1-Planar Drawings of Products of Cycles

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Abstract

We show that the Cartesian product of two cycles $C_m \times C_n$, i.e. the $m \times n$ toroidal grid, admits exactly three distinct simple 1-planar drawings when $5 \leq m \leq n$. These drawings have $(m-2)n$, $m(n-2)$, and $mn-4$ crossings, respectively. As a corollary, we obtain that the 1-planar crossing number of $C_m \times C_n$ with $3 \leq m \leq n$ is $(m-2)n$. This confirms, in the 1-planar setting, a conjecture of Harary, Kainen, and Schwenk. The drawings have the property that the n fundamental cycles C_m and the m fundamental cycles C_n of $C_m \times C_n$ form two classes of pairwise disjoint cycles. In fact, establishing this property is a crucial step in our proof.

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1 Introduction

Among the graphs whose crossing number has been studied most intensely are *products of cycles* $C_m \times C_n$, also known as *toroidal grids*. In 1973 Harary, Kainen and Schwenk [5] conjectured that

$$\mathsf{cr}(C_m \times C_n) = (m-2)n \quad \text{for all } 3 \leq m \leq n.$$

The conjecture has been verified for $C_m \times C_n$ with $3 \leq m \leq 7$ and $m \leq n$, cf. Adamsson and Richter [1] and references therein. Based on techniques introduced in the thesis of Adamsson and used in [1] Glebsky and Salazar [4] proved that the crossing number of $C_m \times C_n$ is as conjectured when $n \geq m(m+1)$.

A 1-planar graph is a graph that can be drawn in the plane or on the sphere such that each edge participates in a crossing with at most one other edge. In a 1-planar drawing we can eliminate touchings of edges and crossings of edges which share a vertex, hence, we can assume that 1-planar drawings only have proper crossings between pairs of independent edges, i.e., they are *simple drawings*. 1-planar graphs are among the most popular classes of beyond-planar graphs. They have been studied in many respects, including colorings, edge density, and recognition complexity. The annotated bibliography [7] contains almost 150 references. Also, see the local crossing number chapter in [11]. 1-planar drawings and more generally drawings minimizing the maximum number of crossings per edge have been considered for products of graphs such as hypercubes [6], products of paths, stars and cycles [10] and 3-dimensional grids [2]. Minimal obstructions against 1-planarity have been studied by Korzhik and Mohar [8], they also show that recognizing 1-planar graphs is NP-complete.

The result of the present paper is:

▶ **Theorem 1.** *For $5 \le m < n$ there are exactly three different simple 1-planar drawings of $C_m \times C_n$ on the sphere. The three drawings have crossing numbers $(m-2)n$, $(n-2)m$, and $mn-4$, respectively. If $m = n$ two of the three drawings coincide.*

Figure 1 shows the respective drawings of $C_5 \times C_7$. In the figure we have drawn the 7 fundamental cycles C_5 in red and the 5 fundamental cycles C_7 in blue. In the drawing in (a) the family of red cycles is nested[1] and the family of blue cycles is disjoint; in the drawing in (b) the family of red cycles is disjoint and the family of blue cycles is disjoint; and in the drawing in (c) both families are nested. The first two drawings are already present in [10]. Of course, if $m = n$ then these two drawings coincide if the coloring is ignored.

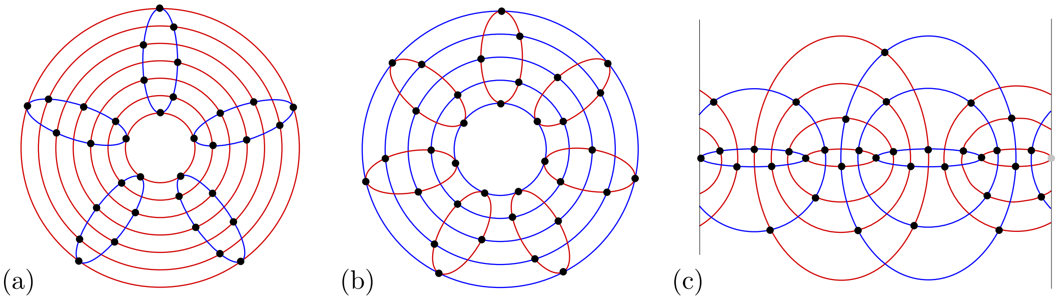

Figure 1 The three non-isomorphic 1-planar drawings of $C_5 \times C_7$. In drawing (c) the left and right boundary are identified.

Topologically the third type of drawings is the most interesting, it comes from the well-known degree 2 map ϕ from the torus T^2 to the sphere S^2 with 4 branch points. To understand this map consider a flat torus in a unit square and label the half-integer points as in Figure 2(right). These points are the corners of four squares half the size. Now take the sphere and label four equidistant points on the equator as w, x, y, z in cyclic order. Finally map the four half-squares of the flat torus to the upper or lower hemisphere of S^2 such that the labeled points on the boundary of the half-squares are mapped to their counterparts on the equator. The two half-squares where w, x, y, z are in counterclockwise order are mapped to the same hemisphere and the other two half-squares are mapped to the other hemisphere.

If we want to use this map to obtain a 1-planar drawing of a product of cycles from a regular mesh on the flat torus we must be careful with the choice of the maps from the square to the hemispheres. And finally we must adjust in one detail. At the branch points the picture will locally be as in Figure 3(a), since two edges incident to the same vertex cross we have to change the figure slightly to make the drawing simple. After the modification we have a vertex where two cycles touch as shown in Figure 3(b).

In the cases $m = 3$ and $m = 4$ there are more than three 1-planar drawings. In these cases the number of 1-planar drawings grows with n. The additional flexibility can be seen

[1] A triple of cycles in a drawing is *nested* if one of the cycles is separating the other two. A family is nested if every triple is nested. A family is disjoint if none of its triples is nested.

