Bichromatic Perfect Matchings with Crossings*

Oswin Aichholzer[1], Stefan Felsner[2], Rosna Paul[1], Manfred Scheucher[2], and Birgit Vogtenhuber[1]

1 **Institute of Software Technology, Graz University of Technology, Austria**
`oaich,ropaul,bvogt@ist.tugraz.at`

2 **Institute for Mathematics, Technical University of Berlin, Germany**
`felsner,scheucher@math.tu-berlin.de`

Abstract

We consider bichromatic point sets with n red and n blue points and study straight-line bichromatic perfect matchings on them. We show that every such point set in convex position admits a matching with at least $\frac{3n^2}{8} - O(n)$ crossings. Moreover, this bound is asymptotically tight since for any $k > \frac{3n^2}{8}$ there exist bichromatic point sets that do not admit any perfect matching with k crossings.

1 Introduction

Let $P = R \cup B$, $|R| = |B| = n$ be a point set in *general position*, that is, no three points of P are collinear. We refer to R and B as the set of red and blue points, respectively. A straight-line matching M of P where every point in R is uniquely matched to a point in B is called a *straight-line bichromatic perfect matching* (all matchings considered in this work are straight-line, so we will mostly omit this term). In this work, we study the existence of bichromatic perfect matchings with a fixed number k of crossings on P, where $0 \leq k \leq \binom{n}{2}$. It is well known and easy to see that there exists a crossing-free (that is, $k = 0$) such matching for every P [7]. Perfect matchings with k crossings on uncolored point sets have been considered in [2]. There it is shown that for every $k \leq \frac{n^2}{16} - O(n\sqrt{n})$, every point set of size $2n$ admits a perfect matching with exactly k crossings and that there exist such point sets where every perfect matching has fewer than $\frac{20n^2}{72}$ crossings. As a direct consequence, there exist bichromatic point sets which do not admit bichromatic perfect matchings with k crossings for $k > \frac{20n^2}{72}$. On the other hand, $2n$ uncolored points in convex position admit perfect matchings with k crossings for all k, where $0 \leq k \leq \binom{n}{2}$ [2]. But when we color the points, the situation changes quite drastically.

Consider a point set P of $2n$ points in convex position (convex point set, for short) with an alternating coloring, that is, every second point along the convex hull is red (and the other points are blue). Moreover, let the number n of red (and blue) points be even. Then the number of crossings in a bichromatic perfect matching M on P is at most $\frac{n(n-2)}{2} = \binom{n}{2} - \frac{n}{2}$. The idea is as follows: Label the points of P as $p_0, p_1, \ldots, p_{2n-1}$ along the boundary of the convex hull. The point p_i cannot be matched to p_{i+n} since both points are of the same color. Hence, for any edge e in such a matching M of P, the number of crossings of e is at most $n-2$. As every crossing involves two edges, the number of crossings in M is at most $\frac{n(n-2)}{2} = \binom{n}{2} - \frac{n}{2}$. This bound is tight, since it is possible to construct a bichromatic perfect matching M on P with exactly $\binom{n}{2} - \frac{n}{2}$ crossings as follows. For $0 \leq i \leq n-1$, match the point p_i to the point p_{i+n+1}, when i is even. Otherwise, match p_i to p_{i+n-1}. Together this leads to the following question.

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▶ **Open Problem 1.** *For which values of k does every bichromatic convex point set $P = R \cup B$, $|R| = |B| = n$, admit a straight-line bichromatic perfect matching with exactly k crossings?*

The above example implies that if $k > \binom{n}{2} - \frac{n}{2}$, there exist bichromatic point sets with n red and n blue points that do not have any bichromatic perfect matching with k crossings. Thus, Open Problem 1 can be true only for $k \leq \binom{n}{2} - \frac{n}{2}$. In this paper, we further improve the bound on k as follows.

▶ **Theorem 1.1.** *For any $k > \frac{3n^2}{8}$, there exists a bichromatic convex point set with n red and n blue points that does not have a straight-line bichromatic perfect matching with k crossings.*

Related work: A survey by Kano and Urrutia [6] gives an overview of various problems on bichromatic point sets, including matching problems. Crossing-free bichromatic perfect matchings have been studied from various perspectives such as their structure [5, 8], linear transformation distance [1], and matchings compatible to each other [3, 4]. Sharir and Welzl [9] proved that the number of crossing-free bichromatic perfect matchings on $2n$ points is at most $O(7.61^n)$. However, not much is known about the number or existence of bichromatic perfect matchings with k crossings, for $k > 0$.

