pseudocircles that cannot be realized by great-circles despite fulfilling all necessary combinatorial properties of great-



(a) (b)

Figure 1: (a) A 4-chromatic non-simple line arrangement. The red subarrangement not intersecting the Moser spindle (highlighted blue) can be chosen arbitrarily. (b) A simple intersecting arrangement of 5 pseudocircles with $\chi = 4$ and $\chi_f = 3$.

circle arrangements (see below for details). We have re-generated the data from [Kra03, Chapter 4.6.4] for arrangements of up to 11 great-circles (cf. [SSS20]) and verified the conjecture also for non-realizable arrangements of the same size, by this confirming it for all arrangements of up to 11 *great-pseudocircles*. *Arrangements of great-pseudocircles* are defined as arrangements of pairwise intersecting pseudocircles where along each pseudocircle, the sequence of the $2n-2$ intersections with the other pseudocircles is $(n-1)$-periodic. Equivalently, the induced subarrangement of every three pseudocircles only has triangular faces.

Results and outline In Section 2 we discuss two infinite families of 3-colorable arrangements. The first is the family of $\triangle$-saturated arrangements of pseudocircles: A plane graph is $\triangle$-*saturated* if every edge is incident to exactly one triangular face, an arrangement is $\triangle$-*saturated* if its arrangement graph is $\triangle$-saturated. The second family is based on a specific construction which replaces a pseudocircle by a bundle of three pseudocircles and preserves 3-colorability.

In Section 3 we extend our study of $\triangle$-saturated arrangements and present an infinite family of arrangements which require 4 colors. We believe that the construction results in infinitely many 4-vertex-critical arrangement graphs. A k-chromatic graph is k-*vertex-critical* if the removal of every vertex decreases the chromatic number. It is k-*edge-critical* if the removal of every edge decreases the chromatic number. One of the arrangements which can be obtained with our construction is Koester's arrangement of 7 circles [Koe85]; see Figure 8(b) in Section 3.1. Koester obtained his example using a "crowning" operation, which actually yields infinite families of 4-edge-critical 4-regular planar graphs. However, except for the initial 7 circles example, these graphs are not arrangement graphs of arrangements of pseudocircles.

In Section 4 we investigate the fractional chromatic number χ_f of arrangement graphs. Roughly speaking, this variant of the chromatic number is the objective value of the linear relaxation of the ILP formulation for the chromatic number[1]. We show that intersecting arrangements of pseudocircles are "close" to being 3-colorable by proving that $\chi_f(\mathscr{A}) \leq 3 + O(\frac{1}{n})$ for any intersecting arrangement $\mathscr{A}$ of n pseudocircles.

[1]The exact definition of the fractional chromatic number is deferred to Section 4

In their work about the fractional chromatic number of planar graphs, Gimbel, Kündgen, Li, and Thomassen conjectured that every 4-chromatic planar graph has fractional chromatic number strictly greater than 3 [GKLT19, Conjecture 3.2]. They argued that a positive answer to this statement would yield an alternative proof of the Four Color Theorem. In Section 5, we present an example of a 4-edge-critical arrangement graph which is fractionally 3-colorable. The example is the basis for constructing an infinite family of 4-regular planar graphs which are 4-edge-critical and fractionally 3-colorable. This disproves the conjecture of Gimbel et al. in a strong form.

We conclude this paper with a discussion in Section 6, where we also propose three strengthened versions of Conjecture 1 which are supported by exhaustive experiments with small arrangements.

2 Families of 3-colorable arrangements of pseudocircles

In this section we present two classes of arrangements of pseudocircles which are 3-colorable.

2.1 △-saturated arrangements are 3-colorable

Recall that an arrangement is *△-saturated* if every edge of the arrangement graph is incident to exactly one triangular face. Figure 2 shows some examples of △-saturated arrangements of pseudocircles. We show that △-saturated arrangements are 3-colorable. This verifies Conjecture 1 for a class of great-pseudocircle arrangements. Note, however that not all △-saturated arrangements are great-pseudocircle arrangements; For example the first two arrangements in Figure 2 are not. To see this, consider the subarrangement of the black, blue, and red pseudocircle in each of the two arrangements.

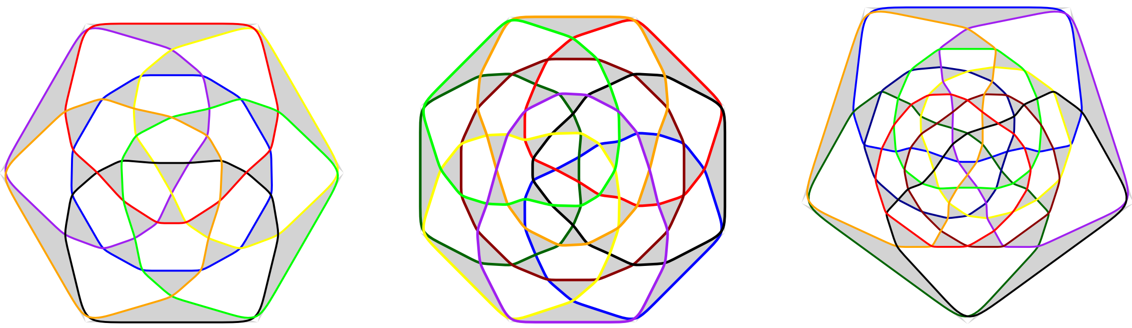
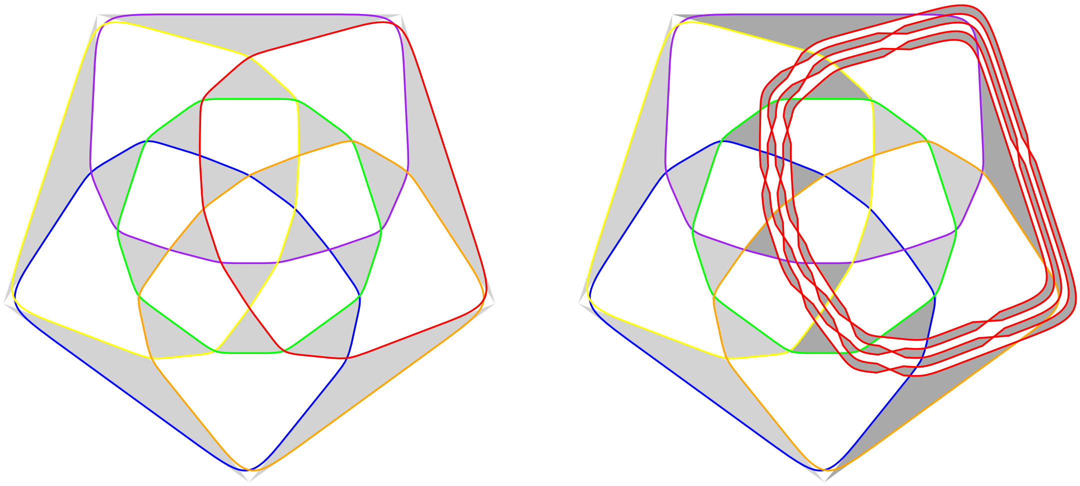


Figure 2: △-saturated intersecting arrangements with 7, 9, and 10 pseudocircles.

Theorem 2. *Every simple △-saturated arrangement $\mathscr{A}$ of pseudocircles is 3-colorable.*

Proof. Let H be a graph whose vertices correspond to the triangles of $\mathscr{A}$ and whose edges correspond to pairs of triangles sharing a vertex of $\mathscr{A}$. This graph H is planar and 3-regular. Moreover, since the arrangement graph of $\mathscr{A}$ is 2-connected, H is bridgeless. Now Tait's theorem, a well known equivalent of the 4-color theorem, asserts that H is 3-edge-colorable, see e.g. [Aig87] or [Tho98]. The edges of H correspond bijectively to the vertices of the arrangement $\mathscr{A}$ and, since adjacent vertices of $\mathscr{A}$ are incident to a common triangle, the corresponding edges of H share a vertex. This shows that the graph of $\mathscr{A}$ is 3-colorable. □

The maximum number of triangles in arrangements of pseudolines and pseudocircles has been studied intensively, see for example [Grü72, Rou86, Bla11] and the recent work [FS21]. By recursively applying the "doubling method", Harborth [Har85], Roudneff [Rou86], and Blanc [Bla11] proved the existence of △-saturated arrangements of n pseudolines for infinitely many values of $n \equiv 0, 4 \pmod 6$. Similarly, a doubling construction for arrangements of (great-)pseudocircles yields infinitely many △-saturated arrangements of (great-)pseudocircles. Figure 3 illustrates the doubling method applied to an arrangement of great-pseudocircles. Note that for $n \equiv 2 \pmod 3$ there is no △-saturated intersecting pseudocircle arrangements because the number of edges of the arrangement graph equals 3 times the number of triangles but the number of edges is $2n(n-1)$ which is not divisible by 3.