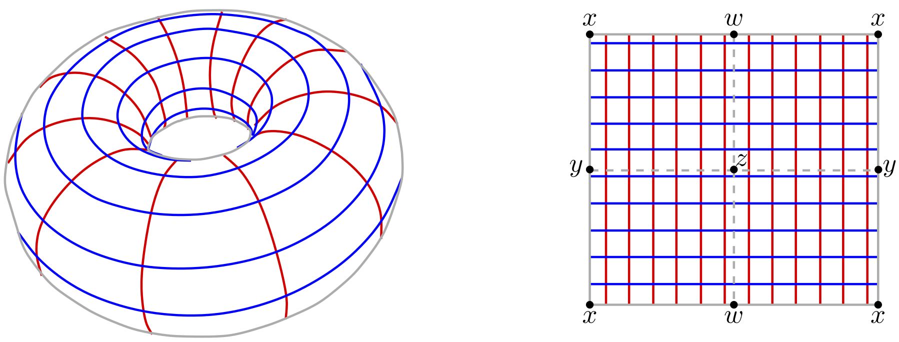

Figure 2 Embeddings of $C_{10} \times C_{12}$ on the torus (left) and on the flat torus (right).

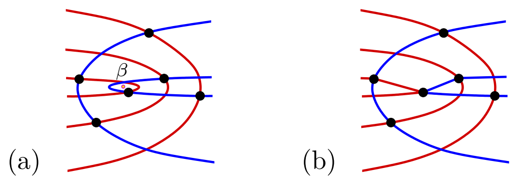

Figure 3 Local modification around branch point β to make the drawing simple.

in the sketches of Figure 17. We expect to still be able to provide a complete classification of the 1-planar drawings of products $C_4 \times C_n$ and $C_3 \times C_n$ in the near future.

Figure 2 shows a crossing free embedding of a product of cycles on two different models of the torus. The coloring of the fundamental cycles has the following primal and dual alternation property:

- The color of edges incident to every vertex is cyclically alternating.
- The color of edges incident to every face is cyclically alternating.

We refer to the quadrangular faces of torus grids (and their boundary cycle) as *tiles*. In the context of 1-planar drawings of $C_m \times C_n$ with $m, n \geq 5$ a tile may be just a 4-cycle of the graph or a simple 4-cycle together with its interior, see Section 2 for the definition.

We present the proof of our characterization in three sections. In Section 2 we show that if in a 1-planar drawing of $C_m \times C_n$ a tile is drawn as a simple curve, then one side of the tile contains at most one vertex of the drawing. The argument is based on connectivity properties.

In Section 3 we study tiles with one interior vertex. The final result is that there are only three possibilities for such a tile and that every tile is drawn as a simple curve. As a byproduct of the proof we obtain that in a 1-planar drawing of $C_m \times C_n$ with $m, n \geq 5$ the red and the blue cycles form two families of pairwise disjoint cycles.

In Section 4 we investigate how to glue the tiles to finally obtain a complete 1-planar drawing. Interestingly a well-chosen seed of two adjacent tiles is already enough to establish in which of the three 1-planar drawings of $C_m \times C_n$ the construction will end.

In the final Section 5 we indicate future directions for research and mention some open problems.

2 Tiles and their interior

In the following proposition we collect some properties of $C_m \times C_n$ with $m, n \geq 5$. We refer to this product of cycles as $G = (X, E)$, and write $C[Y]$ for the cut induced by $Y \subset X$, i.e.,

$C[Y] = \{(y, z) \in E : y \in Y, z \in \bar{Y} = X - Y\}$.

▶ **Proposition 2** (Connectivity properties).

1. *G has no triangles and every 4-cycle of G is a tile.*
2. *The graph G is 4-connected.*
3. *All 4-cuts of G are vertex cuts, i.e., of type $C[x]$ where x is a vertex.*
4. *If $|Y| = 2$, then $|C[Y]| = 6$ if the two vertices form an edge and $|C[Y]| = 8$ otherwise.*
5. *If $|Y| = 3$ then $|C[Y]| \geq 8$ with equality if and only if Y induces a 2-path.*
6. *If $|X \setminus Y| \geq |Y| \geq 4$ then $|C[Y]| \geq 8$ with equality if and only if $|Y| = 4$ and Y consists of the four vertices of a tile.*

Proof. 1. Every odd cycle in $C_m \times C_n$ is non-separating. Hence, a triangle can only occur if $m = 3$ or $n = 3$. A cycle of length four in $C_m \times C_n$ is either separating or a tile. Since we assume $m, n \geq 5$ there is no separating 4-cycle.

2. It is enough to show that for all u, v distinct there are four internally disjoint paths connecting u and v. Just select 3 blue and 3 red cycles such that u and v are two of the nine vertices at an intersection of a blue and a red cycle. It is easy to find the four paths on the $C_3 \times C_3$.

For the remaining items note that if the cut $C[Y]$ can be viewed as a union of cycles of the dual graph. We denote the set of dual edges of $C[Y]$ as $C^*[Y]$. If $C^*[Y]$ contains a non-separating cycle, then $|Y| \geq 5$ and $|C[Y]| \geq 10$. Hence we can view Y as a subset of an infinite grid. If $|Y| = k$ then $C^*[Y]$ is a set of cycles in a grid which enclose k cells. The claimed bounds on $|C[Y]|$ follow by inspection. ◀

In the remainder of this subsection we locally look at a tile respectively a tile, T in a 1-planar drawing of $C_m \times C_n$ with $m, n \geq 5$. The red blue coloring of G imposes an alternating coloring on T. Each of the four vertices of T is incident to two edges of T and two extra edges connecting T to the rest of the graph, these edges will be called the *attachment edges* of T.

In a drawing T can either be *simple*, i.e., a simple curve, or *crossing*, otherwise. If T is simple, then by the Jordan curve theorem T has two sides. For each side we count how many attachment edges are contained in the side and select a side with at most four attachment edges as the small side and refer to the small side as the *interior* of T.

