Coloring Circle Arrangements: New 4-Chromatic Planar Graphs*

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Abstract

Felsner, Hurtado, Noy and Streinu (2000) stated a conjecture that the arrangement graphs of great-circle arrangements have chromatic number at most 3. In this paper, we prove results related to this conjecture.

We show that the conjecture holds in the special case when the arrangement is $\triangle$-*saturated*, i.e., when one color class of the bipartite dual of the arrangement consists of triangles only. Moreover, we extend $\triangle$-saturated arrangements with certain properties to a family of arrangements which are 4-chromatic. Our construction generalizes Koester's construction from 1985.

Last but not least we investigate fractional colorings. We show that arrangements $\mathcal{A}$ of pairwise intersecting pseudocircles are "close" to being 3-colorable by proving that $\chi_f(\mathcal{A}) \leq 3+O(\frac{1}{n})$ where n is the number of pseudocircles. We further construct an infinite family of 4-edge-critical 4-regular planar graphs which are fractionally 3-colorable. This disproves a conjecture by Gimbel, Kündgen, Li and Thomassen (2019) that every 4-chromatic planar graph has fractional chromatic number strictly greater than 3.

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1 Introduction

An arrangement of pseudocircles is a family of simple closed curves on the sphere or in the plane such that each pair of curves intersects at most twice. Similarly, an arrangement of pseudolines is a family of x-monotone curves such that every pair of curves intersects exactly once. An arrangement is *simple* if no three pseudolines/pseudocircles intersect in a common point and *intersecting* if every pair of pseudolines/pseudocircles intersects. Given an arrangement of pseudolines/pseudocircles, the *arrangement graph* is the planar graph obtained by placing vertices at the intersection points of the arrangement and thereby subdividing the pseudolines/pseudocircles into edges.

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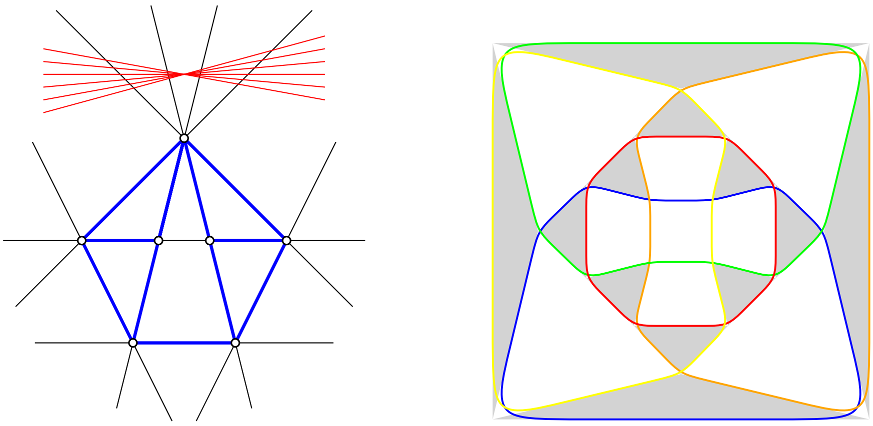
A *(proper) coloring* of a graph assigns a color to each vertex such that no two adjacent vertices have the same color. The *chromatic number* χ is the smallest number of colors needed for a proper coloring. Since both the well-known 4-color theorem and also Brook's theorem imply the 4-colorability of planar graphs with maximum degree 4, the major question is: which arrangement graphs require 4 colors in any proper coloring?

There exist arbitrarily large non-simple line arrangements that require 4 colors; see Figure 1(a). Using a line – great-circle transformation, one gets non-simple arrangements of great-circles with $\chi = 4$. Koester [11] presented a simple arrangement of 7 circles with $\chi = 4$ in which all but one pair of circles intersect, see Figure 3(b). Moreover, there are simple intersecting arrangements that require 4 colors but we do not have an infinite family. It is therefore natural to ask which simple intersecting arrangements of pseudocircles could possibly be 3-colorable.



(a) (b)

Figure 1 (a) A construction of non-simple line arrangements with $\chi = 4$ which contains a Moser spindle as subgraph (highlighted blue). The red subarrangement not intersecting the Moser spindle can be chosen arbitrarly. (b) A simple intersecting arrangement of 5 pseudocircles with $\chi = 4$ and $\chi_f = 3$.