2 Bichromatic Convex Point Sets

Let $\mathcal{C}_{n,n}$ be the collection of all bichromatic convex point sets $P = R \cup B$ with $|R| = |B| = n$. For a point set $P \in \mathcal{C}_{n,n}$, we label the points in P in clockwise direction along the convex hull as $p_0, p_1, \ldots, p_{2n-1}$ and refer to this as the *clockwise ordering*. We will consider all indices modulo $2n$. The number of crossings in any birchromatic perfect matching M_P of P is denoted by $\overline{\text{cr}}(M_P)$. If M_P has the maximum number of crossings among all such matchings of P, then it is called a *max-crossing* matching on P. Among all max-crossing matchings for all $P \in \mathcal{C}_{n,n}$, we are interested in max-crossing matchings which have the minimum number of crossings. We call such a matching a *min-max-crossing matching* of $\mathcal{C}_{n,n}$. Further, from now on, we refer to bichromatic perfect matchings just as *matchings*, unless otherwise stated.

For $P \in \mathcal{C}_{n,n}$ we define the collection of all points of the same color which are consecutive in clockwise order as a *block*. For example, if $R_1 = \{p_a, p_{a+1}, \ldots, p_{a+s}\}$ is a block of red points, then p_{a-1} and p_{a+s+1} are blue, and the next collection will be a block of blue points including p_{a+s+1}. By repeating the process along the boundary of the convex hull of P, we get a collection of blocks $\{R_1, B_1, R_2, B_2, \ldots, R_s, B_s\}$ w.r.t. the clockwise order such that $|R_1 \cup R_2 \cup \ldots \cup R_s| = |B_1 \cup B_2 \cup \ldots \cup B_s| = n$. These blocks are non-empty and alternate in color. A bichromatic convex point set with the collection of blocks $\{R_1, B_1, R_2, B_2, \ldots, R_s, B_s\}$ is called a $2s$*–block* coloring. For example, a $2n$–block coloring is an alternating coloring. If all $2s$ blocks have the same cardinality, then the coloring is called a *balanced* $2s$*–block* coloring. See Figure 2(a) for a bichromatic point set with a balanced 4–block coloring. Strictly speaking, a balanced 4–block coloring can only be achieved if n is even. If n is odd, the cardinalities of some blocks differ by at least ± 1. We call a 4–block coloring with maximum cardinality difference of ± 1 *nearly balanced*. Considering the properties that we discuss in this paper, nearly balanced 4–block colorings and balanced 4–block colorings behave very similarly. So abusing the terminology a bit, we will mostly refer to all of them as balanced 4–block colorings. With these definitions we can now formulate a stronger version of Theorem 1.1.

▶ **Theorem 2.1.** *Let $P \in \mathcal{C}_{n,n}$ have a balanced 4–block coloring and let $\mathrm{M}_P^{\vee}$ be a max-crossing matching of P. Then $\overline{\text{cr}}(\mathrm{M}_P^{\vee}) = \frac{3n^2}{8} - O(n)$. Moreover, $\mathrm{M}_P^{\vee}$ is a min-max-crossing matching of the set $\mathcal{C}_{n,n}$.*

Clearly, Theorem 2.1 implies Theorem 1.1, as any bichromatic convex point set with a balanced 4–block coloring satisfies Theorem 1.1. Theorem 2.1 will follow from Lemma 2.4 and Lemma 2.5 below which are stated and shown in the next sections.

2.1 Min-Max-Crossing Matching for 4–block colorings

In this section, we construct a min-max-crossing matching among all bichromatic convex point sets with a 4–block coloring.