▶ **Lemma 3.** *A simple tile T in a 1-planar drawing of $C_m \times C_n$ has at most one vertex in its interior.*

Proof. Let Y be the set of vertices in the open interior of T. All the edges of $C[Y]$ must either cross one of the four edges of the tile T or they are attachment edges and connect to a vertex of T. By definition of the small side there are at most four attachment edges, hence $|C[Y]| \leq 8$. The connectivity properties imply $|Y| \leq 4$.

We claim that every vertex $y \in Y$ is incident to at most one attachment edge. Indeed two attachment edges connecting to adjacent vertices of T would form a triangle which is impossible (connectivity property 1). Two attachment edges connecting to diagonally opposite vertices of T would form two 4-cycles sharing the two attachment edges. This is impossible since 4-cycles are tile boundaries of G (connectivity property 1), and two tiles of G share at most one edge.

Suppose $|Y| = 1$, i.e., there is a single vertex y interior to T. We already know that y is incident to at most one attachment edge. Up to symmetries, the possible cases are shown in Figure 4.

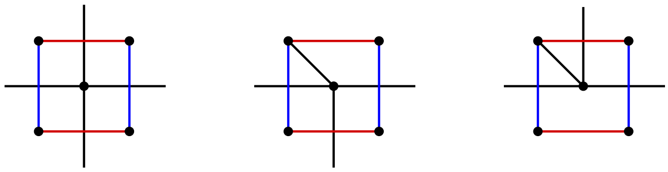

Figure 4 The possibilities for a tile with exactly one vertex in the interior.

If $|Y| = 3$, then these are at most tree attachment edges, one for every vertex of Y. Hence, at most 7 edges can connect Y to the complement $\bar{Y} = X - Y$. Since, $|C[Y]| \geq 8$ (connectivity property 5), this case is not possible.

If $|Y| \geq 4$, then $|C[Y]| \geq 8$. Since we allow at most four attachment edges on the small side we need $|C[Y]| = 8$ which is possible only if $|Y| = 4$ and the four vertices form a 4-cycle. The picture then looks as in Figure 5. There is no coloring of the edges such that every 4-cycle is alternatingly colored. Hence, this is not possible if $m, n \geq 5$. It is, however possible in 1-planar drawings of $C_4 \times C_n$, see Figure 17.

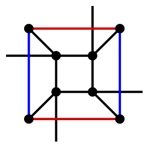

Figure 5 A tile with four internal vertices is only possible if $\min(m, n) \leq 4$.

We now come to the most complex case $|Y| = 2$. From $|C[Y]| \geq 6$ (connectivity property 4) it follows that there are at least two attachment edges. Since each vertex in Y has at most one attachment edge we conclude that we have two attachment edges and $|C[Y]| = 6$. In particular the two vertices of Y are connected by an edge. Two attachment edges connecting to the same vertex of T would be part of a triangle, impossible. The two remaining cases are shown in Figure 6. In the situation shown in Figure 6(b) there is a 5-cycle $(y_1, y_2, x_1, x_2, x_3)$. Contractible cycles of G are part of a bipartite graph, hence, they are of even length. It follows that the 5-cycle is non-contractible and since we have $m, n \geq 5$ it is monochromatic. However, the 5-cycle contains the edges x_1x_2 and x_2x_3 which alternate in color. This yields a contradiction.

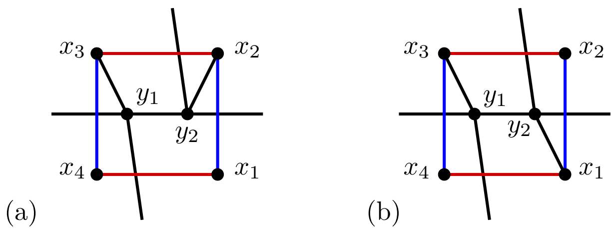

Figure 6 Two tiles with two vertices in the interior.

The case shown in Figure 6(a) is more involved. The four cycle (y_1, y_2, x_2, x_3) must be the second tile containing edge x_2x_3, this establishes the coloring of the edges of this 4-cycle. We now consider different cases for the coloring of the other edges incident to y_1 and y_2. If the edge y_1u crossing x_4x_1 is red, see Figure 7(a), then there is a vertex v such that (y_1, u, v, x_3) is the second tile of edge y_1x_3. This tile separates x_4 and w from x_1. The side of the tile containing x_4 and w has $|Y| \geq 2$ and from the previous cases it must have $|Y| = 2$, however $|Y| = 2$ is only possible if the two vertices are connected. Since the path (w, y_1, x_3, x_4) is monochromatic we deduce that w, x_4 is not an edge. Hence, this case is not possible in a 1-planar drawing of G.

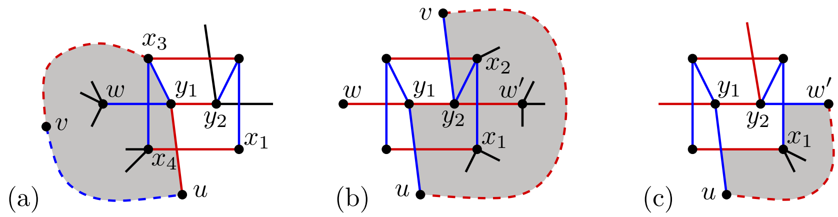

Figure 7 Details for the three subcases of Figure 6(a).

From now on we assume that the edge y_1u crossing x_4x_1 is blue. If the edge y_2v crossing x_2x_3 is also blue, Figure 7(b), then there is a red edge uv which completes the second tile of edge y_1y_2. The tile separates w and w' and has at least three vertices on either side, this is not possible.

Now assume that y_1u is blue and y_2v is red, see Figure 7(c). Again we consider the second tile of edge y_1y_2. The tile is (y_1, y_2, u, w') and has x_1 in its interior. An edge x_1w' would close a non-alternating 4-cycle. An edge x_1u would close a non-monochromatic 5-cycle. The interior of the tile (x_1, x_2, x_3, x_4) contains y_1 and y_2 and cannot contain further vertices. We now get a contradiction because the four edges of x_1 must cross the four edges of the tile, but y_1y_2 is not crossed by an edge of x_1. This shows that this case is also impossible.