(a) (b)

Figure 3: The doubling method applied to an arrangements of 6 great-pseudocircles. The red pseudocircle is replaced by a cyclic arrangement.

The proof of Theorem 2 actually can be extended to a larger class of graphs (cf. Theorem 3). Before stating the result we need some more definitions.

The *medial graph* $M(G)$ of an embedded planar graph G is a graph representing the adjacencies between edges in the cyclic order of vertices and faces, respectively: The vertices of $M(G)$ correspond to the edges in G. Two vertices of $M(G)$ share an edge whenever their corresponding edges in G are adjacent along the boundary of a face of G (and hence consecutive around a vertex; vertices of degree 1 and 2 in G induce loops and multi-edges, respectively, in $M(G)$). Note that every medial graph is a 4-regular planar graph. Vice-versa, every 4-regular planar graph is the medial graph of some planar graph.

In order to see that the latter statement is true for connected graphs, let H be a 4-regular connected embedded planar graph, and consider its dual graph H^*. Since H is 4-regular and hence Eulerian, H^* is a bipartite graph. Next consider the 2-coloring of the faces of H which is induced by the bipartition of H^*, say, with colors gray and white. Pick one of the color classes, e.g., the gray faces, and create a new plane graph G as follows: G has exactly one vertex placed in the interior of every gray face of H, and two vertices u and v of G are connected via an edge if and only if their corresponding gray faces touch at a vertex x. In this case, the edge uv is drawn in G in such a way that it connects u to v by crossing through x and staying within the union of the gray faces corresponding to u and v otherwise. From this construction it is now easy to see that G is a plane graph satisfying $M(G) = H$, and every such graph G is referred to as a *premedial graph* of H. By picking the white instead of the gray faces in the above construction, we would have obtained another premedial graph of H, namely the dual graph G^* of G. While this shows that reconstructing G from $M(G)$ is in general not possible, it can be seen that H determines a primal-dual pair $\{G, G^*\}$ of premedial graphs uniquely up to isomorphism. Figure 4 shows an example of a medial graph and its two premedial graphs.

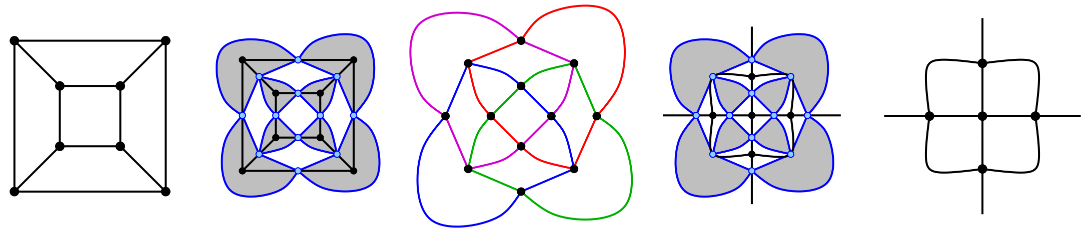


Figure 4: The cube graph (left) and its medial graph $G_{\mathcal{A}}$ (middle). The graph $G_{\mathcal{A}}$ is also the graph of an arrangement of four pseudocircles; as indicated by the edge colors. The second premedial graph of $G_{\mathcal{A}}$ is the octahedron graph (right).

Note that if G is the graph of an arrangement of pseudocircles, then G is 4-regular while its dual graph G^* has vertices of degree ≤ 3. Hence, in this case we can identify G in the pair of premedial graphs given by $M(G)$.

In the other direction, an arrangement graph G has a cubic premedial graph – the graph H in the proof of Theorem 2 – if and only if G is Δ-saturated. Moreover, the proof of Theorem 2 does not require that the 4-regular graph G under consideration is actually an arrangement graph. It just requires it to be 2-connected to ensure that the cubic premedial graph H is bridgeless. Hence the following theorem generalizes Theorem 2, while essentially having the same proof.

Theorem 3. *If G is a 2-connected 4-regular planar graph which has a cubic premedial graph H then $\chi(G) = 3$.*

We remark that 2-connectivity is a crucial condition in Theorem 3 as illustrated in Figure 5.

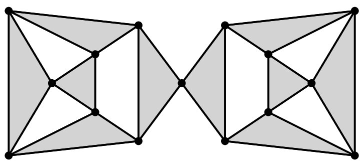


Figure 5: A connected 4-regular planar graph G with a cubic premedial graph and $\chi(G) = 4$

Proof. Let H be the cubic premedial graph of G, i.e., $G = M(H)$. A bridge in H corresponds to a cut vertex of its medial graph G. Since G is assumed to be 2-connected, it follows that H is bridgeless and hence, by Tait's theorem, 3-edge-colorable. Adjacent vertices of G correspond to edges of H that are consecutive in the circular order at a vertex of H. As such pairs of edges receive different colors in the edge-coloring of H, the 3-edge-coloring of H induces a 3-coloring of G. □

It follows from the above discussion that $\chi(M(H))$ is upper bounded by the *chromatic index* $\chi'(H)$, i.e., the minimum number of colors required for a proper edge coloring of H.

Indeed, if v is a vertex of H, then edges incident to v require pairwise distinct colors in an edge coloring, while in $M(H)$ these edges are vertices along the boundary of a facial cycle so that repetitions of colors might be feasible.

2.2 More families of 3-colorable arrangements

We next show how to construct more infinite families of 3-colorable arangements of (intersecting) pseudocircles, great-pseudocircles, or circles, respectively.

Let $\mathscr{A}$ be a 3-colorable arrangement of n pseudocircles and let ϕ be a coloring of $\mathscr{A}$ with colors $0, 1, 2$. We will use the additive structure of $\mathbb{Z}_3$ on the colors.

Fix a pseudocircle C of $\mathscr{A}$ and let V_I and V_O be the sets of vertices of $\mathscr{A}$ inside and outside of C, respectively. Let $\mathscr{A}'$ be the arrangement obtained from $\mathscr{A}$ by adding two parallel pseudocircles C' and C'' along C, i.e., the order in which the three pseudocircles C, C', and C'' cross the other pseudocircles is the same. We can think of the parallel pseudocircles as drawn close to C such that C is the innermost, C' the middle, and C'' the outer of the three pseudocircles. For every vertex $v \in C$, we have the corresponding vertices v' and v'' on C' and C'' respectively. Formally, this correspondence can be stated by saying that vv' and $v'v''$ are edges of $\mathscr{A}'$, and edges vw with $w \in V_O$ of $\mathscr{A}$ are replaced by $v''w$ in $\mathscr{A}'$.

The following defines a 3-coloring ϕ' of $\mathscr{A}'$: For $u \in V_I$ let $\phi'(u) = \phi(u)$; for a triple v, v', v'' of corresponding vertices on the three pseudocircles C, C', C'', let $\phi'(v) = \phi(v)$, $\phi'(v') = \phi(v) + 1$, and $\phi'(v'') = \phi(v) + 2$; finally for $w \in V_O$ let $\phi'(w) = \phi(w) + 2$.

Since we are mostly interested in intersecting arrangements we next describe how to transform $\mathscr{A}'$ into a 3-colorable intersecting arrangement $\mathscr{A}''$. Let e_1 and e_2 be two edges on C in $\mathscr{A}$. Corresponding to each of e_1 and e_2, we have a 2×3 grid in $\mathscr{A}'$; see Figure 6 left. This grid can be replaced by a triangular structure with pairwise crossings of the three pseudocircles, see Figure 6 middle and right. The figure also shows that a 3-coloring of the grid, where the colors in the columns are $0, 1, 2$, or $1, 2, 0$, or $2, 0, 1$, can be extended to the three added crossings. Hence, we obtain a 3-colorable intersecting arrangement $\mathscr{A}''$.

