Antipodal Pairs and Crossing Numbers of Complete Graphs

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This paper makes contributions to the theory of crossing numbers of complete and almost complete graphs. An antipodal pair in a drawing is a pair of vertices such that the stars of the two vertices induce a plane subgraph of the drawing. An optimal drawing of a graph is a drawing with a minimum number of crossings. Hill conjectured that the crossing number of the complete graph is:

$$cr(K_n) = H(n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

We establish connections between antipodal pairs and optimal drawings of complete graphs. In particular we show that Hill's conjecture is true within the class of antipodally shellable drawings.

The graph M_n^t is obtained from the complete graph K_n by removing the edges of a matching of size t. For these graphs we conjecture that all optimal drawings have antipodal pairs. If true this conjecture implies a conjecture of Mohar about the crossing number of M_{2k}^t and a conjecture of the author about the crossing number of M_n^t for odd n.

We also construct a new family of Hill drawings, i.e., a family of drawings of K_n with crossing number equal to $H(n)$. As a consequence of this new construction we obtain that every spherical drawing of K_k is a subdrawing of a Hill drawing of K_n for all $n \geqslant 2k+1$.

1 Introduction

We consider drawings of graphs in the plane with vertices being represented by points and edges by simple curves not containing vertices other than their endpoints. We also assume that intersections are proper crossings and that two edges share at most one point which may be a common vertex or a crossing point. Such drawings are known as *simple drawings*.

A drawing Γ of G is called *optimal* if $cr(\Gamma) = cr(G)$, i.e., if Γ minimizes the number of crossings among all drawings of G. It is known that optimal drawings are simple.

The crossing number of the complete graph has been studied since the late 1950s. British artist Anthony Hill conjectured that

$$cr(K_n) = H(n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$$

Hill's conjecture has been verified with massive computer power for $n \leqslant 14$, see Aichholzer [Aic21].

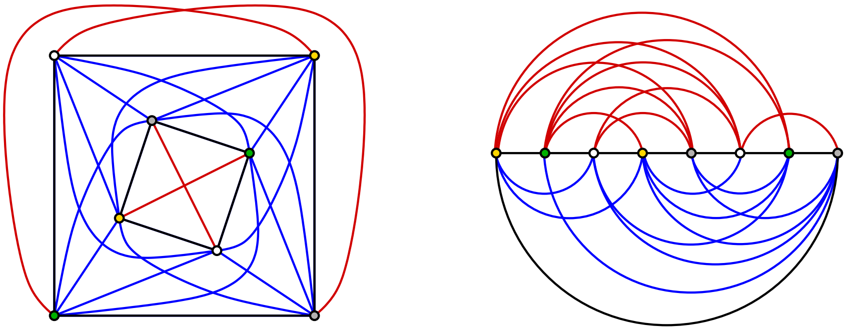

Figure 1: Antipodal pairs (pairs of vertices of the same color) in the can drawing and in the optimal 2-page drawing of K_8.

With $N(v)$ we denote the neighbor set of a vertex v in a graph, i.e., $N(v) = \{w : vw \in E_G\}$. The star $S(v)$ of a vertex v in a graph is the tree consisting of all edges incident to v, i.e., $S(v) = (\{v\} \cup N(v), \{vw : w \in N(v)\})$.

Definition. A pair x, x' of vertices in a drawing is an *antipodal pair* if the drawing induced by the union of the stars of x and x' is crossing free and $N(x) - \{x'\} = N(x') - \{x\} = V_G - \{x, x'\}$.

In other words: the two stars of an antipodal pair induce a plane spanning *doublestar* of minimum degree 2. Each vertex of the two drawings of K_8 in Figure 1 is part of an antipodal pair.

Conjecture 1. *There is some integer N such that every optimal drawing of K_n with $n > N$ has an antipodal pair.*

Remark. Originally the conjecture was formulated without restriction on n. However, Matthew Sunohara found an optimal drawing of K_9, i.e., a drawing of K_9 with 36 crossings, which has no antipodal pair, see Figure 2(a). Joachim Orthaber later pointed out that an optimal drawing of K_9 shown in [Aic21] has the same properties, this drawing is isomorphic to the drawing shown in Figure 2(b). Both drawings have the property that they have no optimal subdrawings of K_8. Oswin Aichholzer performed a search on his database of drawings of complete graphs and provided the following numbers: There are exactly 2 optimal drawings of K_9, 13.180 optimal drawings of K_{11}, and 3 optimal drawings of K_{12} without antipodal pair.

Despite these counterexamples to the original conjecture we think that it may be true that all counterexamples to the conjecture are sporadic and small.

In the next section (Section 2) we discuss how Conjecture 1 relates to Hill's conjecture and point to some additional consequences of the conjecture. In Section 3 we construct a new family of drawings of complete graphs whose crossing number matches the Hill conjecture. In the subsequent Section 4 we survey the existing literature about the conjectures of Hill and Zarankiewicz.

The graph M_n^t is obtained from the complete graph K_n by removing a matching of size t. Mohar [Moh20] investigated spherical drawings of M_n^t for even n, he conjectured that $cr(M_{2k}^t) = H(2k) - t\binom{k-1}{2}$. We conjecture that optimal drawings of M_{2k}^t have *antipodal pairs*. In Section 5 we show that our conjecture implies Mohar's conjecture. We also point to a case where it is known that Hill's conjecture implies a case of Mohar's conjecture and add a new case of this

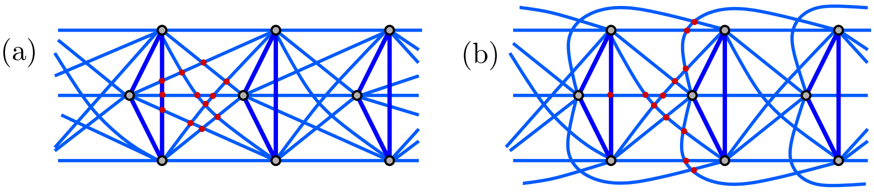

Figure 2: Two optimal drawings of K_9 without antipodal pairs. Both drawings have a threefold rotational symmetry which maps dark triangles to dark triangles. In each of the drawings we have emphasized the 12 crossings in one of the three blocks of the symmetry.

phenomenon. In particular we show that the crossing number of the double-icosahedron M_{12}^6 is 90 as conjectured.

