Crossing Numbers of Random Graphs

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Abstract

The talk will explore the expected behaviour of several different crossing numbers on random graphs. Crossing numbers generally describe the minimal number of crossings between edges in any given drawing of a graph. This definition can be extended with some extra conditions to get different types of crossing numbers with various properties.
Keywords: drawing of a graph, crossing number, rectilinear crossing number, pairwise crossing number, random graph.

1 Introduction and Definitions

In this section we will introduce some key definitions, as well as note some trivial properties that follow instantaneously from the definitions.

A **drawing of a Graph** is an embedding of a graph $G = (V, E)$ with set of vertices V and set of edges E into the $\mathbb{R}^2$ plane. Every vertex is mapped to a unique point in the plane. All edges need to be mapped as continuous curves starting at one vertex and ending at the corresponding other vertex. We also assume no three edges cross in the same point, otherwise all statements are trivial.

The **crossing number** of a given graph G, denoted by $\mathrm{CR}(G)$, is the minimal number of crossings for any drawings of G.
For example we can consider the complete graphs on different numbers of vertices:

G	K_1	K_4	K_5	K_8
$\mathrm{CR}(G)$	0	0	1	18

There are many different optimal arrangements for the minimal number of crossings in a given graph, for example Figure 1 depicts optimal drawings for K_4 and K_5.

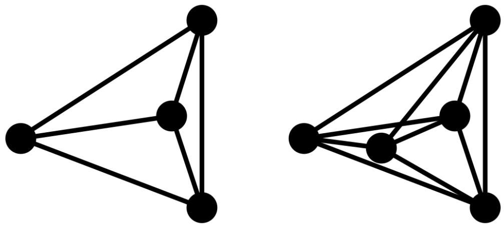

Figure 1: optimal rectilinear drawings of K_4 and K_5

The **rectilinear crossing number** of a given graph G, denoted by LIN-$\mathrm{CR}(G)$, is the minimal number of crossings for any drawings of G with the added condition, that edges must be

drawn as straight lines.
For example:

G	K_1	K_4	K_5	K_8
LIN-CR(G)	0	0	1	19

The **pairwise crossing number** of a given graph G, denoted by PAIR-CR(G), is the minimal number of pairs of edges that cross for any drawings of G. As of now it is unknown if this number is different from the crossing number CR(G).
For example:

G	K_1	K_4	K_5	K_8
PAIR-CR(G)	0	0	1	18

We can now state some trivial properties, that follow instantaneously from the definitions:
For any given graph G, we can always state that

$$\text{PAIR-CR}(G) \leq \text{CR}(G) \leq \text{LIN-CR}(G),$$

since every pair of edges with at least one crossing increases the crossing number by at least one. Of course one can imagine that an optimal drawing of a graph has a pair of edges with more than one crossing, though such a case has not been discovered as of the writing of this paper. We can obviously also only get less or equally many crossings when removing the condition that edges have to be straight line segments.

For any number $n \in \mathbb{N}$ and $p \in [0,1]$ we define the **random graph** $G(n,p)$ as the graph with n vertices and each of the $\binom{n}{2}$ possible edges exists with the probability of p. We also denote the number of expected edges as $e := p\binom{n}{2}$.
So for example $G(n,1) = K_n$.

The crossing numbers of arbitrary complete graphs are not known, though we have fairly good estimates. We are also interested in the ratio of crossings to the maximal number of crossings given by

$$\gamma_{\text{PAIR-CR}} := \lim_{n\to\infty} \frac{\text{PAIR-CR}(K_n)}{\binom{n}{2}^2},\ \gamma_{\text{CR}} := \lim_{n\to\infty} \frac{\text{CR}(K_n)}{\binom{n}{2}^2},\ \gamma_{\text{LIN-CR}} := \lim_{n\to\infty} \frac{\text{LIN-CR}(K_n)}{\binom{n}{2}^2},$$

which are constants that are known to exist.

We want to now formulate similar statements for random graphs, but as the graphs are random we cannot define constants as before. Thus we define the following functions using the expected values for the given crossing numbers:

$$\kappa_{\text{PAIR-CR}}(n,p) = \frac{E[\text{PAIR-CR}(G(n,p))]}{(p\binom{n}{2})^2},\ \kappa_{\text{CR}}(n,p) = \frac{E[\text{CR}(G(n,p))]}{(p\binom{n}{2})^2},$$
$$\kappa_{\text{LIN-CR}}(n,p) = \frac{E[\text{LIN-CR}(G(n,p))]}{(p\binom{n}{2})^2} \quad .$$

For these functions we can always state the obvious inequality;

$$\forall n \in \mathbb{N} \forall p \in [0,1]:\ \kappa_{\text{PAIR-CR}}(n,p) \leq \kappa_{\text{CR}}(n,p) \leq \kappa_{\text{LIN-CR}}(n,p) \quad .$$

This inequality follows from the previous one, which of course also holds when we consider the expected values on random graphs. We will use this inequality to simplify the proofs, since we can show lower and upper bounds to exist for just one of the functions and have them apply to the rest automatically.

2 Main Statements

The paper mainly covers the following five theorems. The first theorem will state some general facts about the behaviour of our functions.

THEOREM 1:

For any fixed n, $\kappa_{\text{PAIR-CR}}$, κ_{CR} and $\kappa_{\text{LIN-CR}}$ are increasing continuous functions in p.

The following theorem consists of three upper bound statements which apply to all three crossing numbers, giving us a total of nine statements. We only prove the three statements for the rectilinear crossing number and use the previous inequality to obtain all four other results.

THEOREM 2:

Let κ_{crossing} represent any one of the three functions $\kappa_{\text{PAIR-CR}}, \kappa_{\text{CR}}$ or $\kappa_{\text{LIN-CR}}$ and let γ_{crossing} be the respective constant defined earlier. Then we can state

1. $\limsup_{n\to\infty} \kappa_{\text{crossing}}(n, c/n) = 0$ for $c \leq 1$

2. $\lim_{c\to 1} \limsup_{n\to\infty} \kappa_{\text{crossing}}(n, c/n) = 0$

3. $\limsup_{n\to\infty} \kappa_{\text{crossing}}(n, c/n) < \gamma_{\text{crossing}}$ for all c

The next set of three theorems will provide some statements about lower bounds for each crossing number.

THEOREM 3:

For any $\varepsilon > 0, p = p(n) = n^{\varepsilon-1}$,

$$\liminf_{n\to\infty} \kappa_{\text{PAIR-CR}}(n, p) > 0.$$

THEOREM 4:

For any $c > 1, p = p(n) = c/n$,

$$\liminf_{n\to\infty} \kappa_{\text{CR}}(n, p) > 0.$$

The last theorem even states exact convergence of $\kappa_{\text{LIN-CR}}$ to the $\gamma_{\text{LIN-CR}}$ which could be considered a suprising result, though it requires strong conditions for our edge probability p.

THEOREM 5:

If $p = p(n) \gg \frac{\ln n}{n}$, then

$$\lim_{n\to\infty} \kappa_{\text{LIN-CR}}(n, p) = \gamma_{\text{LIN-CR}}.$$