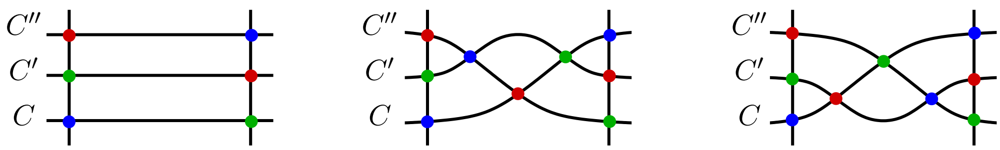


Figure 6: A 2×3 grid (left) and two ways of adding pairwise crossings on the horizontal curves.

Let $\mathscr{A}$ be a 3-colorable arrangement of great-pseudocircles. If we pick e_1 and e_2 as a pair of antipodal edges on C and add the intersections between C, C', and C'' along those two edges, once as in the middle of Figure 6 and once as in the right of the figure, then we obtain an arrangement $\mathscr{A}''$ which is again an arrangement of great-pseudocircles.

Moreover, if $\mathscr{A}$ is an arrangement of (proper) circles, then clearly $\mathscr{A}'$ is again an arrangement of circles. Less obvious but still true is that $\mathscr{A}''$ can also be realized as a circle arrangement. The reason is that the three circles C', C'' can be placed inside an arbitrarily narrow belt centered at C. Figure 7 shows an example of a transformation $\mathscr{A} \to \mathscr{A}' \to \mathscr{A}''$.

The following is a direct consequence of the above-described constructions.

Proposition 4. *Let $\mathscr{A}$ be a 3-colorable arrangement of n (intersecting) pseudocircles, great-pseudocircles, or circles, respectively. Then for any $k \in \mathbb{N}$, arrangement $\mathscr{A}$ can be extended to a 3-colorable arrangement of size $n + 2k$ of the same type.*

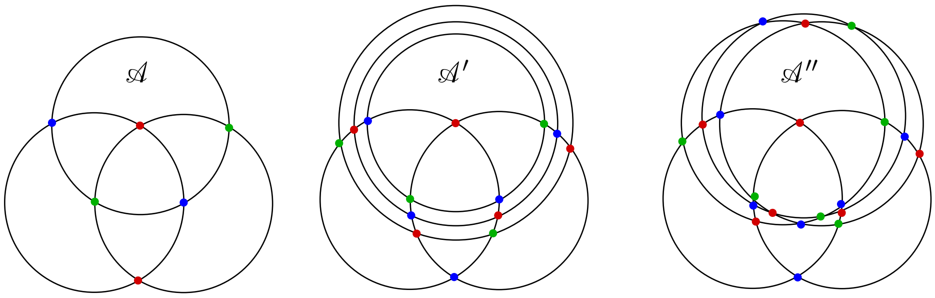


Figure 7: A 3-colorable arrangement $\mathscr{A}$ of circles and the derived arrangements $\mathscr{A}'$ and $\mathscr{A}''$.

3 Constructing 4-chromatic arrangement graphs

In the first part of this section, we describe an operation that extends any △-saturated intersecting arrangement of pseudocircles with a pentagonal cell (which is 3-colorable by Theorem 2) to a 4-chromatic arrangement of pseudocircles by inserting only one additional pseudocircle. This corona extension is somewhat related to Koester's crowning, an operation used to construct an infinite family of 4-regular 4-edge-critical planar graphs [Koe90]. This motivates the study of criticality of the graphs obtained via the corona extension, which is the topic of Subsection 3.2.

3.1 The corona extension

We start with a △-saturated arrangement $\mathscr{A}$ of pseudocircles which contains a pentagonal cell ⬠. By definition, in the 2-coloring of the faces of $\mathscr{A}$, one of the two color classes consists of triangles only; see e.g. the arrangement from Figure 8(a). Since the arrangement is △-saturated, the pentagonal cell ⬠ is surrounded by triangular cells.

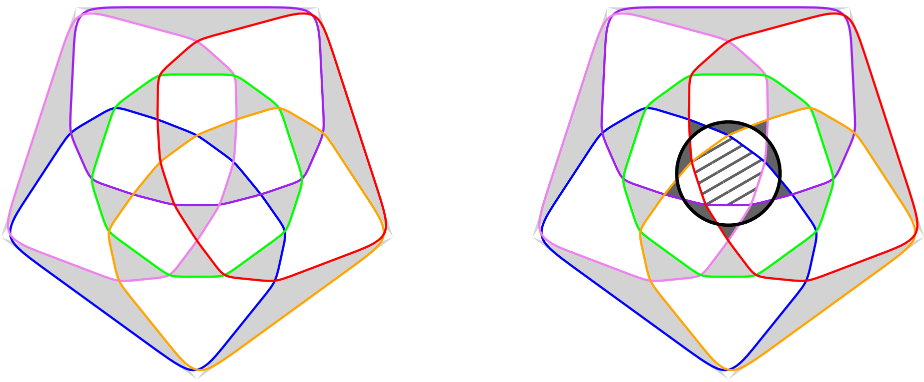
We can now insert an additional pseudocircle enclosing ⬠ so that the new pseudocircle intersects only the 5 pseudocircles which bound ⬠ and does so only at edges incident to vertices of ⬠. Figure 8(b) illustrates this extension for the arrangement from Figure 8(a). In the extended arrangement $\mathscr{A}^+$, one of the two color classes of faces consists of triangles and the pentagon ⬠. We say that $\mathscr{A}^+$ is obtained via a *corona extension*[2] from $\mathscr{A}$. It is interesting to note that the arrangement depicted in Figure 8(b) is Koester's arrangement [Koe85].

To discuss the colorability of the corona extension, we introduce some notation. For a graph G, let $\alpha(G)$ denote the size of any maximum independent set of G. In a proper k-coloring of G, the vertices of every color class form an independent set, and we trivially have $\alpha(G) \geq \frac{|V(G)|}{k}$ for every k-colorable graph.

Lemma 5. *Let G be a 4-regular planar graph. If in the 2-coloring of the faces of G, one of the classes consists of only triangles and a single pentagon, then $\alpha(G) < \frac{|V(G)|}{3}$.*

Proof. Color the faces of $G = (V, E)$ with black and white. Let the black class contain only triangles and one pentagon. Let t be the number of these triangles and let $\alpha := \alpha(G)$. Given an independent set I of cardinality α, we count the number of pairs (v, F), where v is a vertex of I and F is a black face of $\mathscr{A}$ incident to v. There are 2 such faces for every $v \in I$, hence, 2α pairs in total. Since any independent set of G contains at most one vertex of each triangle

[2]The writing of this article has benefited from the corona lockdown in April 2020.



(a) (b)

Figure 8: (a) A △-saturated arrangement of 6 great-circles and (b) the corona extension at its central pentagonal face. The arrangement in (b) is Koester's [Koe85] example of a planar 4-edge-critical 4-regular planar graph.

and at most two vertices of the pentagon, the number of pairs (v, F) is at most $t + 2$. Hence, we have

$$2\alpha \leq t + 2. \tag{1}$$

Since G is 4-regular, there are exactly $|E| = 2|V|$ edges. As every edge is incident to exactly one black face, we also have $|E| = 3t + 5$. This yields the equation

$$3t + 5 = 2|V|. \tag{2}$$

From equation (2), we conclude that t is odd. Therefore we can strengthen equation (1) to

$$2\alpha \leq t + 1. \tag{3}$$

Combining equations (2) and (3) yields $6\alpha \leq 3t + 3 = 2|V| - 2$ and hence $\alpha < \frac{|V|}{3}$. □

Proposition 6. *The corona extension of a △-saturated arrangement of pseudocircles with a pentagonal cell ⬠ is 4-chromatic.*

Proof. From Lemma 5 we know that after the corona extension the inequality $3\alpha(G) < |V(G)|$ holds. This implies that the corona extension of a △-saturated arrangement of pseudocircles with a pentagonal cell ⬠ is not 3-colorable. □

It is remarkable that the argument from the proof of Lemma 5 only holds for pentagons. More precisely, if the class of black faces of G consists of triangles and a single k-gon, then we need $k = 5$ to get $\alpha < |V|/3$.

By iteratively applying the doubling method (cf. Section 2.1) to the arrangements depicted in Figure 2, we obtain △-saturated arrangements of n pseudocircles which have pentagonal cells for infinitely many values of $n \equiv 0, 4 \pmod 6$. Applying the corona extension to the members of this infinite family yields an infinite family of arrangements that are not 3-colorable.

Theorem 7. *There exists an infinite family of simple 4-chromatic arrangements of pseudocircles, each of which is obtained from an intersecting arrangement of pseudocircles by adding only one additional pseudocircle.*

3.2 Criticality

Koester [Koe90] introduced the *crowning* operation and used this operation to construct an infinite family of 4-regular 4-edge-critical planar graphs (cf. Proposition 15 and Figure 14). A particular example of a graph obtained by crowning is the Koester graph of Figure 8(b), which happens to be an arrangement graph of circles.