In Section 6 we argue that in the odd case the crossing number of M_n^t should be $cr(M_{2k+1}^t) = H(2k+1) - t\binom{k}{2} + \binom{t}{2}$. Again we show that this is implied if every optimal drawing of M_{2k+1}^t has an antipodal pair.

2 Antipodal pairs in K_n

While it is nice to have a closed formula for the Hill number $H(n)$ it is more convenient in practice to work with formulas avoiding rounding.

$$H(n) = \begin{cases} \frac{1}{4}k(k-1)^2(k-2) = \binom{k}{2}\binom{k-1}{2} = 6\binom{k}{4} + 3\binom{k}{3} & \text{if } n = 2k \qquad \text{(even)} \\ \frac{1}{4}k^2(k-1)^2 = \binom{k}{2}^2 & \text{if } n = 2k+1 \qquad \text{(odd).} \end{cases}$$

For a uniform statement of the following proposition we define $\Delta(n) = H(n+2) - H(n)$ and note that for n even $\Delta(n) = \Delta(2k) = H(2k+2) - H(2k) = (2k-1)\binom{k}{2}$ and for n odd $\Delta(n) = \Delta(2k+1) = H(2k+3) - H(2k+1) = k^3$.

Proposition 2. *If Γ is a drawing of K_{n+2} with an antipodal pair x, x' and Γ_Θ is the induced drawing of K_n on $V - \{x, x'\}$, then*

$$cr(\Gamma) \geqslant \Delta(n) + cr(\Gamma_\Theta).$$

Before proving the proposition we look at some of its its consequences.

We now restate Conjecture 1 from the introduction with a venturously chosen constant.

Conjecture 3. *Every optimal drawing of K_n with $n > 14$ has an antipodal pair.*

If we assume the conjecture and the induction hypothesis $cr(K_n) = H(n)$, then the proposition implies:

$$cr(K_{n+2}) \geqslant \Delta(n) + H(n) = H(n+2). \qquad (\star)$$

Since $cr(K_n) = H(n)$ is known for all $n \leqslant 14$ we have:

Corollary 4. *Conjecture 3 implies Hill's conjecture ($cr(K_n) = H(n)$ for all n).*

From Inequality $\star$ and the crossing number of small complete graphs we obtain that a minimal counterexample to Hill's conjecture must be an optimal drawing of a complete graph without antipodal pairs.

A drawing of K_n is *antipodally shellable* if the vertices of the graph can be listed $v_1, \dots, v_n$ such that if Γ_i is the drawing induced by $v_1, \dots, v_{n-2i}$ for $i = 0, \dots, \lfloor \frac{n}{2} \rfloor$, then v_{n-2i-1}, v_{n-2i} is an antipodal pair in Γ_i.

Corollary 5. *Crossing minimal antipodally shellable drawings of K_n have $H(n)$ crossings.*

The class of antipodally shellable drawings of complete graphs is a new and fairly large class of drawings where the Hill conjecture is true. In Section 4 we point to some other classes of drawings where the Hill conjecture has been shown.

2.1 Proof of Proposition 2

Let Γ be a drawing of K_{n+2} with an antipodal pair x, x' Let D be the doublestar of the pair x, x' and Γ_Θ be the drawing of K_n obtained by removing x, x', and the edges of D from Γ. We have

$$cr(\Gamma) = \#(\text{crossings on edges of } D) + cr(\Gamma_\Theta).$$

We now aim for a lower bound on the number of crossings on edges of D. The doublestar D has an edge directly connecting x and x'. To simplify the analysis we subdivide this edge xx' with an auxiliary vertex o. The doublestar D now consists of $n+1$ faces of degree 4. Let $x_0, x_1, \dots, x_n$ be a labeling of the vertices of Γ different from x, x' but including o such that for $i = 1, \dots, n$ the vertices x_i, x, x_{i+1}, x' belong to a 4-face of D. An edge $x_i x_j$ of Γ with $j > i$ has at least $c_{ij} = \min\,[j - i - 1\,,\, i + (n - j)]$ crossings with edges of D. For a fixed x_i we are interested in $\sum_j c_{ij}$ where j runs through all values such that $x_i x_j$ is an edge. To simplify the analysis we assume virtual edges between o and all the other vertices x_j. To evaluate $\sum_j c_{ij}$ we recall that with the subdivision vertex the doublestar has $n+1$ vertices of degree 2. For the following Figure 3 may help.

$$\sum_j c_{ij} = \begin{cases} 2\sum_{\ell=0}^{k-1} \ell = 2\binom{k}{2} = k(k-1) & \text{if } n = 2k \quad \text{(even)} \\ \sum_0^k \ell + \sum_0^{k-1} \ell = \binom{k+1}{2} + \binom{k}{2} = k^2 & \text{if } n = 2k+1 \quad \text{(odd)} \end{cases}$$

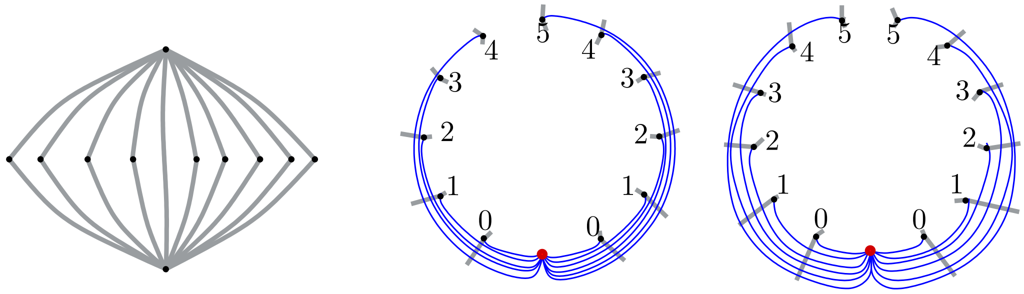

Figure 3: Left: A plane doublestar. Right: The $t = n+1$ degree 2 vertices of a doublestar including o cyclically arranged. The left example shows $t = 12$ and the right examples shows $t = 13$. Labels at vertices reflect how many crossings with the doublestar an edge joining to the red bottom vertex must at least have.

In the even case we have $n + 1 = 2k + 1$ vertices x_i. In the expression $(2k+1)k(k-1)$ we count the crossings of each edge $x_i x_j$ twice and we also count the contribution of the virtual edges of o twice. Hence, $\#(\text{crossings on edges of } D) \geqslant \frac{1}{2}(2k-1)k(k-1) = \Delta(2k) = \Delta(n)$.