In 2000, Felsner, Hurtado, Noy and Streinu [3] studied various properties of arrangement graphs of pseudoline and pseudocircle arrangements, showing multiple interesting results concerning connectivity, Hamiltonicity, and colorability of those graphs. In this work, they also stated the following conjecture:

▶ **Conjecture 1** (Felsner et al. [3]). *The arrangement graph of every simple arrangement of great-circles is 3-colorable.*

While the conjecture is fairly well known (cf. [13, 9, 17] and [18, Chapter 17.7]) there has been almost no progress in the last 20 years. Aichholzer, Aurenhammer, and Krasser verified the conjecture for up to 11 great-circles [12, Chapter 4.6.4].

Results and outline

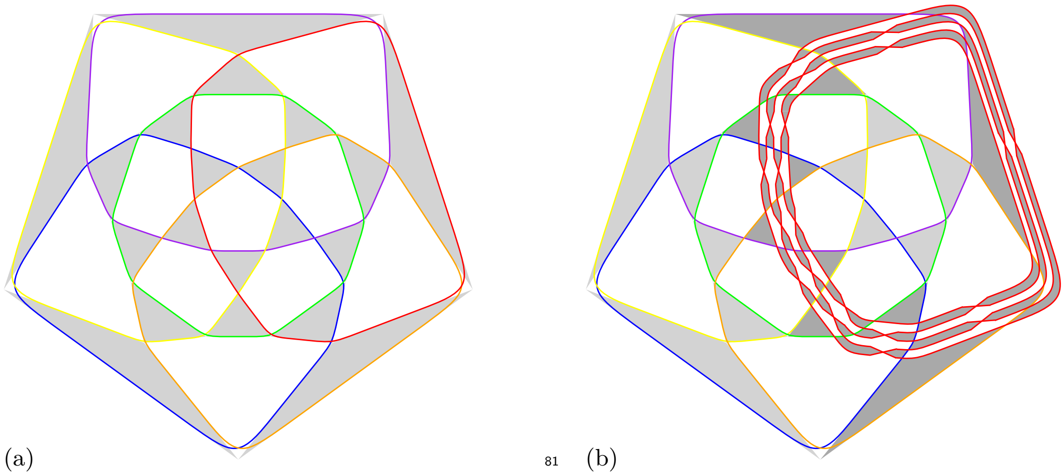
In Section 2 we show that Conjecture 1 holds for △-*saturated* arrangements of pseudocircles, i.e., arrangements where one color class of the 2-coloring of faces consists of triangles only. In Section 3 we extend our study of △-saturated arrangements and present an infinite family of arrangements which require 4 colors. Our construction generalizes Koester's [11] arrangement of 7 circles which requires 4 colors; see Figure 3(b). Moreover, we believe that the construction results in infinitely many 4-vertex-critical arrangement graphs. Koester [11] obtained his example using a "crowning" operation, this operation actually yields infinite families of 4-regular planar 4-critical graphs, however, except for the 7 circles example these graphs are not arrangement graphs.

In Section 4 we investigate the fractional chromatic number χ_f of arrangement graphs. This variant of the chromatic number is the objective value of the linear relaxation of the ILP formulation for the chromatic number. We show that intersecting arrangements of pseudocircles are “close” to being 3-colorable by proving that $\chi_f(\mathcal{A}) \leq 3 + O(\frac{1}{n})$ where n is the number of pseudocircles of $\mathcal{A}$. In Section 5, we present an example of a 4-edge-critical arrangement graph which is fractionally 3-colorable, and use this as a basis for constructing an infinite family of 4-regular planar graphs with the same property. This family disproves Conjecture 3.2 by Gimbel, Kündgen, Li and Thomassen [6] that every 4-chromatic planar graph has fractional chromatic number strictly greater than 3.

Last but not least, we summarize our computational data, report on some new discoveries related to Conjecture 1, and present strengthened versions of Conjecture 1 in Section 6.

2 △-saturated arrangements are 3-colorable

The maximum number of triangles in arrangements of pseudolines and pseudocircles has been studied intensively, see e.g. [7, 14, 2] and [5]. By recursively applying the "doubling method" Harborth [8] and also [14, 2] proved the existence of infinite families of △-saturated arrangements of pseudolines. A doubling construction for pseudocircle arrangements similarly yields infinitely many △-saturated arrangements of great-pseudocircles. Figure 2 illustrates the doubling method applied to an arrangement of great-pseudocircles. It will be relevant later that arrangements obtained via doubling contains pentagonal cells. Note that for $n \equiv 2 \pmod 3$ there is no △-saturated intersecting pseudocircle arrangement because the number of edges of the arrangement graph is not divisible by 3.



(a) (b)

Figure 2 The doubling method applied to an arrangements of 6 great-pseudocircle. The red pseudocircle is replaced by a cyclic arrangement. Triangular cells are shaded gray.