Since crowning and the corona extension show some similarities and both operations can be used to obtain the Koester graph depicted in Figure 8(b), we believe that many of the 4-chromatic arrangements obtained with the corona extension (Theorem 7) are in fact 4-vertex-critical. In the following, we present sufficient conditions to obtain 4-vertex-critical and 4-edge-critical arrangements via the corona extension.

We need some terminology. Let H be a cubic plane graph and let $G = M(H)$ be its medial graph. If H is bridgeless, then $\chi(G) = 3$ by Theorem 3. If in addition H has a pentagonal face $\pentagon_H$, then we can apply the corona extension to G to obtain a 4-regular graph G° with $\chi(G^\circ) = 4$ (Lemma 5). We are interested in conditions on H which imply that G° is 4-vertex-critical or even 4-edge-critical.

With $\pentagon_G$ we denote the pentagon corresponding to $\pentagon_H$ in G. The *connector vertices* of $\pentagon_G$ are the five vertices of the triangles adjacent to $\pentagon_G$ which do not belong to $\pentagon_G$.

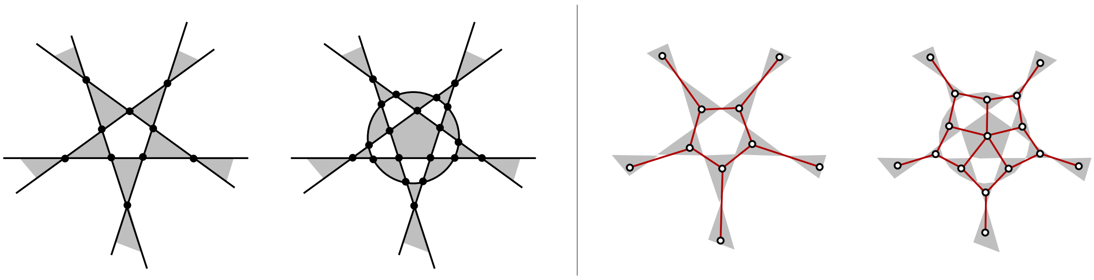


Figure 9: Applying the corona extension at a pentagon of a $\triangle$-saturated 4-regular planar graph.

Consider a 3-edge-coloring φ of H. We call φ *trihamiltonian*, if all three subgraphs induced by edges of two of the three colors of φ induce a Hamiltonian cycle on H. We will prove the following:

Theorem 8. *Let H be a cubic planar graph with a pentagonal face $\pentagon_H$ and a trihamiltonian 3-edge-coloring φ. If G is the medial graph of H and G° is obtained from G by the corona extension at $\pentagon_G$, where $\pentagon_G$ is the pentagonal face of G corresponding to $\pentagon_H$, then G° is 4-vertex-critical. If, additionally, H admits 5-fold rotational symmetry around $\pentagon_H$, then G° is even 4-edge-critical.*

Proof. Recall from Section 2.1 that the 3-edge-coloring φ of H yields a 3-vertex-coloring of the $\triangle$-saturated graph G. Each of the three 2-colored Hamiltonian cycles in H given by φ yields a cycle in G, which covers all the vertices of the respective colors. This is indicated in Figure 10 (left). Each edge of G is contained in exactly one of the 3 cycles, hence we obtain a non-proper 3-edge-coloring of G with the property that every color class is a cycle. The two red, two green and one blue circular arcs indicate the way that these three cycles are closed outside the corona region. Note that the order of connector vertices on the red and green cycle must be as indicated in Figure 10 (left) since each monochromatic cycle is non-crossing. Every

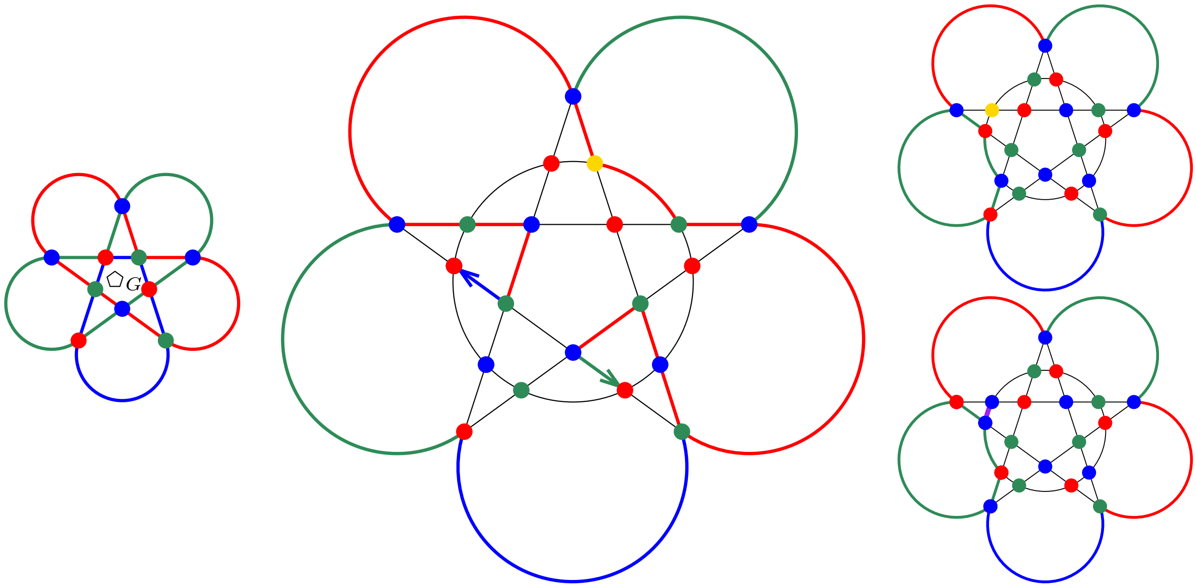


Figure 10: Left: Edges of G have the color that is missing on its incident vertices.
Middle: The golden vertex can move along the red path and arrows in both directions.
Right: The purple edge indicates the critical edge after we change colors on the green cycle.

vertex belongs to two of the three cycles induced by the edge coloring, hence, in Figure 10 arcs of different colors can have multiple intersections and touchings.

Note that Figure 10 (left) has a vertical axis of symmetry which preserves the blue vertices and the blue cycle but exchanges the colors red and green. In the following, we will show how to modify these two colorings (original and reflected) in order to find a collection of 4-colorings of G° that allows to argue for 4-vertex- and 4-edge-criticality in the respective cases.

Figure 10 (middle) shows a 4-coloring φ° of G° with the same coloring of the connector vertices and vertices outside the corona region as in Figure 10 (left). Note that a single vertex is colored with the fourth color (gold). If in a 4-vertex-coloring φ' a vertex is the single golden vertex, we call it the *special vertex* of the coloring. To show that G° is 4-vertex-critical, we need to show that every vertex in the graph is the special vertex of some 4-coloring of G°.

In every 4-coloring with a special vertex v, this vertex is surrounded by all three colors, since we know from Lemma 5 that the graph G° is 4-chromatic. Thus only one of the colors appears twice. Recoloring v with the one of the other colors and the corresponding neighbor w of v with gold makes this neighbor the special vertex of the new coloring. We say that the special vertex *moves* from v to w. To show that an edge e is critical in G°, it suffices to show that there is a 4-coloring with special vertex v which allows such a move from v to w.