In the odd case we have $n + 1 = 2k + 2$ vertices x_i. In the expression $(2k+2)k^2$ we count the crossings of each edge $x_i x_j$ twice and we also count the contribution of the virtual edges of o twice. Hence, $\#(\text{crossings on edges of } D) \geqslant k^3 = \Delta(2k+1) = \Delta(n)$. □

3 A new family of Hill drawings

A *Hill drawing* is a drawing of K_n with $H(n)$ crossings. Since the Hill conjecture is open we have to distinguish between Hill drawings and optimal drawings. There are two classical families of Hill drawings, *can drawings* a.k.a. *cylindrical drawings* and *optimal 2-page drawings*, see Figure 1. Both types of drawings were already known to Hill, see [HH63]. In the following section we survey the known Hill drawings. Before this, however, we introduce a new family of Hill drawings.

Let M_{2k} be the complete graph K_{2k} minus a perfect matching μ. Kynčl [Kyn13] and Mohar [Moh20] construct spherical arc drawings of M_{2k}: Start with a set of k points in general position on the sphere and add their antipodes, the matching μ consists of the pairs of antipodal[1]points, the edges of M_{2k} are realized with geodesic arcs. Let Γ be such a drawing. The edges of Γ form $\binom{k}{2}$ great circles, with each great circle representing four edges involving two antipodal pairs, i.e., edges $ab, b\bar{a}, \bar{a}\bar{b}, \bar{b}a$. Every pair of great circles has two points of intersection which may be a pair of antipodal vertices or two crossings. Hence, the spherical arc drawing Γ of M_{2k} has $M(2k) = \binom{k}{2}\binom{k-2}{2} = 6\binom{k}{4}$ crossings. Mohar conjectures that $cr(M_{2k}) = M(2k)$. Based on known crossing numbers for complete graphs with $n \leqslant 14$ we can prove this conjecture for $k \leqslant 7$ (Theorem 9).

Let $z, \bar{z}$ be an antipodal pair of Γ. We construct a drawing $\Gamma_\star$ of K_{2k-1} by adding for each antipodal pair $p, \bar{p}$ the *great arc* (half of great circle) $p\bar{p}$ which contains $\bar{z}$ and by removing $\bar{z}$ together with all its edges. An example of the construction for $k = 6$ is shown in Figure 4. Every crossing of Γ has a corresponding crossing in $\Gamma_\star$ in addition the $k-1$ great arcs of $\Gamma_\star$ pairwise cross in the point of the sphere where $\bar{z}$ was located. Hence

$$cr(\Gamma_\star) = cr(\Gamma) + \binom{k-1}{2} = M(2k) + \binom{k-1}{2} = 6\binom{k}{4} + \binom{k-1}{2} = H(2k-1).$$

This shows that $\Gamma_\star$ is a Hill drawing.

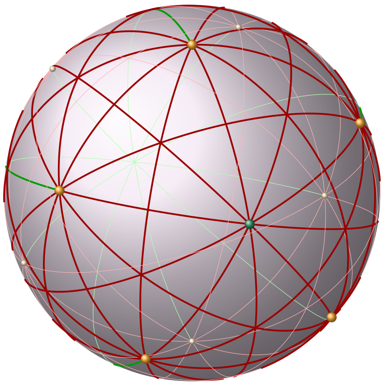

Figure 4: A drawing of K_{11} with $H(11) = 100$ crossings. The green edges are the great arcs. The vertices are vertices of a regular icosahedron.

Let $\Gamma_\star$ of be a drawing of K_n with $n = 2k-1$ obtained with the above construction. We use $\Gamma_\star$ as the construction basis for a Hill drawings $\Gamma_\star^+$ with $n = 2k$: In $\Gamma_\star$ add a new vertex z^+

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[1] Note that we use the attribute antipodal in two ways. A meaning of antipodal has been established with Definition , however, on the sphere an antipodal drawing is a geodesic drawing where the points come in pairs $p, \bar{p}$ which are antipodal in the geometry of the sphere.

very close to the antipode z of the removed vertex $\bar{z}$ and use geodesics for the edges, this yields the drawing $\Gamma_\star^+$. In the original drawing Γ the edges incident to z and $\bar{z}$ both participated in $(k-1)\binom{k-2}{2}$ crossings: there are $k-1$ choices for a point r to select a great circle $zr, r\bar{z}, \bar{z}\bar{r}, \bar{r}z$ and $\binom{k-2}{2}$ choices for a great circle $ab, b\bar{a}, \bar{a}\bar{b}, \bar{b}a$ with $\{r,z\} \cup \{a,b\} = \varnothing$. The two great circles cross at an edge incident to z and at the antipodal edge incident to $\bar{z}$. The edges of the new vertex z^+ have the same number of crossings with edges not incident to z. The great circle spanned by zz^+ splits each antipodal pair, hence, the great circle has $k-1$ vertices on each side and the stars of z and z^+ have $2\binom{k-1}{2}$ crossings. Since $(k-1)\binom{k-2}{2} + 2\binom{k-1}{2} = (k-1)\binom{k-1}{2}$ this shows that the crossing number of the drawing $\Gamma_\star^+$ of K_{2k} is

$$cr(\Gamma_\star^+) = cr(\Gamma_\star) + (k-1)\binom{k-1}{2} = H(2k-1) + (k-1)\binom{k-1}{2} = \frac{1}{4}k(k-1)^2(k-2) = H(2k)$$

and $\Gamma_\star^+$ is a Hill drawing for $n = 2k$ even.

Theorem 6. *The Hill drawings $\Gamma_\star$ and $\Gamma_\star^+$ of K_{2k-1} and K_{2k} respectively are distinct from the known Hill drawings.*

Proof. In both drawings the great arcs replacing the vertex $\bar{z}$ of M_{2k} are a set of $k-1$ pairwise crossing edges. Previously known Hill drawings of K_n have no set of pairwise crossing edges of size $\lfloor \frac{n-1}{2} \rfloor$. $\square$

In the construction of the initial drawing Γ of M_{2k} we can choose fhe first k points arbitrary, therefore, we get the following surprising fact:

Fact 7. Every spherical arc drawing of K_k is an induced subdrawing of a Hill drawing of K_{2k-1} and of K_{2k}.