This concludes the proof of Lemma 3. ◀

3 Tiles with one interior vertex

We now know that in a 1-planar drawing of $C_m \times C_n$ with $m, n \geq 5$ the tiles either have zero or one interior vertex. In this section we study tiles with one interior vertex. The final result is that there are three possibilities for such a tile, they are shown in Figure 14.

▶ **Lemma 4.** *Let T be a simple tile in a 1-planar drawing of $C_m \times C_n$ and v in its interior. If two edges of v cross opposite edges of T, then the colors of these two edges of v are equal.*

Proof. Every pair of a red and a blue edge incident to v extends to a tile. Suppose two edges of v leaving T through opposite edges have distinct colors. Then these two edges belong to a tile containing at least two vertices on either side, see Figure 8, by Lemma 3 this is not possible. ◀

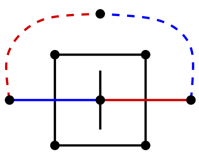

Figure 8 Illustrating the proof of Lemma 4.

In the proof of the following lemma we use a non-local argument. This is necessary because 1-planar drawings of "twisted tori" (4-regular vertex-transitive graphs whose edges can be partitioned into a family of red cycles and a family of blue cycles) may contain tiles with crossing edges.

▶ **Lemma 5.** *Every tile in a 1-planar drawing of $C_m \times C_n$ is simple.*

Proof. Suppose that in the drawing there is a tile with crossing edges. Our argument will allow the exchange of colors, hence, we can assume that the blue edges of the tile cross, see Figure 9(a). The second tile of each of the blue edges either has the red edge x_1x_2 or the edge x_3x_4 as attachment edge for its interior vertex. Up to symmetry there are two cases, either they have the same interior red edge or they have distinct interior red edges. These are the cases shown in Figure 9(b) and Figure 9(c).

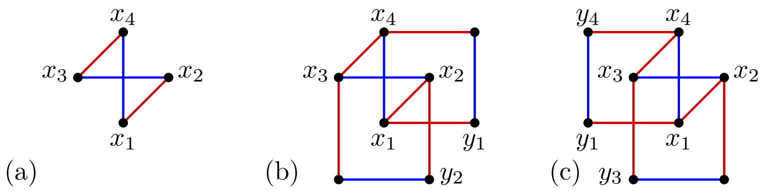

Figure 9 Illustrating the proof of Lemma 5.

The first case (Figure 9(b)) is easy. The second tile incident to edge x_1x_2 must contain either vertices y_1 and y_2 or x_3 and x_4 in its interior. This is not consistent with Lemma 3.

In the second case, there is no local argument. In Figure 9(c) consider the second tile T of edge y_1x_1. This tile must enclose y_3 as its only interior vertex. The second red edge of y_3 cannot be an attachment edge to T because the red cycle C of edge x_3y_3 shares the vertices x_4 and y_4 with the cycles of the blue edges of T. The second red edge of y_3 is not allowed to cross one of the blue edges, otherwise a violation of Lemma 4 is unavoidable. It follows that the second red edge of y_3 is crossing the second red edge of T.

The essentially same argument can be iterated. In Figure 10(a) consider the second tile T of edge xz. This tile x, x', z', z must enclose y' as its only interior vertex. The second red edge $y'y''$ of y' cannot be an attachment edge to T because the red cycle C of edge yy' shares the vertices x_4 and y_4 with the cycles of the blue edges of T. The edge $y'y''$ is not allowed to cross one of the blue edges, otherwise a violation of Lemma 4 is unavoidable. It follows that $y'y''$ is crossing $x'z'$ as shown in Figure 10(b).

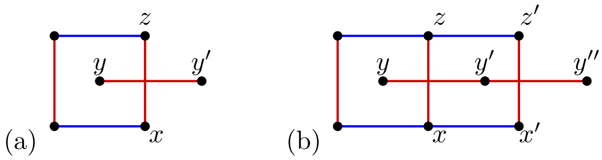

Figure 10 Illustrating the continuation argument for the second case (Figure 9(c)) in Lemma 5.

The cycle C is closed, hence, at some iteration of the argument we must have $y'y'' = y_4x_4$ (here we mix the labelings from Figure 9(c) and Figure 10(b)). This implies that one endpoint of the edge $x'z'$ is interior to the tile x_1, y_1, y_4, x_4. Since the tile already contains x_3 this violates Lemma 3. ◀

▶ **Lemma 6.** *If T is a tile with an interior vertex and an attachment edge in a 1-planar drawing of $C_m \times C_n$, then up to symmetries and exchanging colors T looks as shown in Figure 11.*

Proof. In Figure 4 we have two cases where the interior vertex has an attachment edge. Assuming that the attachment edge is blue and respecting Lemma 4 we are left with the four cases shown in Figure 12.

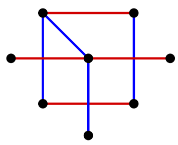

Figure 11 Up to symmetries and exchanging colors the unique tile with an interior vertex and one attachment edge.

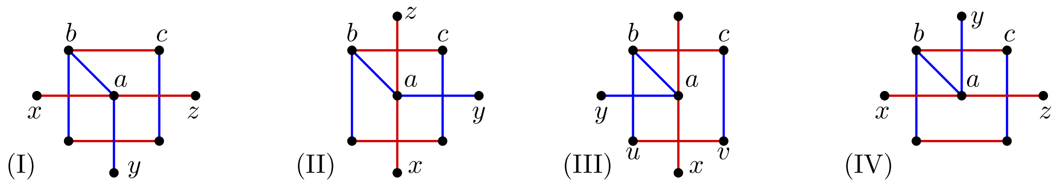

Figure 12 The four cases.

The lemma says that case (I) is the only possible case, hence, we have to show that the other cases are impossible.

In case (II) consider the tile that contains edges ab and bc. If ax is the second red edge of the tile, then the second blue edge is xc and the tile contains two vertices in its interior, i.e., it violates Lemma 3. If az is the second red edge of the tile, then az and bc form a pair of crossing edges in the tile, i.e., the tile violates Lemma 5.