Starting from φ°, we can make the special vertex move along the red path (see Figure 10 (middle)), changing the colors of green and blue vertices along the way. To see this, remember that the green and blue vertices on the red arcs have 2 red neighbors (in G and thus in G°), so the blue and green color are the ones to move along. At one of the endpoints of the red path in Figure 10 (middle), there are two blue neighbors, thus the next neighbor to move to is the

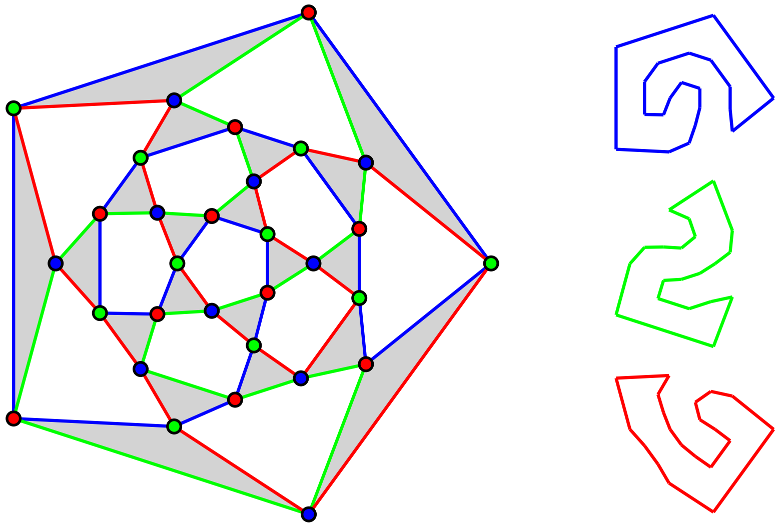


Figure 11: A 3-coloring of the great-circle arrangement from Figure 8(a). The three cycles obtained by removing each of the color classes are depicted on the right.

red neighbor. At the other endpoint, there are two green neighbors, so again the red neighbor is the next neighbor to move to. This is indicated by arrows in Figure 10 (middle).

Move the golden vertex along the two red branches and the extending steps. Then turn to the symmetric (reflected) coloring and do the symmetric moves. We claim that together this yields a collection of colorings of G° such that every vertex is the special (golden) vertex of one of them. For the vertices inside or on the new circle, this is easily checked from Figure 10 (middle). The vertices outside of the corona region are colored with 3 colors. The blue and green ones lie on the red arcs and are therefore reached when moving the golden vertex along the red arcs. The red ones lie on the green paths and will therefore be reached if we start from the reflected coloring, because then these same vertices would be green and lie on the corresponding red arcs. Thus 4-vertex-criticality is established.

Now suppose that H has a 5-fold symmetry fixing $\pentagon_H$. This symmetry carries over to G and G°. Thus it is sufficient to show that any edge can be rotated to an edge that we have covered already. The only edges which are not covered by the moves along the extended red paths of Figure 10 (middle) and its rotations are the small edges on the new circle that are inside the triangles of G next to $\pentagon_G$. In Figure 10 (right), we show an additional extension of the coloring of the connector vertices to the interior. Exchanging the colors of the blue and red vertices of the green cycle in the bottom left makes it possible to 3-color the graph by making the special vertex blue, if the purple edge is omitted. Since the purple edge is a representative of the last rotational orbit we did not cover yet, this yields 4-edge-criticality. $\square$

Next, we present some examples of the application of Theorem 8. Let H be a cubic planar graph which has a unique 3-edge-coloring up to permutations of the colors. Then this coloring is trihamiltonian, since if a graph induced by two colors has more than one component, we can change the two colors on this component alone and construct a different coloring. Thus for any pentagon in H, the resulting graph G° is 4-vertex-critical.

The class of uniquely 3-edge-colorable cubic graphs is well understood. Fowler [Fow98, Theorem 2.8.5] characterized them as the graphs that can be obtained from K_4 by successively

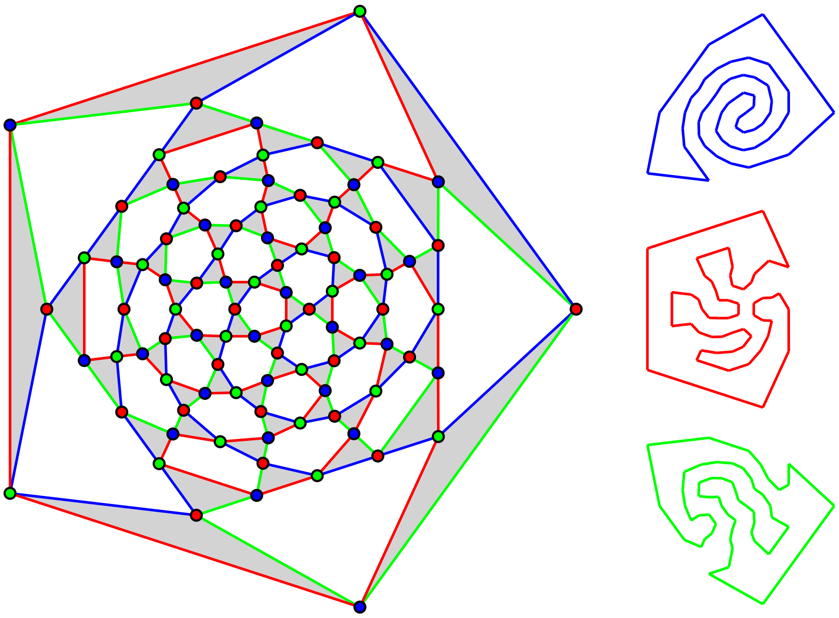


Figure 12: A 3-coloring of a △-saturated great-circle arrangement. The three cycles obtained by removing each of the color classes are depicted on the right.

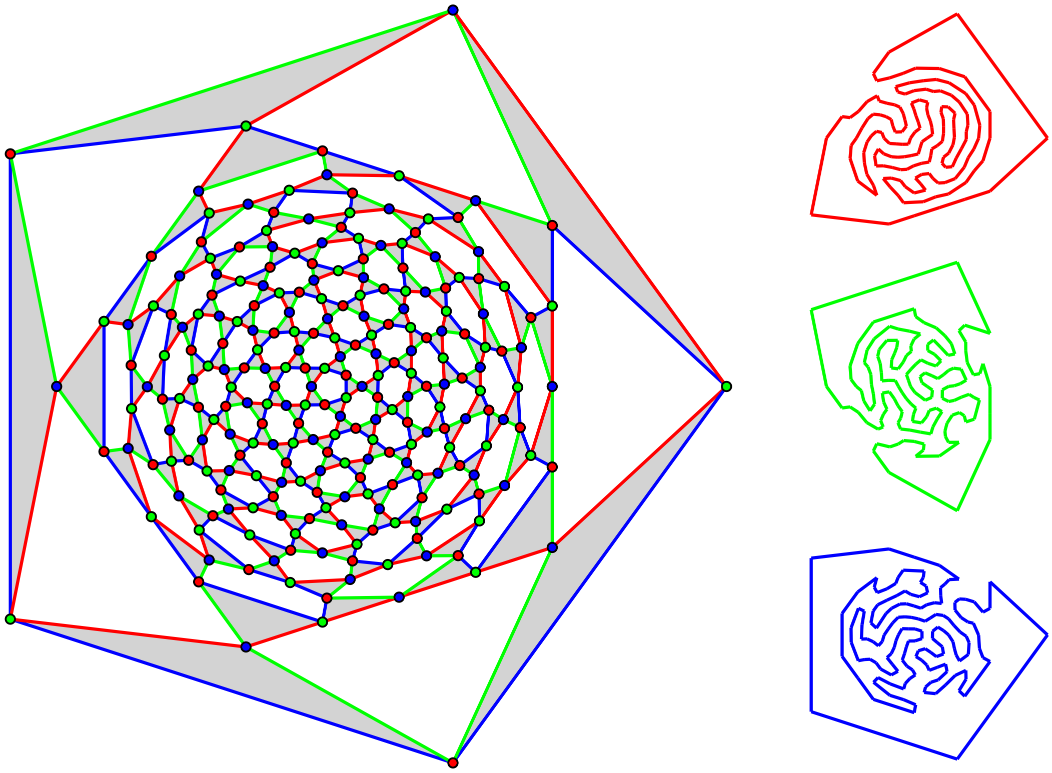


Figure 13: This great-circle arrangement of 16 great-pseudocircles has been discovered by Simmons [Sim73]. The three cycles obtained by removing any of the color classes are depicted on the right.

replacing a vertex by a triangle. These are the duals of stacked triangulations, which are the uniquely 4-colorable planar graphs [Fow98, Conjecture 1.2.1][3].

Additionally, Figures 11, 12, and 13 show △-saturated arrangements of 6, 10, and 16 great-pseudocircles respectively, that admit 5-fold rotational symmetry. The arrangement graphs are shown with 3-colorings which correspond to trihamiltonian 3-edge-colorings of their respective premedial graphs. The theorem implies that the corona extension at the outer pentagon of these arrangements yields 4-edge-critical graphs. We are aware of three more △-saturated arrangements of 6, 7, and 9 pseudocircles respectively, which have 4-edge-critical corona extensions. For these arrangements, however, the 4-edge-criticality is not implied by our theorem. All data is available on the supplementary website [FS].