For later use we also remark that with a corresponding construction every simple drawing Λ of M_{2k} can be used to generate a drawing $\Lambda_\star$ of K_{2k-1} and a drawing $\Lambda_\star^+$ of K_{2k} such that:

$$cr(\Lambda_\star) = cr(\Lambda) + \binom{k-1}{2} \quad \text{and} \quad cr(\Lambda_\star^+) = cr(\Lambda_\star) + (k-1)\binom{k-1}{2}. \tag{$\star\star$}$$

4 Status of the conjectures of Hill and Zarankiewicz

Hill drawings. The artist Anthony Hill was the first who found drawings of K_n with $H(n)$ crossings. His drawings can be described using a can of tomato soup. Place half the vertices evenly spaced on the top rim of the can, and the other half on the bottom rim. Vertices on the same rim are connected using only the top respectively the bottom of the can. Edges between the two rims are routed as geodesics (shortest connecting curves) on the cylinder of the can. Here we denote these drawings as *can drawings* other authors refer to them as *cylindrical drawings*. In [ÁAF$^+$14b] the authors introduced new families of Hill drawings which are obtained by small modifications of can drawings.

Can drawings can be realized as antipodal spherical arc drawings, see Figure 5. Mohar [Moh20] describes a construction of many pairwise non-isomorphic spherical antipodal Hill drawings. With the spherical representations of can drawings they share the property that they contain a perfect matching of pairwise disjoint great arcs. Streltsova and Wagner [SW25] prove that a point configuration on the sphere in general position (no three points on a common great circle) induces a spherical arc Hill drawing exactly if the point configuration is *coneighborly*, i.e., every open halfsphere contains at least $\lfloor (n-2)/2 \rfloor$ of the n points.

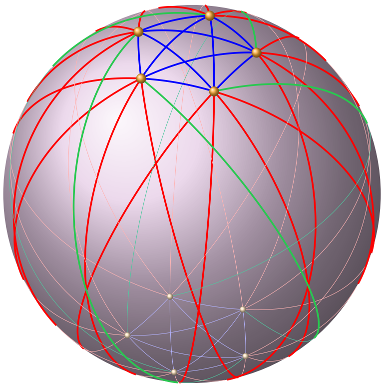

Figure 5: A spherical antipodal realization of the can drawing of K_{10}.

A second construction of drawings of K_n with $H(n)$ crossings is known as the 2-page drawing. A drawing appeared in the classical paper by Harary and Hill [HH63], the formal description and analysis is by Blažek and Koman [BK64]. The authors of [ÁAF$^+$13a] prove the Hill conjecture for the class of 2-page drawings and investigate properties of all crossing minimal 2-page drawings. They show that in the even case there is a unique 2-page Hill drawing while in the odd case there are exponentially many. Arroyo, Richter, and Sunohara [ARS26] have a complete classification of optimal 2-page drawings.

Hill conjecture. The conjecture has been verified by Guy [Guy72] for $n \leqslant 10$. With computer support the bound was pushed to $n \leqslant 12$ by Pan and Richter [PR07], and to $n \leqslant 14$ by Aichholzer [Aic21]. While the question remains open in general, there has been substantial progress in the past two decades. In two independent papers [ÁFM05] and [LVWW04] it was observed that bounds on the number of $\leqslant k$-sets can be used in lower bounds for the rectilinear crossing number of complete graphs. This idea for a lower bound was adapted in [ÁAF$^+$13a] for more general drawings and the concept of $\leqslant\leqslant k$-edges was introduced to simplify the involved formulas. With this toolkit it was shown that 2-page drawings of K_n have at least $H(n)$ crossings. In subsequent years the technique was used to show that several other classes of drawings of K_n have at least $H(n)$ crossings. Here is a list of the most relevant classes with references:

- x-monotone drawings ([ÁAF$^+$13b] and [BFK15]).
- cylindrical drawings, x-bounded drawings, and shellable drawings ([ÁAF$^+$14a]).
- bishellable drawings ([ÁAF$^+$18]).

Streltsova and Wagner [SW25] show that the Hill conjecture holds for the class of spherical arc drawings of complete graphs. Their proof is based on a connection with $\leqslant k$-sets in three dimensions.

The Zarankiewicz conjecture. Turán's brick factory problem has its origin in the early 1940s (see [Tur77]). It asks for the crossing number of complete bipartite graphs. Zarankiewicz claimed a proof that

$$cr(K_{m,n}) = Z(m,n) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

but Ringel found a gap. In fact Zarankiewicz argument was based on the unproven assumption that an optimal drawing of $K_{m,n}$ contains a plane doublestar (antipodal pair). It follows that minimal counterexamples to the Zarankiewicz conjecture do not contain antipodal pairs (see [Sch18, Thm. 1.13]). Interestingly there exist optimal drawings of $K_{5,n}$ without antipodal pairs, see [HMS14, Fig. 3].

Kleitman [Kle71] proved that $cr(K_{m,n}) = Z(m,n)$ for $m \leqslant 6$ and all n. He also proved that the smallest counterexample to the Zarankiewicz conjecture must occur for odd m and n. With computer assistance the conjecture was confirmed for $K_{7,7}$, $K_{7,8}$, and $K_{7,9}$ by Woodall [Woo93].

5 Antipodal pairs in M_{2k}^t

For $n \in \mathbb{N}$ and $0 \leqslant t \leqslant \lfloor \frac{n}{2} \rfloor$ the graph M_n^t is obtained from the complete graph K_n by deleting a matching μ consisting of t edges, in particular $M_n^0 = K_n$ and $M_{2k}^k = M_{2k}$. A pair x, y of vertices such that xy belongs to μ is a *matching pair* of M_n^t.

Define $M(2k,t) = H(2k) - t\binom{k-1}{2} = 6\binom{k}{4} + (k-t)\binom{k-1}{2}$.

Conjecture 8 (Mohar [Moh20] Conj. 5)**.** *The crossing number of M_{2k}^t equals $M(2k,t)$.*

With the following theorem we shed light on the $t = k$ case of Mohar's conjecture.