In case (III), consider the tile that contains edges ab and ax. If bc is the second red edge of the tile, then the second blue edge is xc and the tile contains vertex v in its interior. An edge vx would close a triangle, hence, the two missing edges of v must both cross xc, a contradiction. If the second red edge of the tile is not bc, then the second tile of edge ab contains az and bc, i.e., a crossing pair of edges, this violates Lemma 5.

In case (IV) consider the tile containing edges ab and bc. If ax is the second red edge of the tile, then the second blue edge is xc and the tile contains two vertices in its interior, i.e., it violates Lemma 3.

It remains to consider the subcase of (IV) where a, b, c, z form a tile, i.e., cz is a blue edge. This again requires a non-local argument. Consider the second tile incident to the edge cv, this tile T must contain z. The second tile incident to edge az must contain the vertex c. It follows that the tile with z in the interior looks just as the tile with a in the interior, see Figure 13. This argument can be iterated. With the labeling as in the figure the iteration generates a red cycle C containing the edges $z_i z_{i+1}$, i.e., some z_k has to be equal to a.

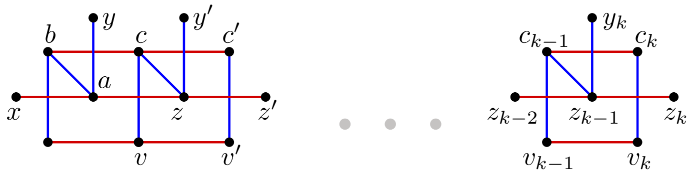

Figure 13 Iterating the argument generates a strip of identical tiles.

Cycle C is only intersected by blue edges, hence it is a simple cycle. The blue cycle C' containing vertex z_i has a vertex v_i on one side of the Jordan curve C and a vertex y_{i+1} on the other side, hence it has to cross the Jordan curve a second time. This, however, comes with a second vertex in $C \cap C'$, a contradiction. ◀

▶ **Lemma 7.** *Every crossing in a 1-planar drawing of $C_m \times C_n$ involves a red and a blue edge.*

Proof. Suppose that the drawing contains a pair of red crossing edges xz and yy'. Consider a tile of edge xz, this tile has one of the vertices of yy' in its interior. By symmetry we choose y'. From Lemma 6 we know that a tile with an interior attachment edge has no monochromatic crossing. Hence, the four edges of y' cross the four edges of the tile. From Lemma 4 we know that the second red edge $y'y''$ of y' is crossing the second red edge $x'z'$ of the tile. We obtain a situation as shown in Figure 10. Iterating the argument we obtain a red path $y, y', y'', \dots$ which must eventually close to a cycle C. Each vertex of C is enclosed by a tile, together these tiles form a strip S containing C.

Now consider the two blue edges at y, they belong to a blue cycle C' and have vertices on both sides of the strip S. Hence, C' has to cross the strip a second time, however, when a blue cycle crosses S it has a vertex in the strip, i.e., a crossing with C. This shows that $|C \cap C'| \geq 2$, a contradiction. ◀

In this section we have established that in a 1-planar drawing of $C_m \times C_n$ with $m, n \geq 5$ a tile with one interior vertex looks like one of the tiles shown in Figure 14.

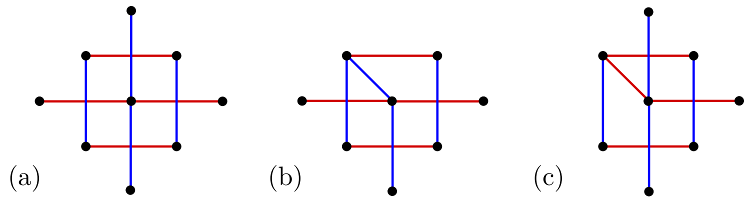

Figure 14 The three cases of a tile with one interior vertex.

4 Assembling the tiles

In the previous sections we investigated the local structure of 1-planar drawings of a product $C_m \times C_n$ of cycles with $m, n \geq 5$. We established that a tile must be drawn as a simple curve, it may contain at most one vertex in its interior, and if it contains a vertex, then it looks like one of the tiles shown in Figure 14. In the following we call a tile as shown in Figure 14(a) an *x-tile*, in the other two cases we have an *f-tile*. Sometimes we emphasize the color of the internal attachment edge and denote the f-tile in (b) and (c) as *f-tile* and *f-tile* respectively.

In this section we investigate how to glue the tiles to finally obtain a complete 1-planar drawing. Interestingly, a well-chosen seed of two adjacent tiles is already enough to establish in which of the three 1-planar drawings of $C_m \times C_n$ the construction will end.

For all $m, n \geq 3$ the graph $C_m \times C_n$ contains a subdivision of $K_{3,3}$ as a subgraph. Hence, $C_m \times C_n$ is non-planar and a 1-planar drawing will contain a crossing. The tiles incident to the edges of a crossing must be x-tiles or f-tiles. After placing these incident tiles there are new crossing edges which require additional x- or f-tiles. As long as only x-tiles are used in the growth process the graph that has been laid out has two components. We conclude:

▶ **Fact.** *A 1-planar drawing of $C_m \times C_n$ must contain an f-tile.*

Now fix $m, n \geq 5$ and let $G = C_m \times C_n$. We start the construction of a 1-planar drawing of G with an f-tile. We have the choice between an f-tile and a f-tile. We take a tile f-tile with a blue attachment edge and vertices labeled as in Figure 15(a). The attachment edge

x_0y_0 belongs to two tiles. Let T be the tile which is also incident to y_0y_1. Tile T is also incident to one of the red edges of x_0. Hence T is either x_0, y_0, y_1, x_1 or it is x_0, y_0, y_1, x_{-1}. The two cases are shown in Figure 15(b) and (c). In both cases we have already added -in light color- another tile.