▶ **Lemma 2.2.** *Let $P \in \mathcal{C}_{n,n}$ have a 4–block coloring with blocks R_1, B_1, R_2, B_2 and let $\mathrm{M}_P^\vee$ be a max-crossing matching on P. Then for any block $X \in \{R_1, R_2, B_1, B_2\}$, the edges emanating from X form a* crossing family*, that is, every pair of these edges forms a crossing.*

Proof. Let e and f be two edges of $\mathrm{M}_P^\vee$ such that $e = r_i b_{i'}$ and $f = r_j b_{j'}$ where $r_i, r_j \in X$ with $i < j$. For the sake of contradiction, assume that e and f do not cross. By replacing e and f by $e' = r_i b_{j'}$ and $f' = r_j b_{i'}$, we get a new matching, say M'_P. As all points are in convex position, the edges crossing e and f will also cross the new edges e' and f'. Also, the edges crossing exactly one of e and f will cross exactly one of e' and f'. In addition, e' and f' cross each other. Hence the number of crossings of M'_P is strictly larger than the number of crossings of $\mathrm{M}_P^\vee$, a contradiction. ◀

Recall the clockwise ordering of the points in P. Without loss of generality, assume that $R_1 = \{p_1, p_2, \ldots, p_{|R_1|}\}$. Let $\mathrm{M}_P^\vee$ be a max-crossing matching of P. Assume that in $\mathrm{M}_P^\vee$, the point $p_i \in R_1$ is matched to a point (say p'_i) in B_2, and $p_j \in R_1$ is matched to a point (say p'_j) in B_1 such that $i < j$. By the clockwise ordering of P, $i < j < j' < i'$ and hence $p_i p'_i$ does not cross $p_j p'_j$. This is a contradiction by Lemma 2.2. Thus, in $\mathrm{M}_P^\vee$, if $p_i \in R_1$ is matched to a point in B_2, then all $p_j \in R_1$ with $i < j$ must be matched to a point in B_2. More precisely, there exists an integer a_1 for R_1 such that all $p_i \in R_1$ with $i \le a_1$ are matched to B_1 and all $p_i \in R_1$ with $i > a_1$ are matched to B_2. In a similar way we can show that all $p_i \in B_1$ with $i \le |R_1| + |B_1| - a_1$ have to be matched to R_2 and the remaining unmatched points have to be matched to R_1. In other words, there exists an integer a_1 such that first a_1 points of R_1 are matched to the last a_1 points of B_1 (as a crossing family). The remaining last $|R_1| - a_1$ points of R_1 have to be matched to the first $|R_1| - a_1$ points of B_2 (as a crossing family). This fixes the remaining edges of $\mathrm{M}_P^\vee$. That is, the remaining last $n - |B_1| - |R_1| + a_1$ points of B_2 must be matched, as a crossing family, to the first $n - |B_1| - |R_1| + a_1$ points of R_2 (see Figure 1). Finally, match the remaining points as a crossing family. Hence, to get a max-crossing matching on P, it is sufficient to determine the optimal value of a_1.

▶ **Lemma 2.3.** *Let $P \in \mathcal{C}_{n,n}$ have a 4–block coloring with blocks R_1, B_1, R_2, and B_2. Let M_P be a matching on P such that the first x points of the set R_1 are matched to the last x points of B_1, as a crossing family. Then M_P is a max-crossing matching on P iff $x = \frac{1}{2}(|R_1| + |B_1| - \frac{n}{2})$.*

Proof. Consider a matching M_P with the above property. Assume that $|R_1| = r_1 \ge \frac{n}{2}$ and $|B_1| = b_1 \ge \frac{n}{2}$. The number of pairs of non-crossing edges in M_P is obtained by $(r_1 - x)(b_1 - x) + x(n - r_1 - b_1 + x) = r_1 b_1 - 2x b_1 - 2x r_1 + nx + 2x^2$. As we want to find the value of x that gives the maximum number of crossings, we calculate the value of x such that $f(x) = (n - 2r_1 - 2b_1)x + 2x^2 + r_1 b_1$ attains the minimum. This is achieved by setting the first derivative of $f(x)$ to zero as $f''(x) > 0$. We get $f'(x) = 0$ for $x = \frac{1}{2}(r_1 + b_1 - \frac{n}{2})$. ◀

By the proof of Lemma 2.3 the minimum number of pairs of non-crossing edges in $\mathrm{M}_P^\vee$ is $r_1 b_1 - \frac{1}{2}(r_1 + b_1 - \frac{n}{2})^2$. This gives the exact structure and the number of crossings in $\mathrm{M}_P^\vee$.