Theorem 9. *If $\mathrm{cr}(K_{2k-1}) = H(2k-1)$ (Hill conjecture for $n = 2k-1$), then $\mathrm{cr}(M_{2k}^k) = M(2k,k)$ (Mohar conjecture for $(n,t) = (2k,k)$).*

Proof. Let Λ be an optimal drawing of M_{2k}^k. With the construction of Section 3 we obtain a drawing $\Lambda_\star$ of K_{2k-1}. The crossing numbers of Λ and $\Lambda_\star$ are related by Equation ($\star\star$), i.e., $cr(\Lambda_\star) = cr(\Lambda) + \binom{k-1}{2}$. The Hill conjecture implies $cr(\Lambda_\star) \geqslant H(2k-1)$, hence, $cr(\Lambda) \geqslant H(2k-1) - \binom{k-1}{2} = \frac{(k-1)(k-2)}{4}\big((k-1)(k-2)-2\big) = 6\binom{k}{4} = M(2k,k)$. $\square$

Since Hill's conjecture has been verified for $n \leqslant 14$ we now know that $cr(M_{10}) = 30$, $cr(M_{12}) = 90$, and $cr(M_{14}) = 210$.

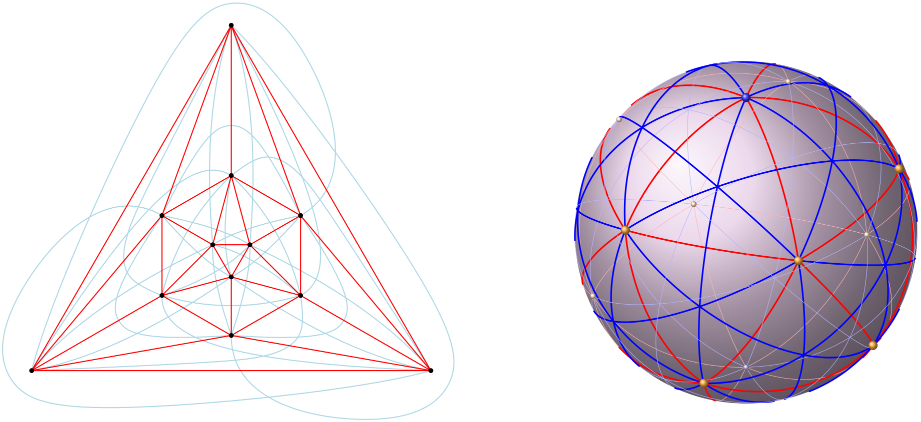

Figure 6: Two optimal drawings of M_{12}. The graph is called the double-icosahedron because both the red and the blue edges induce an icosahedron graph on the 12 vertices.

The crossing number of the complete graph with an edge removed has been studied in [OWH14]. There it is shown that if Hill's conjecture is true, then the crossing number of $K_n \backslash e$ equals $H(n) - \binom{\lfloor \frac{n-1}{2} \rfloor}{2}$, i.e., they show that Hill's conjecture implies the $t = 1$ case of Mohar's conjecture.

An antipodal pair in a drawing of M^t_{2k} can either be a pair x, x' such that neither of the vertices belongs to a matching pair, i.e., both x and x' have degree $2k - 1$, or x, x' is a matching pair and both x and x' have degree $2k - 2$.

Conjecture 10. *Every optimal drawing of* M^t_{2k} *has an antipodal pair.*

We will show that our conjecture implies Mohar's conjecture. The key to the argument is the following proposition:

Proposition 11. *Let* Γ *be a drawing of* M^t_{2k+2} *with an antipodal pair* x, x' *and let* Γ_Θ *be the subdrawing of* Γ *induced by* $V - \{x, x'\}$*, then*

- *if* x, x' *is a matching pair, then* Γ_Θ *is a drawing of* M^{t-1}_{2k} *and*
$$cr(\Gamma) \geqslant \big(k(k-1)^2 - (t-1)(k-1)\big) + cr(\Gamma_\Theta).$$
- *if* x, x' *both have degree* $2k + 1$ *in* Γ*, then* Γ_Θ *is a drawing of* M^t_{2k} *and*
$$cr(\Gamma) \geqslant 6\binom{k}{3} + 3\binom{k}{2} - t(k-1) + cr(\Gamma_\Theta).$$

Note that

- $M(2k+2, t) - M(2k, t-1) = 6\binom{k}{3} + (k - t + 1)(k - 1) = k(k-1)^2 - (t-1)(k-1)$ and
- $M(2k+2, t) - M(2k, t) = 6\binom{k}{3} + (k-t)(k-1) + \binom{k}{2} = 6\binom{k}{3} + 3\binom{k}{2} - t(k-1)$.

Hence by induction on k and t we get:

Corollary 12. *Conjecture 10 (antipodal pairs) implies Conjecture 8 (Mohar's).*

This directly implies that if Mohar's conjecture is wrong, then a minimal counterexample must be an optimal drawing of a M^t_{2k} without antipodal pairs. Moreover, crossing minimal antipodally shellable drawings of M^t_{2k} have exactly $M(2k, t)$ crossings.

5.1 Proof of Proposition 11

Let Γ be a drawing of M^t_{2k+2}. In this proof we consider the two types of antipodal pairs separately.

Case matching. Let x, x' be an antipodal matching pair in Γ. Let D be the doublestar of x, x' and let Γ_Θ be the drawing of M^{t-1}_{2k} obtained by removing x, x' and the edges of D from Γ. We have
$$cr(\Gamma) = \#(\text{crossings on edges of } D) + cr(\Gamma_\Theta).$$

Let X be the set of vertices of degree 2 of the doublestar, $|X| = 2k$. A vertex $x \in X$ is either adjacent to all the other vertices of X or it has a matching partner in X. There are $2k - 2(t-1)$ vertices of the first kind. The edges adjacent to such a vertex contribute at least $\sum_1^{k-2} \ell + \sum_1^{k-1} \ell = \binom{k-1}{2} + \binom{k}{2} = (k-1)^2$ crossings with edges of the doublestar. The $2(t-1)$ vertices of the second kind contribute at least $2\sum_1^{k-2} \ell = (k-1)(k-2)$ crossings with edges of the doublestar.

Since every crossing is counted twice we see that the total number crossings on edges of D equals $(k - (t-1))(k-1)^2 + (t-1)(k-1)(k-2) = k(k-1)^2 - (t-1)(k-1)$.