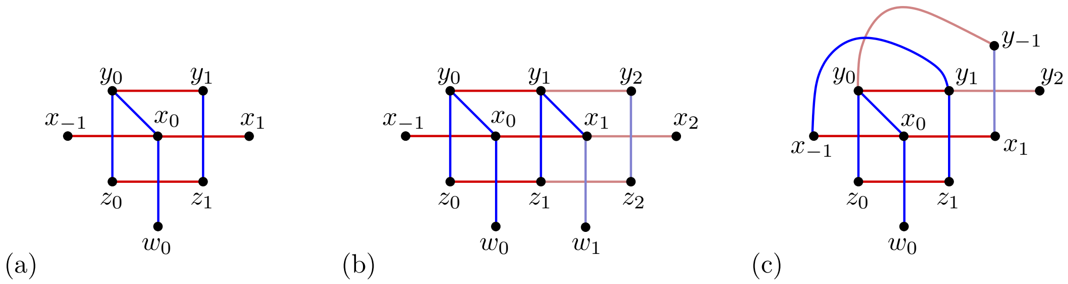

Figure 15 An f-tile (a) and its two possible extensions with a tile $x_0, y_0, y_1, *$.

In Figure 15(b) the second tile of edge y_1z_1 must have x_1 as interior vertex so that we obtain a second f-tile y_1, y_2, z_2, z_1. We will see that these two f-tiles side by side will completely determine the 1-planar drawing of G. In the sequel the path $y_0y_1y_2$ will be called the *ridge* of the configuration.

In Figure 15(c) the second tile of edge x_0y_0 uses edge x_0x_1 and the second red edge of y_0. This tile must enclose y_1 and turns out to be a f-tile. We call such a pair of a blue and a red f-tile a *swirl* and will see that again this local configuration of two f-tiles completely determines the 1-planar drawing of G.

4.1 1-planar drawings with a ridge

We continue referring to Figure 15(b) and the labeling there. By looking at the two tiles of edge x_1y_1 we see that x_2y_2 must be a blue edge. Now, the second tile of edge y_2z_2 must have x_2 in its interior. With the natural labeling we get another f-tile y_2, y_3, z_3, z_2 with a crossing of the blue edge y_3z_3 and the red edge x_2x_3 on the right. This can be iterated and we get a sequence of f-tiles $y_i, y_{i+1}, z_{i+1}, z_i$ with interior vertex x_i. The ridge path also extends $y_0y_1y_2 \dots y_iy_{i+1} \dots$. This red path however must close to a red cycle, i.e. a cycle of length m. So that $y_m = y_0$, it should be obvious that there is no problem with closing the cycle.

At this point three red cycles are completed, the ridge cycle C_y as well as C_x and C_z. From each blue cycle we have a path w_ix_i, y_iz_i of length 3.

The second tile of edge z_iz_{i+1} must contain w_i, let its extra vertices be a_i and a_{i+1} . Adding these tiles for all i yields a new red cycle C_a. Now if $n = 5$, then, since we have five vertices on the blue path $w_ix_i, y_iz_ia_i$, we must add the blue edge a_iw_i. This makes the tile $z_i, z_{i+1}, a_{i+1}, a_i$ an f-tile. Adding all red edges w_iw_{i+1}, i.e., the red cycle C_w completes the construction in this case.

If $n > 5$, then the tile $w_ix_i, y_iz_ia_i$ must be an x-tile and we get new blue edges w_iu_i crossing $a_{i+1}a_i$. This procedure of adding a new cyclic strip of tiles attached to the extreme red cycle can be repeated until the length of all the blue paths becomes $n - 1$. In this case we then have to close the blue cycles with attachment edges, thus generating a second red ridge cycle.

The result of this construction is a 1-planar drawing of $C_m \times C_n$ where only the edges of the two red ridge cycles are uncrossed, i.e., there are $mn - 2m = m(n - 2)$ crossings. The family of red cycles is nested while the family of blue cycles is disjoint. An instance of this type of drawing is shown in Figure 1(a).

Starting with an f-tile, adding a second f-tile to initialize a ridge would yield the very same construction scheme, just change the colors and the parameters m and n. The result would be a 1-planar drawing with $(m-2)n$ crossings, a nested family of blue cycles, and a disjoint family of red cycles. An instance of this type of drawing is shown in Figure 1(b).

4.2 1-planar drawings with a swirl

Figure 16(a) shows a nicer and more symmetric picture of the situation of Figure 15(c).

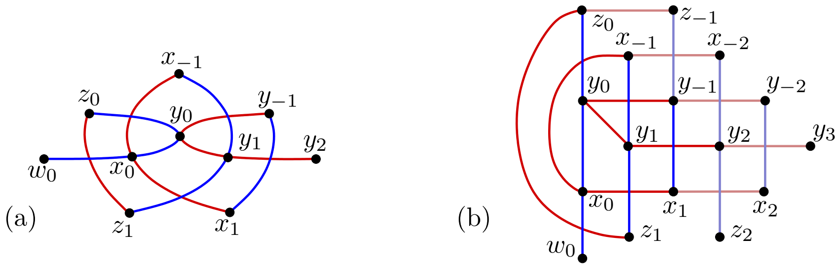

Figure 16 (a) A symmetric picture of the swirl. (b) Adding tiles expanding the y and x paths before closing them to C_y and C_x.

We now discuss vertices and tiles that extend the seed in a unique way. Figure 16(b) illustrates the following description. The second tile of edge y_0z_0 is using the red edges y_0y_{-1} and z_0z_{-1} and encloses x_{-1}. It must be an x-tile, otherwise we get a pair of a red and a blue path sharing two vertices. The second tile of edge $x_{-1}y_1$ is using the red edges y_1y_2 and $x_{-1}x_{-2}$ and encloses y_{-1}. It must be an x-tile because otherwise we get a red 4-cycle or a pair of a red and a blue path would share two vertices. Now the second tile of edge x_1y_{-1} is using the red edges edge $y_{-1}y_{-2}$ and x_1x_2 and encloses y_2. The only option for this tile to not be an x-tile is that $m = 5$ and edge y_2y_{-2} closes the cycle C_y.