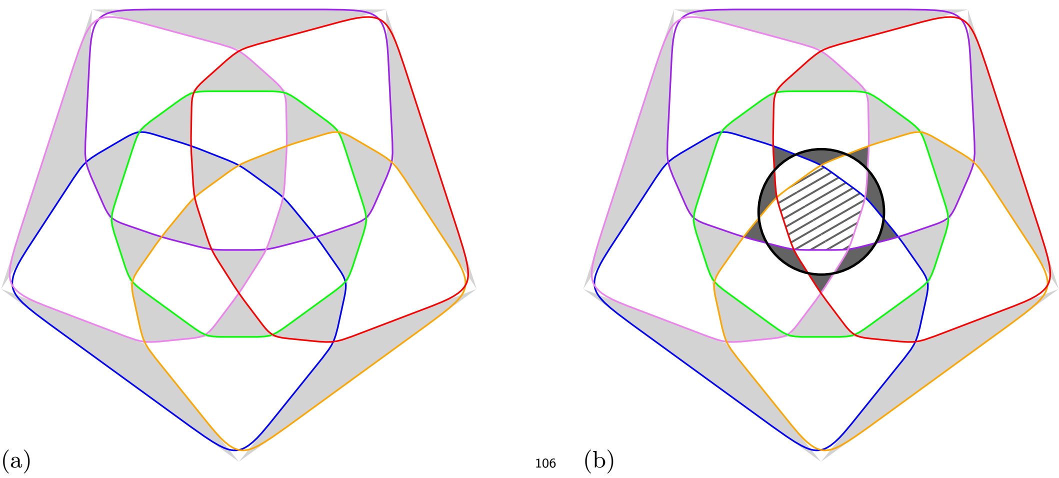
▶ **Theorem 2.** *Every △-saturated arrangement $\mathcal{A}$ of pseudocircles is 3-colorable.*

Proof. Let H be a graph whose vertices correspond to the triangles of $\mathcal{A}$ and whose edges correspond to pairs of triangles sharing a vertex of $\mathcal{A}$. This graph H is planar, 3-regular and bridgeless. Hence, Tait's theorem, a well known equivalent of the 4-color theorem, asserts that H is 3-edge-colorable, see e.g. [1] or [16]. The edges of H correspond bijectively to the vertices of the arrangement $\mathcal{A}$ and, since adjacent vertices of $\mathcal{A}$ are incident to a common triangle, the corresponding edges of H share a vertex. This shows that the graph of $\mathcal{A}$ is 3-colorable. ◀

3 Constructing 4-chromatic arrangement graphs

In this section, we describe an operation that extends any △-saturated intersecting arrangement of pseudocircles with a pentagonal cell (which is 3-colorable by Theorem 2), to a 4-chromatic arrangement of pseudocircles by inserting one additional pseudocircle. This operation generalizes the non-4-colorable arrangement graph constructed by Koester.

The corona extension We start with a △-saturated arrangement of pseudocircles which contains a pentagonal cell ⬠. By definition, in the 2-coloring of the faces one of the two color classes consists of triangles only; see e.g. the arrangement from Figure 3(a). Since the arrangement is △-saturated, the pentagonal cell ⬠ is surrounded by triangular cells. As illustrated in Figure 3(b) we can now insert an additional pseudocircle close to ⬠. This newly inserted pseudocircle intersects only the 5 pseudocircles which bound ⬠, and in the so-obtained arrangement one of the two dual color classes consists of triangles plus the pentagon ⬠. It is interesting to note that the arrangement depicted in Figure 3(b) is precisely Koester's arrangement [10, 11].



(a) (b)

Figure 3 (a) A △-saturated arrangement of 6 great-circles and (b) the corona extension at its central pentagonal face. The arrangement in (b) is Koester's [10] example of a 4-critical 4-regular planar graph.

The following proposition plays a central role in this section. Due to space constraints, we defer its proof to Appendix A.

▶ **Proposition 3.** *The corona extension of a △-saturated arrangement of pseudocircles with a pentagonal cell ⬠ is 4-chromatic.*

By applying the corona extension to members of the infinite family of △-saturated arrangements with pentagonal cells (cf. Section 2), we obtain an infinite family of arrangements that are not 3-colorable.