Case edge. Let x, x' be an antipodal pair in Γ such that both are not covered by the matching, in particular xx' is an edge. We subdivide the edge xx' with an auxiliary vertex o, such that the doublestar has $2k+1$ vertices of degree 2, let X be this set of vertices. We assume virtual edges from o to all the other vertices of X. In X we have $2k-2t+1$ vertices which are adjacent to all the other edges of X. The edges incident to each of these vertices contribute at least $2\sum_1^{k-1}\ell = k(k-1)$ crossings with D. The remaining $2t$ vertices belong to matching pairs, their edges contribute at least $\sum_1^{k-1}\ell + \sum_1^{k-2}\ell = (k-1)^2$ crossings. Since every crossing is counted twice and we have counted $k(k-1)$ crossings for the virtual edges of o we have at least $(2k-2t-1)\binom{k}{2} + t(k-1)^2 = (2k-4)\binom{k}{2} + 3\binom{k}{2} - t(k-1) = 6\binom{k}{3} + 3\binom{k}{2} - t(k-1)$ crossings with edges of D. $\square$

6 Antipodal pairs in M^t_{2k+1}

In this section we focus on the graphs M^t_n with n odd. In Subsection 6.2 we discuss constructions of spherical arc drawings for these graphs which lead us to the following conjecture.

Conjecture 13. *The crossing number of* M^t_{2k+1} *is* $M(2k+1,t) = H(2k+1) - t\binom{k}{2} + \binom{t}{2}$.

Again for this class of graphs we conjecture that antipodal pairs are unavoidable in optimal drawings.

Conjecture 14. *Every optimal drawing of* M^t_{2k+1} *has an antipodal pair.*

We will show that Conjecture 14 implies Conjecture 13. The key to the argument is the following proposition:

Proposition 15. *Let* Γ *be a drawing of* M^t_{2k+3} *with an antipodal pair* x, x' *and let* Γ_Θ *be the subdrawing of* Γ *induced by* $V - \{x, x'\}$*, then*

- *if* x, x' *is a matching pair, then* Γ_Θ *is a drawing of* M^{t-1}_{2k+1} *and*
$$cr(\Gamma) \geqslant \big((2k+1)\tbinom{k}{2} - (t-1)(k-1)\big) + cr(\Gamma_\Theta).$$

- *if* x, x' *both have degree* $2k+1$ *in* Γ*, then* Γ_Θ *is a drawing of* M^t_{2k+1} *and*
$$cr(\Gamma) \geqslant (k^3 - tk) + cr(\Gamma_\Theta).$$

Note that

- $M(2k+3,t) - M(2k+1,t-1) = k^3 - t\binom{k+1}{2} + (t-1)\binom{k}{2} + \binom{t}{2} - \binom{t-1}{2} = k^3 - \binom{k+1}{2} - (t-1)(k-1) = (2k+1)\binom{k}{2} - (t-1)(k-1)$ and
- $M(2k+3,t) - M(2k+1,t) = k^3 - t\binom{k+1}{2} + t\binom{k}{2} = k^3 - tk.$

Hence by induction on k and t we get:

Corollary 16. *Conjecture 10 (antipodal pairs) implies Conjecture 8 (Mohar's).*

This directly implies that if Mohar's conjecture is wrong, then a minimal counterexample must be an optimal drawing of a M^t_{2k} without antipodal pairs. Moreover, crossing minimal antipodally shellable drawings of M^t_{2k} have exactly $M(2k,t)$ crossings.

6.1 Proof of Proposition 15

Let Γ be a drawing of M_{2k+3}^t. In this proof we consider the two types of antipodal pairs separately.

Case matching. Let x, x' be an antipodal matching pair in Γ. Let D be the doublestar of x, x' and let Γ_Θ be the drawing of M_{2k+1}^{t-1} obtained by removing x, x' and the edges of D from Γ. We have

$$cr(\Gamma) = \#(\text{crossings on edges of } D) + cr(\Gamma_\Theta).$$

Let X be the set of vertices of degree 2 of the doublestar, $|X| = 2k+1$. A vertex $x \in X$ is either adjacent to all the other vertices of X or it has a matching partner in X. There are $2k+1-2(t-1)$ vertices of the first kind. The edges adjacent to such a vertex contribute at least $2\sum_1^{k-1} \ell = 2\binom{k}{2}$ crossings with edges of the doublestar. The $2(t-1)$ vertices of the second kind contribute at least $\sum_1^{k-1} \ell + \sum_1^{k-2} \ell = (k-1)^2$ crossings with edges of the doublestar. Since every crossing is counted twice we see that the total number crossings on edges of D equals $(2k+1-2(t-1))\binom{k}{2} + (t-1)(k-1)^2 = (2k+1)\binom{k}{2} - (t-1)(k-1)$.

Case edge. Let x, x' be an antipodal pair in Γ such that both are not covered by the matching, in particular xx' is an edge. We subdivide the edge xx' with an auxiliary vertex o, such that the doublestar has $2k+2$ vertices of degree 2. Let X be the set of vertices of degree 2 of the doublestar. We assume virtual edges from o to all the other vertices of X. In X we have $2k+2-2t$ vertices which are adjacent to all the other edges of X. The edges incident to each of these vertices contribute at least $\sum_{\ell=1}^{k-1} \ell + \sum_{\ell=1}^{k} \ell = k^2$ crossings with D. The remaining $2t$ vertices belong to matching pairs, their edges contribute at least $2\sum_1^{k-1} \ell = k(k-1)$ crossings. Since every crossing is counted twice and we have counted $2k^2$ crossings for the virtual edges of o we have at least $(k-t)k^2 + tk(k-1) = k^3 - tk$ crossings with edges of D. □

6.2 Odd complete graphs with a removed matching

In this section we look at almost antipodal spherical drawings drawings of M_{2k+1}^t. This provides motivation for Conjecture 14.

Recall the antipodal spherical arc drawings of $M_{2k} = M_{2k}^k$ with $M(2k, k) = 6\binom{k}{4}$ crossings from Section 3. Let Γ be such a drawing. We obtain a drawing of M_{2k}^{k-1} by adding in Γ a half great arc from some p to its antipode $\bar{p}$. Since a great arc $p\bar{p}$ has a crossing with the great circle $C_{q,r}$ of points q, r distinct from p the great arc has $\binom{k-1}{2}$ crossings. Hence, we obtain a drawing of M_{2k}^{k-1} with $6\binom{k}{4} + \binom{k-1}{2} = M(2k, k-1)$ crossings.