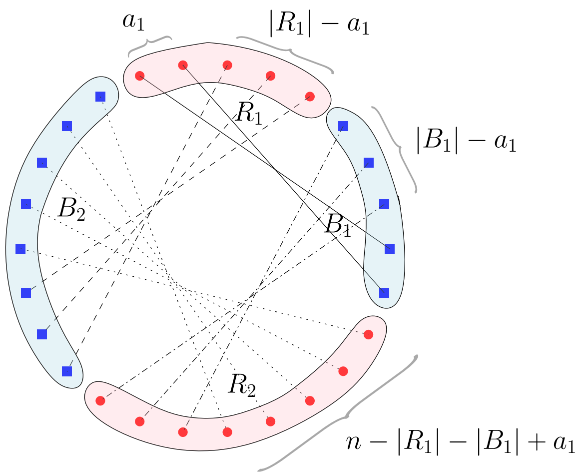

Figure 1 Structure of a max-crossing matching in a 4–block coloring.

The number of crossings $\overline{\text{cr}}(\mathrm{M}_P^\vee)$ can be obtained by subtracting the number of non-crossing edge pairs from the theoretical maximum number of the crossing edge pairs in $\mathrm{M}_P^\vee$. That is, $\overline{\text{cr}}(\mathrm{M}_P^\vee) = \binom{n}{2} - (r_1 b_1 - \frac{1}{2}(r_1 + b_1 - \frac{n}{2})^2)$. Next we check which coloring, among all 4–block colorings, produces a min-max-crossing matching for all 4–block colorings. Full proofs for statements marked with ($\star$) can be found in the full version of this paper.

▶ **Lemma 2.4.** *($\star$) Let $P \in \mathcal{C}_{n,n}$ have a balanced 4–block coloring and let $\mathrm{M}_P^\vee$ be a max-crossing matching of P. Then $\overline{\text{cr}}(\mathrm{M}_P^\vee) = \frac{3n^2}{8} - O(n)$. Moreover, $\mathrm{M}_P^\vee$ is a min-max-crossing matching for all the 4-block colored point sets of size $2n$.*

Proof sketch. The minimum number of pairs of non-crossing edges in $\mathrm{M}_P^\vee$ is given by $h(r_1, b_1) = r_1 b_1 - \frac{1}{2}(r_1 + b_1 - \frac{n}{2})^2$. Since we want to find the values of r_1 and b_1 that minimizes the number of crossings among max-crossing matchings of the 4–block colorings, we need to maximize $h(r_1, b_1)$ for $r_1, b_1 \geq \frac{n}{2}$. By analyzing the partial derivatives of the function h, it follows that h attains its maximum when $r_1 = b_1 = \frac{n}{2}$. This shows that the number of crossings in a max-crossing matching is minimized on a balanced 4–block coloring. If n is a multiple of 4, then $\overline{\text{cr}}(\mathrm{M}_P^\vee) = \frac{3n^2}{8} - \frac{n}{2}$. Otherwise, $\overline{\text{cr}}(\mathrm{M}_P^\vee)$ is either $\lfloor \frac{3n^2}{8} - \frac{n}{2} \rfloor$ or $\lceil \frac{3n^2}{8} - \frac{n}{2} \rceil$. These values are obtained for $r_1 = b_1 = \lceil \frac{n}{2} \rceil$ and $x = \frac{1}{2}(r_1 + b_1 - \frac{n}{2})$ by analyzing the structure of matchings in each of the remaining cases. ◀

We remark that the balanced 4–block coloring is not the only coloring that gives the min-max-crossing matching. See Figure 2(a) for the matching constructed in the above proof and Figure 2(b) for a different example.

2.2 Min-Max-Crossing Matching for all colorings

In the following, we extend Lemma 2.4 to all bichromatic convex point sets. Let $P \in \mathcal{C}_{n,n}$. For any point $v \in P$, the point $w \in P$ is called the antipodal point of v, if the line through v and w partitions P into two equal sized halves (this is possible as we have an even number of points). If the antipodal points v and w are of the same color, then they are called *monochromatic antipodal* points (in short *m-antipodal* points) and if they have different colors then they are called *bichromatic antipodal* points (in short *b-antipodal* points).