▶ **Theorem 4.** *There exists an infinite family of 4-chromatic arrangements of pseudocircles.*

Koester [11] defines a related construction which he calls *crowning* and constructs his example by two-fold crowning of a graph on 10 vertices. He also uses crowning to generate an infinite family of 4-regular 4-critical graphs. In the full version of our paper, we will present sufficient conditions to obtain a 4-vertex-critical arrangement via the corona extension. We conclude this section with the following conjecture:

▶ **Conjecture 5.** *There exists an infinite family of arrangement graphs of arrangements of pseudocircles that are 4-vertex-critical.*

4 Fractional colorings

In this section, we investigate fractional colorings of arrangements. A *b-fold coloring* of a graph G with m colors is an assignment of a set of b colors from $\{1, \ldots, m\}$ to each vertex of G such that the color sets of any two adjacent vertices are disjoint. The *b-fold chromatic number* $\chi_b(G)$ is the minimum m such that G admits a b-fold coloring with m colors. The *fractional chromatic number* of G is $\chi_f(G) := \lim_{b\to\infty} \frac{\chi_b(G)}{b} = \inf_b \frac{\chi_b(G)}{b}$. With α being the independence number and ω being the clique number, the following inequalities holds:

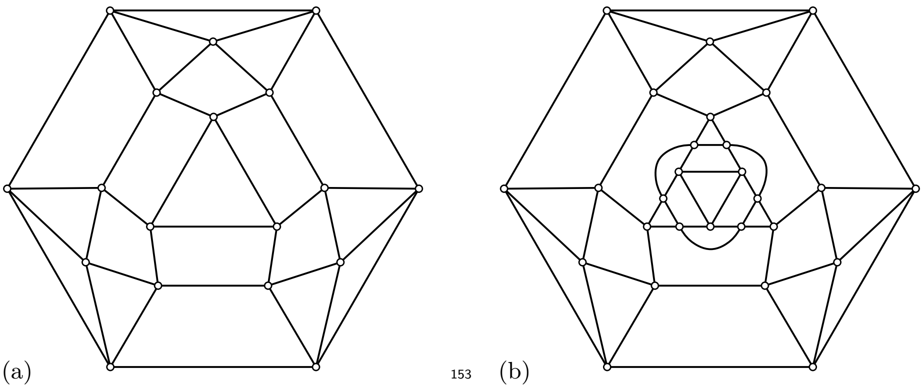
$$\max\left\{\frac{|V|}{\alpha(G)}, \omega(G)\right\} \leq \chi_f(G) \leq \frac{\chi_b(G)}{b} \leq \chi(G). \tag{1}$$

▶ **Theorem 6.** *Let G be the arrangement graph of an intersecting arrangement $\mathcal{A}$ of n pseudocircles, then $\chi_f(G) \leq 3 + \frac{6}{n-2}$.*

Sketch of the proof. Let C be a pseudocircle of $\mathcal{A}$. After removing all vertices along C from the arrangement graph G we obtain a graph which has two connected components A (vertices in the interior of C) and B (vertices in the exterior). Let C' be a small circle contained in one of the faces of A, the Sweeping Lemma of Snoeyink and Hershberger [15] asserts that there is a continuous transformation of C' into C which traverses each vertex of A precisely once. In particular, when a vertex is traversed, at most two of its neighbors have been traversed before. Hence, we obtain a 3-coloring of the vertices of A by greedily coloring vertices in the order in which they occur during the sweep. An analogous argument applies to B. Taking such a partial 3-coloring of G for each of the n pseudocircles of $\mathcal{A}$, we obtain for each vertex a set of $n-2$ colors, i.e., an $(n-2)$-fold coloring of G. The total number of colors used is $3n$. The statement now follows from inequality (1). ◀

5 4-edge-critical planar graphs which are fractionally 3-colorable

From our computational data (cf. [4]) , we observed that some of the arrangements such as the 20 vertex graph depicted in Figure 1(b) have $\chi = 4$ and $\chi_f = 3$, and therefore disprove Conjecture 3.2 by Gimbel et al. [6]. Moreover, we determined that there are precisely 17 4-regular 18-vertex planar graphs with $\chi = 4$ and $\chi_f = 3$, which are minimal in the sense that there are no 4-regular graphs on $n \leq 17$ vertices with $\chi = 4$ and $\chi_f = 3$. Each of these 17 graphs is 4-vertex-critical and the one depicted in Figure 4(a) is even 4-edge-critical.



(a) (b)

Figure 4 (a) A 4-edge-critical 4-regular 18-vertex planar graph with $\chi = 4$ and $\chi_f = 3$ and (b) the crowning extension at its center triangular face.

Starting with a triangular face in the 4-regular 4-edge-critical graph depicted in Figure 4(a) and repeatedly applying the Koester's crowning operation [11] as illustrated in Figure 4(b) (which by definition preserves the existence of a facial triangle), we deduce the following theorem, a formal proof of which is found in Appendix B.