A collection of k great arcs, one for each pair of antipodal points, is called *strong* if they are pairwise disjoint. Given a strong collection of great arcs we can add a subset of $k-t$ of them to the drawing Γ of M_{2k} to obtain a drawing of M_{2k}^t with $6\binom{k}{4} + (k-t)\binom{k-1}{2} = M(2k, t)$ crossings.

Can drawings of K_{2k} can be realized as spherical arc drawings on a set consisting of k pairs of antipodal points and these drawings have $H(2k) = M(2k, 0)$ crossings. This implies the existence of point sets which admit a strong collection of great arcs. Mohar [Moh20] gives a construction rule which produces many different sets of points which admit a strong collection of great arcs.

Let us turn to the odd case. The following lemma was suggested by Mohar [Moh20].

Lemma 17. *Let Γ be a spherical arc drawing of K_{2k} with $H(2k)$ crossings based on a set P of k points and their antipodes. If z is a generic point on the sphere then by adding z and the geodesic arcs to the other $2k$ vertices we obtain a drawing Γ^+ of K_{2k+1} with $H(2k+1)$ crossings.*

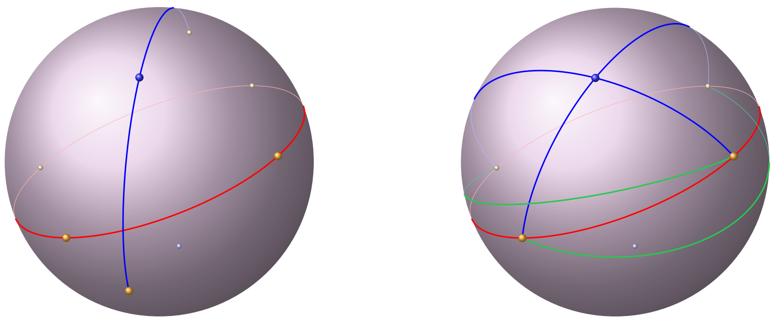

Figure 7: Left: The blue half circle consists of edges zp and $z\bar{p}$, it has one crossing with $C_{q,r}$. Right: The two green great arcs are on different sides of $C_{p,q}$, hence, together they contribute a single crossing with one of the four blue edges.

Proof. Let S_z denote the star of z in the drawing of K_{2k+1}. Since $H(2k+1)-H(2k) = (k-1)\binom{k}{2}$ we have to show that the edges of S_z contribute $(k-1)\binom{k}{2}$ crossings to the drawing. For every $p \in P$ the two edges pz and $z\bar{p}$ form one half of a great circle. This half great circle has a single intersection with every great circle $C_{q,r}$ disjoint from p, see Figure 7 left. We have k choices of p and $\binom{k-1}{2}$ choices of $\{q, r\}$, hence a total of $k\binom{k-1}{2} = (k-2)\binom{k}{2}$ crossings.

The remaining crossings are the crossings between edges of S_z and great arcs of the drawing Γ. Recall that the great arcs of Γ are strong, i.e., pairwise disjoint. We claim that we can assign exactly one crossing to every pair p, q of points of P. Let $C = C_{p,q}$ and note that one of the hemispheres defined by C contains z and the four edges from z to the four points $p, q, \bar{p}, \bar{q}$ on C (see Figure 7 right). On C the points of the pair $p, \bar{p}$ and the pair $q, \bar{q}$ alternate. It follows that the great arcs of p and q must be contained in different hemispheres of C, otherwise, they would be crossing. If the great arc of $p\bar{p}$ is contained in the hemisphere of z it has a crossing with one of the edges edge zq or $z\bar{q}$, otherwise, the great arc $q\bar{q}$ has a crossing with one of zp or $z\bar{p}$. This proves the claim.

Hence, the total number of crossings on S_z is $(k-2)\binom{k}{2} + \binom{k}{2}$ and the drawing of K_{2k+1} has $H(2k+1)$ crossings. □

Let Γ^+ be a drawing of K_{2k+1} with $H(2k+1)$ crossings which is based on an antipodal spherical arc drawing of K_{2k} as in the lemma. How many crossings can be saved by removing a matching of size t from the drawing? Of course we remove t great arcs. A great arc $p\bar{p}$ has a single crossing with each $C_{q,r}$ not containing p, this makes $\binom{k-1}{2}$ crossings on the arc. A single great arc can also have crossings with up to $k-1$ edges of the star S_z. Since the collection of arcs is strong, however, we know that if the great arc $p_1\bar{p}_1$ has a crossing with zp_2 or $z\bar{p}_2$, then there is no crossing between the great arc $p_2\bar{p}_2$ and the edges zp_1 and $z\bar{p}_1$. It follows that by removing t great arcs from the drawing Γ^+ we get rid of at most $t\binom{k-1}{2} + \sum_{\ell=1}^{t}(k-\ell)$ crossings.

We have $t\binom{k-1}{2} + \sum_{\ell=1}^{t}(k-\ell) = t\binom{k-1}{2} + tk - \binom{t+1}{2} = t(\binom{k-1}{2} + (k-1)) + (t - \binom{t+1}{2}) = t\binom{k}{2} - \binom{t}{2}$. This implies that a drawing of M^t_{2k+1} obtained along these lines has at least $H(2k+1) - t\binom{k}{2} + \binom{t}{2} = M(2k+1, t)$ crossings.

Proposition 18. *For all k and $0 \leqslant t \leqslant k$ there are spherical arc drawings of M^t_{2k+1} with $M(2k+1, t)$ crossings.*

This proposition motivates the conjecture that $cr(M^t_{2k+1}) = M(2k+1, t)$ (Conjecture 13). We provide two different constructions to prove the proposition.

Construction 1 (Can drawings)**.** In the even case $n = 2k$ can drawings can be realized as antipodal spherical arc drawings with a strong collection of great arcs between the rims, see Figure 8 (left). By extending this drawing with a new vertex z on one of the rims close to an existing vertex as in Figure 8 (right) we get a drawing of K_{2k+1}. In this drawing we have a collection of t great arcs which have $\sum_{\ell=1}^{t}(k-\ell)$ crossings with S_z. As shown above this implies that we obtain drawings of M_{2k+1}^{t} with $M(2k+1,t)$ crossings for all $t = 0,\ldots,k$. An example of this construction for $k = 3$ is shown in Figure 9 IV, vertex z is blue.