We conclude this section with the following conjecture:

Conjecture 9. *There exists an infinite family of simple arrangement graphs of* 4*-edge-critical arrangements of pseudocircles.*

Relaxing the condition of the conjecture to 4-regular planar graphs, this is a known result of Koester (see Proposition 15).

4 Fractional colorings

In this section, we investigate fractional colorings of arrangements. A *b-fold coloring* of a graph G with m colors is an assignment of a set of b colors from $\{1, \dots, m\}$ to each vertex of G such that the color sets of any two adjacent vertices are disjoint. The *b-chromatic number* $\chi_b(G)$ is the minimum m such that G admits a b-fold coloring with m colors. The *fractional chromatic number* of G is $\chi_f(G) := \lim\limits_{b\to\infty} \frac{\chi_b(G)}{b} = \inf\limits_{b} \frac{\chi_b(G)}{b}$. With $\alpha(G)$ being the independence number of G and $\omega(G)$ being the clique number of G, the following inequalities hold:

$$\max\left\{\frac{|V|}{\alpha(G)}, \omega(G)\right\} \le \chi_f(G) \le \frac{\chi_b(G)}{b} \le \chi(G). \tag{4}$$

The fractional chromatic number forms a natural lower bound for the chromatic number of graphs. While the chromatic number of quite some intersecting arrangements of pseudocircles is four, at least their fractional chromatic number is always close to three:

Theorem 10. *Let G be the arrangement graph of a simple intersecting arrangement $\mathscr{A}$ of n pseudocircles, then $\chi_f(G) \le 3 + \frac{6}{3n-2} = 3 + \frac{2}{n} + o\left(\frac{1}{n}\right)$. In particular, if v denotes the number of vertices of G, then $\chi_f(G) \le 3 + \frac{2}{\sqrt{v}} + o\left(\frac{1}{\sqrt{v}}\right)$.*

Proof. Fix an arbitrary circle $C \in \mathscr{A}$ and let $V_C \subseteq V_G$ be the vertex set of C. Let $V_G \setminus V_C = V_I \cup V_O$, where V_I and V_O are the sets of vertices inside or outside of C, respectively.

Claim 1. The graphs $G[V_I]$ and $G[V_O]$ are 3-colorable.

Proof. We prove the claim for $G[V_I]$, the proof for $G[V_O]$ is analogous. Let $C_0 \notin \mathscr{A}$ be a tiny circle in some face of the arrangement in the interior of C. The Sweeping Theorem of Snoeyink and Hershberger [SH91, Theorem 3.1] asserts that there exists a sweep which continuously

[3] This theorem in the thesis is called a conjecture, since it is first proved to be equivalent to the Fiorini-Wilson-Fisk Conjecture, which is proved much later as the main result of the thesis.

transforms C_0 into C such that at any time $\mathscr{A} \cup C_0$ is an arrangement of pseudocircles. Let $t = |V_I|$ and let $\pi = (v_1, v_2, \ldots, v_t)$ be the ordering of the vertices of V_I induced by this sweep, i.e., v_i for $i \in \{1, \ldots, t\}$ is the i-th vertex met by the sweep-pseudocircle C_0. Orient each edge of $G[V_I]$ from the vertex of smaller index to the vertex of larger index. Note that on every pseudocircle $C' \in \mathscr{A}$ this orientation induces at most two directed paths that share the starting point, the first vertex of C' met by C_0. At every vertex $v \in V_I$ two pseudocircles cross and v has at most one predecessor on each of the two pseudocircles (here we use the fact that $\mathscr{A}$ is an intersecting arrangement, and hence every pseudocircle of $\mathscr{A}$ different from C intersects both the interior and the exterior of C). Hence, in the acyclic orientation of G defined above, every $v \in V_I$ satisfies $\operatorname{indeg}(v) \leq 2$. Thus, the greedy algorithm with the ordering π yields a 3-coloring of $G[V_I]$. $\square$

Let us pause to note that just on the basis of this first claim we get $\chi_f(G) \leq 3 + \frac{6}{n-2}$ which is not too far from the bound given in the theorem. Indeed if for each pseudocircle C of the arrangement we use 3 colors to color $V \setminus V_C$, then every vertex receives $n-2$ colors, whence we obtain a b-coloring with $b = n-2$ using $3n$ colors in total, i.e., $\chi_{n-2}(G) \leq 3n$.

Claim 2. The graph $G[V_C]$ is 2-colorable.

Proof. Let F be a face of the planar graph $G[V_C]$. Each vertex of F is a crossing of C with some $C' \neq C$ and each $C' \neq C$ contributes 0 or 2 vertices to the boundary of F. This shows that every face of $G[V_C]$ is even whence $G[V_C]$ is a bipartite graph. $\square$

Claim 3. For every weighting $w : V_G \to [0, \infty)$ there is an independent set I of G such that $w(I) \geq (\frac{1}{3} - \frac{2}{9n}) w(V_G)$.

Proof. Let $C \in \mathscr{A}$ be a pseudocircle with minimal weight $w(V_C)$. Let I_1, I_2, I_3 and J_1, J_2, J_3 denote the 3 color classes of a proper 3-coloring of $G[V_I]$ and $G[V_O]$, respectively (Claim 1). For $(i,j) \in \{1,2,3\}^2$, let $I_{i,j} := I_i \cup J_j$ and let $X_{i,j} \subseteq V_C$ denote the set of vertices on C with no neighbor in $I_{i,j}$. The subgraph $G[X_{i,j}]$ of $G[V_C]$ is 2-colorable (Claim 2). Let $X_{i,j}^1, X_{i,j}^2$ denote the color classes of such a coloring, and define independent sets $I_{i,j,k} := I_{i,j} \cup X_{i,j}^k$ in G for $k = 1, 2$.

With $\mathbf{I}$ we denote the random independent set $I_{i,j,k}$ with (i,j,k) being chosen from the uniform distribution on $\{1,2,3\} \times \{1,2,3\} \times \{1,2\}$. In the following we bound the expected weight $\mathbb{E}(w(\mathbf{I}))$.

For every vertex $x \in V_C$, we have $x \in X_{i,j}$ if and only if none of the two neighbors x^O, x^I of x in V_O respectively V_I lie in I_i respectively J_j. Since i, j and k are sampled independently, we conclude

$$\mathbb{P}(x \in \mathbf{I}) = \frac{1}{2}\mathbb{P}(x \in X_{i,j}) = \frac{1}{2}\mathbb{P}(x^O \notin I_i)\mathbb{P}(x^I \notin J_j) = \frac{1}{2}\Big(\frac{2}{3}\Big)^2 = \frac{2}{9}.$$

This implies that

$$\mathbb{E}(w(\mathbf{I})) = \mathbb{E}(w(I_i \cup J_j)) + \mathbb{E}(w(X_{i,j}^k)) = \frac{1}{3}(w(V_G) - w(V_C)) + \frac{2}{9} \cdot w(V_C) = \frac{1}{3}w(V_G) - \frac{1}{9}w(V_C).$$

Since C was chosen as a pseudocircle of minimum weight, and since $\sum_{C' \in \mathscr{A}} w(V_{C'}) = 2w(V_G)$, we conclude that $w(V_C) \leq \frac{2}{n} w(V_G)$ and hence $\mathbb{E}(w(\mathbf{I})) \geq (\frac{1}{3} - \frac{2}{9n}) w(V_G)$. Since $\mathbf{I}$ is ranging in the independent sets of G, this implies the existence of an independent set with total weight at least $(\frac{1}{3} - \frac{2}{9n}) w(V_G)$. $\square$

It is well known that the fractional chromatic number can be obtained as the optimal value of the linear program

$$\min \mathbf{1} \cdot x \quad \text{subject to} \quad Mx \geq \mathbf{1}, \quad x \geq 0$$

where M is the incidence matrix of vertices versus independent sets. The dual of the program is $\max \mathbf{1} \cdot w$ subject to $M^T w \leq \mathbf{1}$, $w \geq 0$. Here w can be interpreted as a weighting on the vertices. If w is an optimal weighting for this program, then $\chi_f(G) = w(V_G)$. With Claim 3 we get $1 \geq \mathbb{E}(w(\mathbf{I})) \geq (\frac{1}{3} - \frac{2}{9n})w(V_G)$. Hence, $\chi_f(G) \leq \frac{1}{\frac{1}{3} - \frac{2}{9n}} = 3 + \frac{6}{3n-2}$. □

We note that for 4-vertex-critical graphs G, the following simple bound on the fractional chromatic number further improves the bound given in Theorem 10.