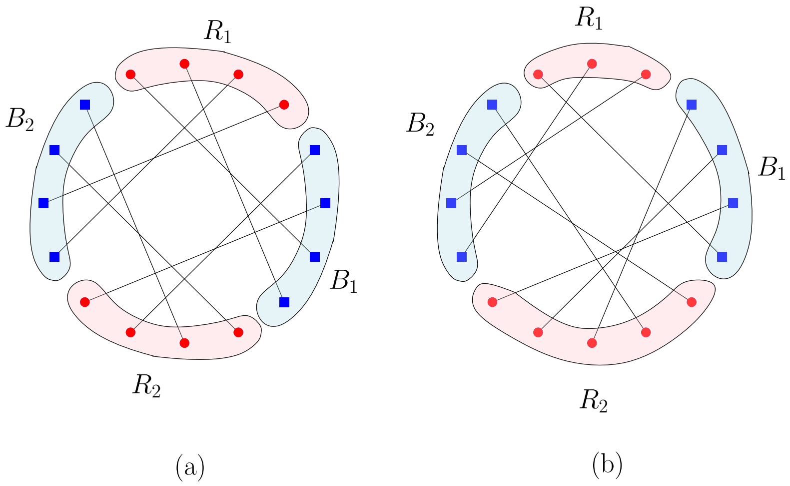

Figure 2 A balanced 4–block coloring (a) and a slightly unbalanced 4–block coloring (b) on 16 points and max-crossing matchings on them, each with 20 crossings.

▶ **Lemma 2.5.** *($\star$) Let $P \in \mathcal{C}_{n,n}$ and let $\mathrm{M}_P^{\vee}$ be a max-crossing matching of P. Then $\overline{\mathrm{cr}}(\mathrm{M}_P^{\vee}) \geq \frac{3n^2}{8} - O(n)$. Moreover, if $Q \in \mathcal{C}_{n,n}$ has a balanced 4–block coloring and $\mathrm{M}_Q^{\vee}$ is a max-crossing matching of Q, then $\mathrm{M}_Q^{\vee}$ is a min-max-crossing matching of the set $\mathcal{C}_{n,n}$.*

To prove Lemma 2.5, we make use of the following well-known theorem.

▶ **Theorem 2.6** (Ham sandwich theorem [10]). *For any bichromatic point set $P = R \cup B$ there exists a halfplane H such that $|R \cap H| = \lfloor \frac{|R|}{2} \rfloor$ and $|B \cap H| = \lfloor \frac{|B|}{2} \rfloor$.*

Proof sketch of Lemma 2.5. For a point set $P \in \mathcal{C}_{n,n}$, construct a new bichromatic point set with 4–block coloring using the following steps S1-S4.

S1: Remove all b-antipodal points from P, name the obtained bichromatic point set as S.

S2: Partition the set S into 4 groups as follows. First, partition S into two halves, say L and R, each having an equal number of red and blue points. Using the ham sandwich theorem partition one of the two halves such that each partition has an equal number of red and blue points. This partition can be duplicated on the other half as S consists only of m-antipodal points. Thus, we get 4 groups, each having an equal number of red and blue points. They are labeled as R_U, R_L, L_L, L_U w.r.t. clockwise order (see Figure 3).

S3: Add all the removed b-antipodal points back to S to get P and the partition of S induces a partition $R_{PU}, R_{PL}, L_{PL}, L_{PU}$ in P. Here the number of red (blue) points of R_{PU} and the number of blue (red) points of L_{PL} are equal. The same holds for R_{PL} and L_{PU}. Sort the points in R_{PU} and L_{PL} such that all the red points appear before the blue points w.r.t. the clockwise order. Then sort the points in R_{PL} and L_{PU} such that all the blue points appear before the red points. This gives a bichromatic point set K with the partition $R_{KU}, R_{KL}, L_{KL}, L_{KU}$. See Figure 4.