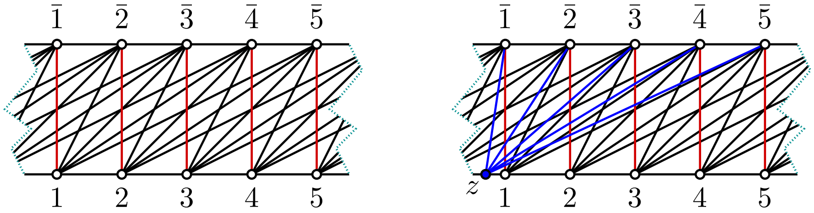

Figure 8: The crossing pattern of edges on the cylinder of a can drawing, red edges represent the great arcs. On the right the drawing was extended with a new vertex z. The number of intersections of the star edges of z with great arcs is decreasing from $k-1$ to 0 in unit steps.

Construction 2 (Intersecting great arcs)**.** The basis of this construction are the new Hill drawings of Section 3. We start with a drawing Γ of M_{2k+1}^{k} which is obtained from the antipodal spherical arc drawing of M_{2k} with $M(2k) = \binom{k}{2}\binom{k-2}{2}$ crossings by adding a new point z in general position and geodesic edges to all $2k$ points. From the proof of Lemma 17 we obtain that $cr(\Gamma) = \binom{k}{2}\binom{k-2}{2} + (k-2)\binom{k}{2} = M(2k+1,k)$.

We use Γ to construct a drawing Γ^{+} of K_{2k+1}. For each antipodal pair $p, \bar{p}$ we introduce the unique great arc passing through $\bar{z}$. This makes k great arcs pass through a single point, i.e., the drawing has $\binom{k}{2}$ crossings between great arcs at $\bar{z}$. The star S_z, however, has no crossings with great arcs. Removing any t of the great arcs reduces the number of crossings between the great arcs by $\sum_{\ell=1}^{t}(k-\ell)$. Hence, we obtain drawings of M_{2k+1}^{t} with $M(2k+1,t)$ crossings. An example of this construction for $k = 3$ is shown in Figure 9 I, vertex z is blue.

7 Conclusion

Table 1 shows the values of $M(n,t)$ for $5 \leqslant n \leqslant 15$. The values in row 1 are the Hill numbers they appear in the paper of Harary and Hill [HH63] and in the earlier publication of Guy [Guy60]. The entries in the columns for even $n = 2k$ have been proposed by Mohar [Moh20], see Conjecture 8. The values in the columns for odd $n = 2k+1$ are new to this paper, as is Conjecture 13. The green entries represent cases where we know that $M(n,t)$ equals $cr(M_n^t)$. The green values in row 1 (Hill conjecture) have been obtained by Guy [Guy72] (for $n \leqslant 10$), Pan and Richter [PR07] (for $n = 11, 12$), and Aichholzer [Aic21] (for $n = 13, 14$). The green values in row 2 are a consequence of the established cases of the Hill conjecture, see Ouyang, Wang, and Huang [OWH14]. The entries for M_{10}^{5}, M_{12}^{6}, and M_{12}^{7} are a consequence of Theorem 9. Entries with $n \leqslant 9$ have been checked partly by hand and partly by computer.

There are some indications that a proof of $cr(M_{2k}) = M(2k,k)$ might be easier than a proof of the Hill conjecture for the complete graph: first the implication of Theorem 9, second the simple closed form of $M(2k,k) = 6\binom{k}{4}$, and third the simple description of antipodal spherical drawings realizing the conjectured crossing number of M_{2k}. These drawings have the property

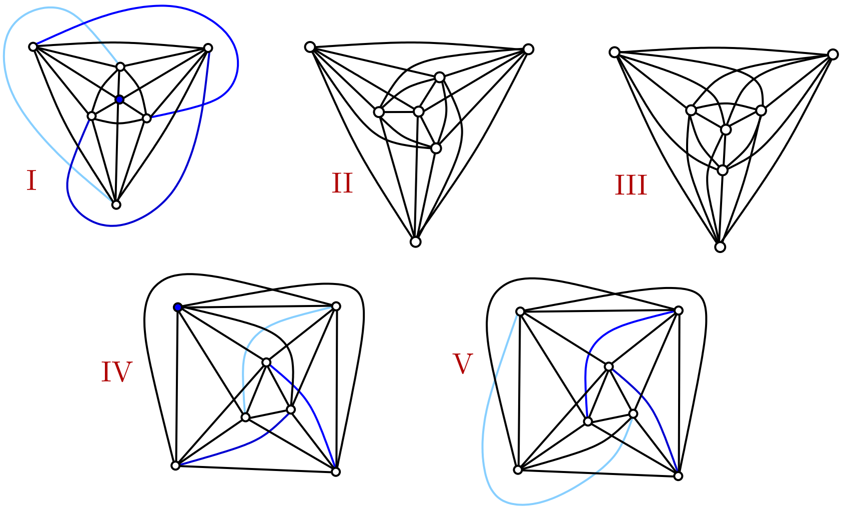

Figure 9: The five weak isomorphism classes of optimal drawings drawings of $K_7 = M_7^0$, i.e., with $cr(K_7) = 9$ crossings. Nr. IV is the can and also the 2-page drawing. Removing the blue edges in order from light to dark yields optimal drawings of M_7^t for $t = 1, 2, 3$.

n	5	6	7	8	9	10	11	12	13	14	15
$H(n) =$ $M(n,0)$	1	3	9	18	36	60	100	150	225	315	441
$M(n,1)$	0	2	6	15	30	54	90	140	210	300	420
$M(n,2)$	0	1	4	12	25	48	81	130	196	285	400
$M(n,3)$		0	3	9	21	42	73	120	183	270	381
$M(n,4)$				6	18	36	66	110	171	255	363
$M(n,5)$						30	60	100	160	240	346
$M(n,6)$								90	150	225	330
$M(n,7)$										210	315

Table 1: Values of $M(n,t)$ for $5 \leqslant n \leqslant 15$. The green entries represent cases where we know that $M(n,t)$ equals $cr(M_n^t)$

that the drawing induced by the six vertices of any three edges of the matching has no crossing; in other words any induced M_6 (octahedron graph) in the drawing is plane. We think that this property holds for all optimal drawings of M_{2k}:

Conjecture 19. *In an optimal drawing of M_{2k} every induced octahedron graph is plane.*

Conjecture 19 implies $cr(M_{2k}) = M(2k, k)$, i.e., the $t = k$ case of Conjecture 8.

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