Proposition 11. *If G is a 4-vertex-critical graph on v vertices, then $\chi_f(G) \leq 3 + \frac{3}{v-1}$.*

Proof. We show that G admits a $(v-1)$-fold coloring using $3v$ colors, which will imply $\chi_{v-1}(G) \leq 3v$ and hence $\chi_f(G) \leq \frac{\chi_{v-1}(G)}{v-1} \leq \frac{3v}{v-1} = 3 + \frac{3}{v-1}$.

The coloring can be obtained as follows: For every vertex $x \in V(G)$, fix a proper 3-coloring $c_x : V(G) \setminus \{x\} \to \{C_{1,x}, C_{2,x}, C_{3,x}\}$ of the vertices in $G - x$ (which exists since G is 4-vertex-critical). Here, $\{C_{1,x}, C_{2,x}, C_{3,x}\}$ is a set of 3 colors chosen such that these color-sets are pairwise disjoint for different vertices x.

We now define a $(v-1)$-fold coloring of G by assigning to every $w \in V(G)$ the following set of $v-1$ colors $\{c_x(w) | x \in V(G), x \neq w\}$. Since every c_x is a proper coloring of G, these color-sets are disjoint for adjacent vertices in G. Furthermore, the coloring uses only colors in $\{\, C_{1,x}, C_{2,x}, C_{3,x} \mid x \in V(G) \,\}$, so $3v$ colors in total, and this proves the above claim and concludes the proof. □

4.1 Arrangements with dense intersection graphs

Given an arrangement $\mathscr{A}$ of pseudocircles, the *intersection graph* of $\mathscr{A}$ is the simple graph $H_{\mathscr{A}}$ with the pseudocircles in $\mathscr{A}$ as the vertex-set in which two distinct pseudocircles $C_1, C_2 \in \mathscr{A}$ share an edge if and only if they cross. Using this notion, we see that intersecting arrangements of pseudocircles are exactly the arrangements whose intersection graph is a complete graph. Looking at Theorem 10, we were able to show that the fractional chromatic number of such arrangements is close to 3. In this section we discuss possible generalizations of this result by extending this bound to arrangements $\mathscr{A}$ for which $H_{\mathscr{A}}$ is sufficiently dense. In particular we have the following question.

Question 1. *For $k \in \mathbb{N}$, let $\chi_{\geq k}$ denote the supremum of $\chi_f(G)$ over all arrangement graphs G of arrangements $\mathscr{A}$ of pseudocircles such that the minimum degree $\delta(H_{\mathscr{A}})$ is at least k. Is it true that $\chi_{\geq k} \to 3$ for $k \to \infty$?*

In the following, we show two weaker statements related to this question. The first one shows that if we require the minimum degree in the intersection graph of an arrangement to be sufficiently large compared to n, then we can indeed conclude that the fractional chromatic number of the arrangement graph is close to 3. The second statement answers a relaxed version of Question 1 by showing that for large minimum degree in the intersection graph, the inverse independence ratio $\frac{|V(G)|}{\alpha(G)}$ of the arrangement graph G approaches 3.

Theorem 12. *Let $d > \frac{1}{2}$ and $n \in \mathbb{N}$. Let $\mathscr{A}$ be a simple arrangement of n pseudocircles such that $\delta(H_{\mathscr{A}}) \geq dn$. Then for the arrangement graph G of $\mathscr{A}$, we have $\chi_f(G) \leq \frac{3}{2d-1}$.*

Proof. The proof is similar to the one of Theorem 10, and we borrow the notations from that proof. For a fixed pseudocircle $C \in \mathscr{A}$, we further define $D_C \subseteq V_G \setminus V_C$ as the union of $V_{C'}$ over all $C' \in \mathscr{A}$ for which C' and C are disjoint. Also the following two claims hold for every choice of $C \in \mathscr{A}$, with word-to-word the same proofs as for the according claims in the proof of Theorem 10.

Claim 1. The graph $G - (V_C \cup D_C)$ is 3-colorable.

Claim 2. The graph $G[V_C]$ is 2-colorable.

Claim 3. For every weighting $w : V_G \to [0, \infty)$ there is an independent set I of G such that $w(I) \geq (\frac{1}{3} - \frac{2(1-d)}{3})w(V_G)$.

Proof. Fix $C \in \mathscr{A}$ as a pseudocircle minimizing $w(V_C) + 3w(D_C)$. In the following we fix some notation analogous to the one in the proof of Theorem 10: We denote by I_1, I_2, I_3 and J_1, J_2, J_3 the color classes of a 3-coloring of $G[V_I] - (V_I \cap D_C)$ and $G[V_O] - (V_O \cap D_C)$, respectively (which exist by Claim 1). For $(i, j) \in \{1, 2, 3\}^2$, we denote again $I_{i,j} := I_i \cup J_j$ and by $X_{i,j} \subseteq V_C$ the set of vertices on C with no neighbor in $I_{i,j}$. Let $X^1_{i,j}, X^2_{i,j}$ denote the color classes of a 2-coloring of $G[X_{i,j}] \subseteq G[V_C]$, and define independent sets $I_{i,j,k} := I_{i,j} \cup X^k_{i,j}$ in G for $k = 1, 2$.

Again we let $\mathbf{I}$ denote the random set $I_{i,j,k}$ where (i, j, k) is chosen uniformly at random from $\{1, 2, 3\} \times \{1, 2, 3\} \times \{1, 2\}$.

Every vertex $x \in V_C$ belongs to $X_{i,j}$ for at least 4 different choices of (i, j). If x has neighbors in I_a and J_b, then it belongs to $X_{i,j}$ for $i \in \{1, 2, 3\} \setminus \{a\}$ and $j \in \{1, 2, 3\} \setminus \{b\}$. Therefore,

$$\mathbb{E}(w(X^k_{i,j})) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} \sum_{i',j',k'} w(X^{k'}_{i',j'}) = \frac{1}{18} \sum_{i',j'} w(X_{i',j'}) \geq \frac{1}{18} \cdot 4w(V_C) = \frac{2}{9} w(V_C).$$

This implies that

$$\begin{aligned}\mathbb{E}(w(\mathbf{I})) = \mathbb{E}(w(I_i \cup J_j)) + \mathbb{E}(w(X^k_{i,j})) &\geq \frac{1}{3}(w(V_G) - w(V_C \cup D_C)) + \frac{2}{9} \cdot w(V_C) \\ &= \frac{1}{3} w(V_G) - \frac{1}{9}(w(V_C) + 3w(D_C))\end{aligned}$$

Since C was chosen as a pseudocircle minimizing $w(V_C) + 3w(D_C)$, we have $w(V_C) + 3w(D_C) \leq \frac{1}{n} \sum_{C' \in \mathscr{A}} (w(V_{C'}) + 3w(D_{C'}))$. Let v be a vertex in the intersection of two pseudocircles C_1 and C_2. For $i = 1, 2$ pseudocircle C_i is disjoint from at most $(n-1) - dn = (1-d)n - 1$ other pseudocircles. Hence, v is in at most $2(1-d)n - 2$ sets $D_{C'}$ and we get

$$\begin{aligned}&\sum_{C' \in \mathscr{A}} (w(V_{C'}) + 3w(D_{C'})) \leq \sum_{C' \in \mathscr{A}} w(V_{C'}) + 3 \sum_{C' \in \mathscr{A}} w(D_{C'}) \leq \\ &2 \sum_{v \in V(G)} w(v) + (2(1-d)n - 2)3 \sum_{v \in V(G)} w(v) \leq 6(1-d)n \cdot w(V_G).\end{aligned}$$

Consequently $w(V_C) + 3w(D_C) \leq 6(1-d)w(V_G)$ and $\mathbb{E}(w(\mathbf{I})) \geq (\frac{1}{3} - \frac{2(1-d)}{3})w(V_G)$. This implies the existence of an independent set with total weight at least $(\frac{1}{3} - \frac{2(1-d)}{3})w(V_G)$. $\square$

Just as in the proof of Theorem 10 we express the fractional chromatic number as the optimal value of the linear program $\max \mathbf{1} \cdot w$ subject to $M^T w \leq \mathbf{1},\ w \geq 0$ where M is the incidence matrix of vertices versus independent sets. As previously, Claim 3 now directly yields that $\chi_f(G) \leq 1/(\frac{1}{3} - \frac{2(1-d)}{3}) = \frac{3}{2d-1}$. This concludes the proof of Theorem 12. □