This procedure of adding new tiles which extends one of the two branches of a red path that will eventually be closed to be the cycle C_y or the cycle C_x can be repeated until the path of C_y contains m vertices and must be closed. Note that when the red cycle has to be closed we have a set of m blue paths 'orthogonal' to the red paths.

Let us first assume that $m = 2k+1$ is odd. In this case, the edge that will close the cycle is $y_{-k}y_k$. This edge is the attachment edge of an f-tile $y_{-(k-1)}y_{-k}x_kx_{k-1}$. The edge x_kx_{-k} must close the red cycle C_x, and the edge z_kz_{-k} must close the red cycle C_z. Together with some of the blue edges of the orthogonal blue paths, this creates a swirl. The uncrossed blue edge of the swirl is $y_{-k}x_k$.

The even case $m = 2k+2$ is very similar. In this case, the edge that will close the cycle is $y_{-k}y_{k+1}$. This edge is the attachment edge of an f-tile $y_{k+1}y_kx_{-k}x_{-(k+1)}$. The edge $x_kx_{-(k+1)}$ must close the red cycle C_x, and the edge $z_{k+1}z_{-k}$ must close the red cycle C_z. Together with some of the blue edges of the orthogonal blue paths, this creates a swirl. The uncrossed blue edge of the swirl is $y_{k+1}x_{-(k+1)}$.

To summarize: If we have a swirl with touching cycles C and C', then we can assume that these cycles are flat in the sense that they contain no vertex in their interior and both of them participate in a second swirl where they again touch a flat cycle of the other color.

If two cycles involved in touchings at swirls cross, then, because they are flat, they have two common vertices. Since any pair of cycles of different color must have exactly one common vertex It follows that we have exactly four circles involved in touchings at swirls. They can be arranged cyclically with alternating colors such that adjacent cycles touch: C_1, C_2, C_3, C_4, C_1.

When all tiles involving edges of these 4-cycles have been laid out, then the rest of the 1-planar drawing of $C_m \times C_n$ is uniquely determined. In fact, we can lay out tile by tile such that whenever a tile is placed, we know that it must be an x-tile. This is because otherwise two paths of the same color are joined, but there is a flat C_i that shares a vertex with both of them.

Starting from the touching with C_1 we can enumerate the red edges crossing C_2 and C_4. If e_1 and e_1' are the first edges in these two enumerations, then the drawing must contain a strip of tiles such that one side of the strip is a red path from e_1 to e_1' the other side of the strip is one of the paths on C_1 between the two touchings and the inner vertices of the tiles in the strip are the vertices of C_1 on the other path between the touchings. The next strip would contain a strip of tiles such that one side of the strip is a red path from e_2 to e_2' the other side contains the vertices that had been interior in the previous strip and the vertices on the path between e_1 to e_1' are the interior vertices of the tiles. Iterating this, we fill one of the two sides of the C_1, C_2, C_3, C_4 cycle in a grid like fashion. The other side is treated the same way.

The result of this construction is a 1-planar drawing of $C_m \times C_n$ where almost all edges are crossed, the only uncrossed edges are the two attachment edges of the f-tiles at the four swirls, i.e., there are $mn - 4$ crossings. The family of red cycles, as well as the family of blue cycles, is nested. An instance of this type of drawing is shown in Figure 1(c).

5 Conclusion and future work

We have shown that for $5 \leq m < n$ there are exactly three simple 1-planar drawings of $C_m \times C_n$. If $m = n$, then two of the three drawings coincide. In the first part of the proof the local structure of such drawings is analyzed, and in the second part it is shown that very small local seeds (drawings of six vertices and the edges induced on them) already determine the whole 1-planar drawing of $C_m \times C_n$.

If we allow cycles C_3 and C_4 as factors of a product of cycles, then in the first part of the proof we have more cases and indeed additional local structures become possible. The additional flexibility can be seen in the sketches of Figure 17. These sketches make it evident, that for growing n the number of 1-planar drawings of $C_3 \times C_n$ and $C_4 \times C_n$ also grows. We still expect to be able to provide a complete classification of the 1-planar drawings of these products in the near future.

There are two natural ways of going beyond $C_m \times C_n$. First, one can consider $C_k \times C_m \times C_n$. From our result it easly follows that already $K_2 \times C_m \times C_n$ is not 1-planar for $5 \leq m \leq n$ and we are in reach to extend this to $3 \leq m \leq n$.

Second, for $t \in \mathbb{Z}_m$, the *twisted torus* $\mathcal{T}_t(m, n)$ is the graph with vertex set $\mathbb{Z}_m \times \mathbb{Z}_n$ such that (i, j) is adjacent to $(i+1, j)$ modulo m, for all $(i, j) \in \mathbb{Z}_m \times \mathbb{Z}_n$ and to $(i, j+1)$, modulo n, for $0 \leq j \leq n-2$, while $(i, n-1)$ is adjacent to $(i+t, 0)$ for all $i \in \mathbb{Z}_m$. In particular, $\mathcal{T}_0(m, n) \cong C_m \times C_n$. Hence, twisted tori generalize products of cycles. In twisted tori a red and a blue cycle may have more than one vertex in common. Thus, in the classification of the local structure additional cases with new challenges have to be considered. Figure 18 shows a 1-planar drawing of a twisted torus where all crossings are between edges of the same color.