S4: Define the matchings M_P and M_K on P and K, respectively, as follows. For any pair $(X, Y) \in \{(R_{PU}, L_{PL}), (R_{PL}, L_{PU}), (R_{KU}, L_{KL}), (R_{KL}, L_{KU})\}$, the points in X are matched to points in Y such that any two of the matching edges emanating from the same colored points on X cross each other. Hence the matching edges of X give two crossing families, where the size of each family is determined by the number of points in X of each color. By our construction, the size of the crossing families is the same in both P and K. But in P, these crossing families cross each other and in K, these

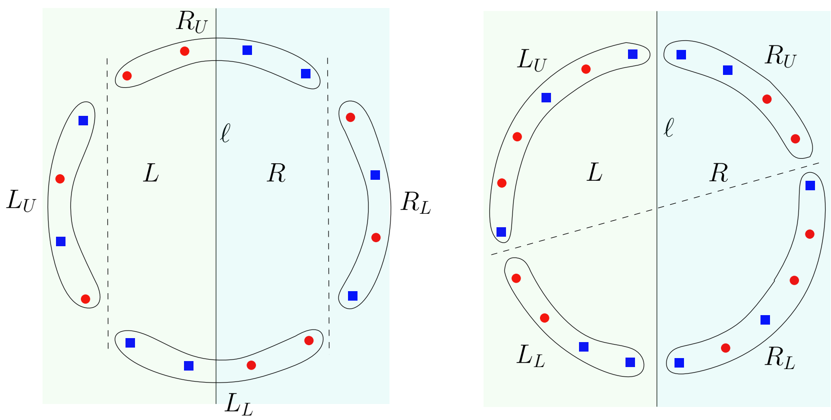

Figure 3 A bichromatic point set S with 16 points (left) and 20 points (right). In both cases, dotted lines represent a partition w.r.t. the ham sandwich theorem on the set R and L.

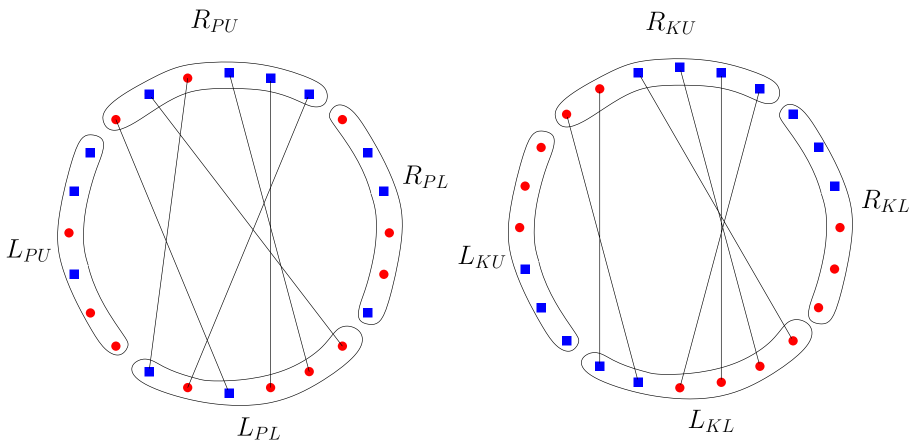

Figure 4 An example of bichromatic point set P (left) and the corresponding K (right) with 24 points. Also, the partial matching M_P and M_K.

crossing families do not cross each other. Hence, $\overline{\text{cr}}(M_P) \geq \overline{\text{cr}}(M_K)$. By construction, $K \in \mathcal{C}_{n,n}$ has a 4–block coloring. But it might not be a balanced 4–block coloring. Thus, by Lemma 2.4, $\overline{\text{cr}}(M_K) \geq \overline{\text{cr}}(\text{M}_Q^{\vee})$. Also, the constructed matching M_P might not be a max-crossing matching of P. Hence, if $\text{M}_P^{\vee}$ is a max-crossing matching for P, then $\overline{\text{cr}}(\text{M}_P^{\vee}) \geq \overline{\text{cr}}(M_P)$. This implies $\overline{\text{cr}}(\text{M}_P^{\vee}) \geq \overline{\text{cr}}(\text{M}_Q^{\vee})$, which completes the proof. ◀

As mentioned, Theorem 2.1 now follows directly from Lemma 2.4 and Lemma 2.5.

3 Conclusion

We showed that for any $k > \frac{3n^2}{8}$ there exists a bichromatic point set with n red and n blue points that does not admit any bichromatic perfect matching with k crossings. By straight-forward calculations, we can show that, for n even, bichromatic convex point sets containing n red and n blue points with alternating coloring cannot have a bichromatic perfect matching with one or two crossings. In ongoing work, we study the range $k \in [3, \frac{3n^2}{8}]$ for bichromatic convex point sets and also work on extending our results to bichromatic point sets in general (non-convex) position.

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