Proposition 13. *Let G be the arrangement graph of a simple arrangement $\mathscr{A}$ of pseudocircles with $\delta(H_{\mathscr{A}}) \geq 2$. Then we have $\frac{|V(G)|}{\alpha(G)} \leq 3 + \frac{3}{\delta(H_{\mathscr{A}})-1}$.*

Proof. Let C_0 and C_1 be pseudocircles not belonging to $\mathscr{A}$, such that C_0 contains all pseudocircles of $\mathscr{A}$ and C_1 in its exterior, while C_1 has all pseudocircles in $\mathscr{A}$ and C_0 in its interior. By the Sweeping Theorem of Snoeyink and Hershberger [SH91, Theorem 3.1] there is a linear ordering $v_1, \ldots, v_{|V(G)|}$ of the vertices of G such that each pseudocircle $C \in \mathscr{A}$ contains a unique vertex $v_C \in V_C$ with precisely 2 predecessors on C in this ordering, while all vertices in $V_C \setminus \{v_C\}$ are preceded by at most one other vertex on C. It is now clear that the graph $G' := G - \{v_C | C \in \mathscr{A}\}$ is 2-degenerate (since in the induced acyclic orientation of G', every vertex has at most one in-edge on each of its two circles, and so the maximum in-degree in this orientation is at most 2). Hence G' is properly 3-colorable by the greedy algorithm. Thus, $\alpha(G) \geq \alpha(G') \geq \frac{1}{3}(|V(G)| - |\mathscr{A}|)$. Since $\delta(H_{\mathscr{A}}) = k \geq 2$, every pseudocircle contains at least $2k$ vertices, and hence we have $|V(G)| \geq k|\mathscr{A}|$. We finally conclude that

$$\alpha(G) \geq \frac{1}{3}\left(|V(G)| - \frac{1}{k}|V(G)|\right) = \frac{k-1}{3k}|V(G)|.$$

□

5 Fractionally 3-colorable 4-edge-critical planar graphs

On the basis of the *database of pseudocircles* [FS] we could compute χ and χ_f exhaustively for small arrangements[4]. We found the arrangement depicted in Figure 1(b) with $\chi = 4$ and $\chi_f = 3$. This is a counterexample to Conjecture 3.2 in Gimbel et al. [GKLT19].

Extending the experiments to small 4-regular planar graphs we found that there are precisely 17 4-regular planar graphs on 18 vertices with $\chi = 4$ and $\chi_f = 3$. They are minimal in the sense that there are no 4-regular graphs on $n \leq 17$ vertices with $\chi = 4$ and $\chi_f = 3$. Each of these 17 graphs is 4-vertex-critical and the one depicted in Figure 15(a) is even 4-edge-critical.

Starting with a triangular face in the 4-edge-critical 4-regular graph of Figure 15(a) and repeatedly applying Koester's crowning operation as illustrated in Figure 15(b) (which by definition preserves the existence of a facial triangle), we can deduce the following theorem.

Theorem 14. *There exists an infinite family of* 4*-edge-critical* 4*-regular planar graphs G with fractional chromatic number $\chi_f(G) = 3$.*

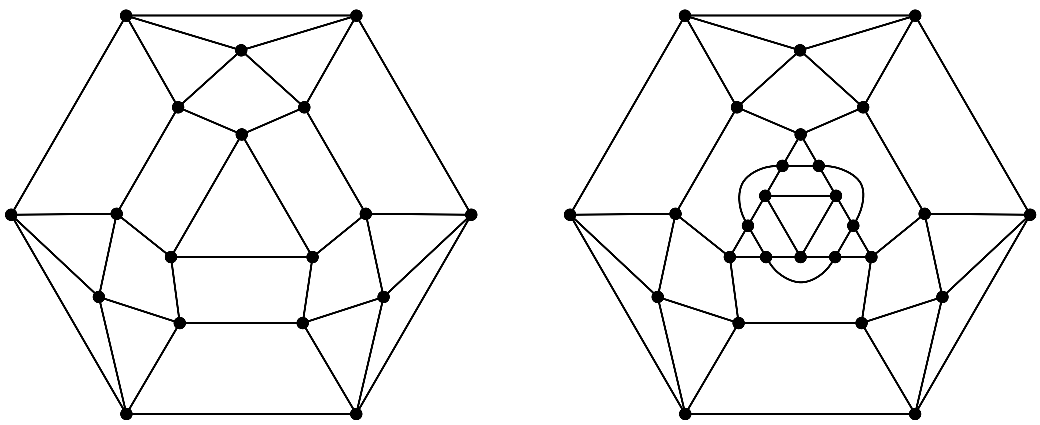
We prepare the proof of the above result with some background on Koester's crowing operation from [Koe90]. For a 4-regular plane graph G and a face $\pentagon$ of odd degree in G, we denote by *Crown*$(G, \pentagon)$ the plane graph obtained by applying the crowning operation to $\pentagon$ in G. Figure 14 shows how to apply the crowning to a triangle and a pentagon respectively, the general case should be deducible. Koester proved the following:

[4] Computing the fractional chromatic number of a graph is NP-hard in general [LY94]. For our computations we formulated a linear program which we then solved using the MIP solver Gurobi.

Figure 14: Crowning of a triangle and a pentagon.

Proposition 15 ([Koe90]). *Let G be a 4-regular plane graph with a facial triangle T. If G is 4-edge-critical, then so is $Crown(G,T)$.*

Via the following lemma, we can use Koester's crowning operation to extend the example from Figure 15(a) to an infinite family of 4-regular 4-edge-critical planar graphs with fractional chromatic number 3.



(a) (b)

Figure 15: (a) A 4-edge-critical 4-regular 18-vertex planar graph with $\chi = 4$ and $\chi_f = 3$ and (b) the crowning extension at its center triangular face.

Lemma 16. *Let G be a 4-regular plane graph with a facial triangle T. If $\chi_f(G) = 3$, then $\chi_f(Crown(G,T)) = 3$.*

Proof. If $\chi_f(G) = 3$, then it follows from the representation of $\chi_f(G)$ as the optimal value of a rational linear program that there exists $b \in \mathbb{N}$ such that G has a b-coloring using $3b$ colors. For every vertex $v \in V(G)$, let $c(v) \in \binom{[3b]}{b}$ be the assigned sets of colors. Let $T = uvw$, then we know that $c(u), c(v), c(w)$ must be pairwise disjoint and hence form a partition of $\{1, \ldots, 3b\}$. Let $c(u) = A_1$, $c(v) = A_2$, and $c(w) = A_3$. It is easy to see that the subgraph of $Crown(G,T)$ induced by the vertices u, v, w and the nine new vertices in $V(Crown(G,T)) \setminus V(G)$ is 3-colorable such that the colors of u, v, w are pairwise distinct. By appropriately replacing the 3 colors by A_1, A_2, A_3 we obtain a b-coloring of $Crown(G,T)$ with $3b$ colors. This proves $\chi_f(Crown(G,T)) \leq 3$, now $\chi_f(Crown(G,T)) = 3$ follows because $Crown(G,T)$ contains a triangle. □

Starting with a facial triangle in the 4-regular 4-edge-critical graph of Figure 15 and repeating the crowning operation (which by definition preserves the existence of a facial triangle), by Lemma 16 and Proposition 15 we obtain an infinite family of 4-edge-critical 4-regular planar graphs G with fractional chromatic number $\chi_f(G) = 3$. This proves Theorem 14.

6 Discussion

With Theorem 2 we gave a proof of Conjecture 1 for △-saturated great-pseudocircle arrangements. While this is a very small subclass of great-pseudocircle arrangements, it is reasonable to think of it as a "hard" class for 3-coloring. The rationale for such thoughts is that triangles restrict the freedom of extending partial colorings. Our computational data indicates that sufficiently large intersecting pseudocircle arrangements that are *diamond-free*, i.e., no two triangles of the arrangement share an edge, are also 3-colorable. Computations also suggest that sufficiently large great-pseudocircle arrangements have *antipodal colorings*, i.e., 3-colorings where antipodal points have the same color. Based on the experimental data we propose the following strengthened variants of Conjecture 1.

Conjecture 17. *The following three statements hold:*

(a) Every simple diamond-free intersecting arrangement of $n \geq 6$ pseudocircles is 3-colorable.

(b) Every simple intersecting arrangement of sufficiently many pseudocircles is 3-colorable.

(c) Every simple arrangement of $n \geq 7$ great-pseudocircles has an antipodal 3-coloring.

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