▶ **Theorem 7.** *There exists an infinite family of 4-critical 4-regular planar graphs G with fractional chromatic number $\chi_f(G) = 3$.*

6 Discussion

With Theorem 2 we gave a proof of Conjecture 1 for △-saturated great-pseudocircle arrangements. While this is a very small subclass of great-pseudocircle arrangements it is reasonable to think of it as a class which is hard for 3-coloring. The rational for such thoughts is that triangles restrict the freedom of extending partial colorings. Our computational data indicates that sufficiently large intersecting pseudocircle arrangements that are *diamond-free*, i.e., no two triangles of the arrangement share an edge, are also 3-colorable. Computations also suggest that sufficiently large great-pseudocircle arrangements have *antipodal colorings*, i.e., 3-colorings where antipodal points have the same color. Based on the experimental data we propose the following strengthened variants of Conjecture 1.

▶ **Conjecture 8.** *The following three statements hold.*

(a) Every diamond-free intersecting arrangement of $n \geq 6$ pseudocircles is 3-colorable.
(b) Every intersecting arrangement of sufficiently many pseudocircles is 3-colorable.
(c) Every arrangement of $n \geq 7$ great-pseudocircles has an antipodal 3-coloring.

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A Proof of Proposition 3

The 4-colorability of corona extensions follows from the following lemma and inequality (1).

▶ **Lemma 9.** *Let G be a 4-regular planar graph. If in the 2-coloring of the faces of G, one of the classes consists of only triangles and a single pentagon, then* $\alpha(G) < \frac{|V(G)|}{3}$.

Proof. Color the faces of $G = (V, E)$ with black and white. Let the black class contain only triangles and one pentagon. Let $\triangle$ be the number of these triangles and let $\alpha := \alpha(G)$. Given an independent set I of cardinality α, we count the number of pairs (v, F), where v is a vertex of I and F is a black face of $\mathcal{A}$ incident to v. There are 2 such faces for every $v \in I$, hence, 2α pairs in total. Since any independent set of G contains at most one vertex of of each triangle and at most two vertices of the pentagon, we have

$$2\alpha \leq \triangle + 2. \quad (2)$$

Since G is 4-regular, it has exactly $|E| = 2|V|$ edges. As every edge is incident to exactly one black face, we also have $|E| = 3\triangle + 5$. This yields the equation

$$3\triangle + 5 = 2|V|. \quad (3)$$

From equation (3), we conclude that $\triangle$ is odd. Therefore we can strengthen equation (2) to

$$2\alpha \leq \triangle + 1. \quad (4)$$

Combining equations (3) and (4) then gives $6\alpha \leq 3\triangle + 3 = 2|V| - 2$ and hence $\alpha < |V|/3$. ◀

B Proof of Theorem 7

When he invented the crowning operation, Köster [11] proved the following.

▶ **Proposition 10.** *Let G be a 4-regular plane graph with a facial triangle T. If G is 4-edge-critical, then so is $G \circ T$.*

We further have the following observation.

▶ **Lemma 2.1.** *Let G be a 4-regular plane graph with a facial triangle T. If $\chi_f(G) = 3$, then $\chi_f(G \circ T) = 3$.*

Proof. Suppose $\chi_f(G) = 3$, then it follows from the representation of $\chi_f(G)$ as the optimal value of a rational linear program that there exists $b \in \mathbb{N}$ such that G has a $(3b, b)$-coloring. For every vertex $v \in V(G)$, let $c(v) \in \binom{[3b]}{b}$ be the assigned sets of colors. Let $T = uvw$, then we know that $c(u), c(v), c(w)$ must be pairwise disjoint and hence form a partition of $\{1, \ldots, 3b\}$. Therefore, possibly after relabelling the colors, we may assume that $c(u) = \{1, \ldots, b\} =: A_1, c(v) = \{b+1, \ldots, 2b\} =: A_2, c(w) = \{2b+1, \ldots, 3b\} =: A_3$. It is easy to see that the subgraph of $G \circ T$ induced by the vertices u, v, w and the nine new vertices in $V(G \circ T) \setminus V(G)$ is properly 3-vertex-colorable and hence admits a proper coloring with the color-set $\{A_1, A_2, A_3\}$ such that u, v and w are assigned, respectively, the colors A_1, A_2, A_3. It is now obvious that joining this coloring of the nine additional vertices to the coloring c of G defines a $(3b, b)$-coloring of $G \circ T$. This proves $\chi_f(G \circ T) \leq 3$, and clearly $\chi_f(G \circ T) \geq 3$ since $G \circ T$ contains a triangle. ◀

Starting with a facial triangle in the 4-regular 4-edge-critical graph depicted in Figure 4 and repeating the crowning operation (which by definition preserves the existence of a facial triangle), we deduce the theorem.