A group is *(1-)planar* if it admits a connected (1-)planar Cayley graph. The finite planar

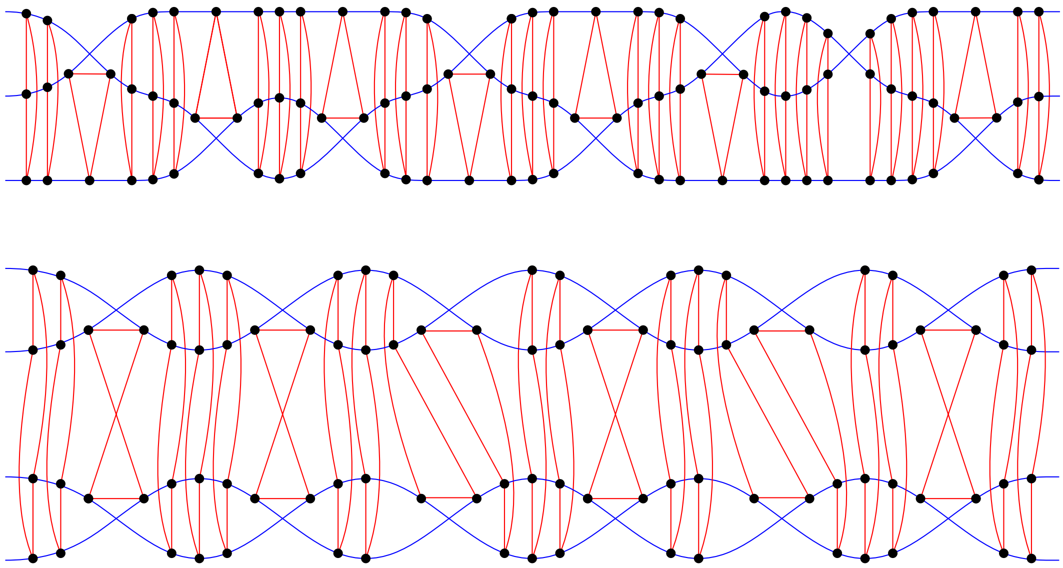

Figure 17 Sections of 1-planar drawings of $C_3 \times C_n$ and $C_4 \times C_n$ for large n which do not fall in one of the categories shown in Figure 1.

groups where characterized a long time ago [9], but the characterization of 1-planar groups is widely open. Understanding 1-planar drawings or the absence thereof for $C_k \times C_m \times C_n$ and twisted tori would be a natural step towards a structural understanding of 1-planar abelian groups, since their Cayley graphs typically contain graphs of these types. In particular, our theorem implies that $\mathbb{Z}_m \times \mathbb{Z}_n$ is 1-planar.

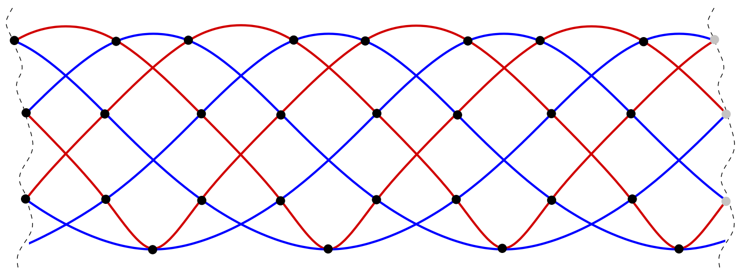

Figure 18 A 1-planar drawing of the twisted torus $\mathcal{T}_1(7,4)$. the left and the right boundary have to be identified.

Finally, our investigations may help with some other problems related to 1-planarity. Recall that in the drawings of products of cycles with swirls all but eight edges are involved in crossings. It is then tempting to ask whether there are 1-planar drawings where every edge is crossed. A first answer is obvious, a cycle of length 2 (mod 4) will do. However, if we request that the graph admits a cycle double cover consisting of 4-cycles or that the drawing is crossing minimal, then the answer is not obvious. The authors of [3] have shown that there exist arbitrary large graphs which admit crossing minimal drawings, in which every edge participates in exactly one crossing. The graphs they construct, however, have many vertices of degree 2. What if the minimum degree is at least 3?

References

1. Jay Adamsson and R. Bruce Richter. Arrangements, circular arrangements and the crossing number of $C_7 \times C_n$. *J. Comb. Theory B*, 90(1):21–39, 2004. doi:10.1016/J.JCTB.2003.05.001.
2. Vida Dujmović, David Eppstein, and David R. Wood. Structure of graphs with locally restricted crossings. *SIAM J. Discrete Math.*, 31(2):805–824, 2017. doi:10.1137/16M1062879.
3. Stefan Felsner, Michael Hoffmann, Tomas Hruz, Tillmann Miltzow, Hannah Rotzoll, Shengzhe Wang, and Emo Welzl. On local and global crossing numbers. In Abstracts 41st Europ. Workshop on Comp. Geom. (EuroCG '25), 42:1–42:8. `html://kam.mff.cuni.cz/conferences/eurocg2025/booklet2025.pdf`.
4. Lev Yu. Glebsky and Gelasio Salazar. The crossing number of $C_m \times C_n$ is as conjectured for $n \geq m(m+1)$. *J. Graph Theory*, 47(1):53–72, 2004. doi:10.1002/JGT.20016.
5. Frank Harary, Paul C. Kainen, and Allen J. Schwenk. Toroidal graphs with arbitrarily high crossing numbers. *Nanta Math.*, 6(1):58–67, 1973.
6. Aard Hoekstra. Crossing numbers of hypercubes. Master's thesis, Technische Universität Berlin, 2025.
7. Stephen G. Kobourov, Giuseppe Liotta, and Fabrizio Montecchiani. An annotated bibliography on 1-planarity. *Comput. Sci. Rev.*, 25:49–67, 2017. doi:10.1016/j.cosrev.2017.06.002.
8. Vladimir P. Korzhik and Bojan Mohar. Minimal obstructions for 1-immersions and hardness of 1-planarity testing. *J. Graph Theory*, 72(1):30–71, 2013. doi:10.1002/JGT.21630.
9. Heinrich Maschke. The representation of finite groups, especially of the rotation groups of the regular bodies of three- and four-dimensional space, by Cayley's color diagrams. *Am. J. Math.*, 18:156–194, 1896. doi:10.2307/2369680.
10. Marine Musulyan. Local crossing numbers of the product of planar graphs and cycles. Master's thesis, California State University, Northridge, Northridge, May 2019.
11. Marcus Schaefer. The graph crossing number and its variants: a survey. *Electron. J. Comb.*, DS21:90, 2013. URL: `www.combinatorics.org/ojs/index.php/eljc/article/view/DS21`.