# *Crossing Numbers of Random Graphs*

**Joel Spencer,[1] Géza Tóth[2,3]***
[1] *Courant Institute, New York University, 251 Mercer Street, New York, New York 10012*
[2] *Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, Massachusetts 02139*
[3] *A. Rényi Institute of Mathematics, Reáltanoda u. 13-15, 1052 Budapest, Hungary; e-mail: geza@renyi.hu*

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**ABSTRACT:** The *crossing number* of $G$ is the minimum number of crossing points in any drawing of $G$. We consider the following two other parameters. The *rectilinear crossing number* is the minimum number of crossing points in any drawing of $G$, with straight line segments as edges. The *pairwise crossing number* of $G$ is the minimum number of pairs of crossing edges over all drawings of $G$. We prove several results on the expected values of these parameters of a random graph. © 2002 Wiley Periodicals, Inc. Random Struct. Alg., 21: 347–358, 2002

## 1. INTRODUCTION

A *drawing* of a graph $G$ is a mapping which assigns to each vertex a point of the plane and to each edge a simple continuous arc connecting the corresponding two points. We assume that in a drawing no three edges (arcs) cross at the same point, and the edges do not pass through any vertex. The *crossing number* $\text{CR}(G)$ of $G$ is the minimum number of crossing points in any drawing of $G$. We consider the following two variants of the crossing number. The *rectilinear crossing number* $\text{LIN-CR}(G)$ is the minimum number of crossing points in any drawing of $G$, with straight line segments as edges. The *pairwise crossing number* or *pair-crossing number,* $\text{PAIR-CR}(G)$, of $G$ is the minimum number of crossing pairs of edges over all drawings of $G$.

Correspondence to: G. Tóth.
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Clearly, PAIR-CR$(G) \leq$ CR$(G) \leq$ LIN-CR$(G)$.

Bienstock and Dean [6, 17] constructed a series of graphs with crossing number 4, whose rectilinear crossing numbers are arbitrary large. On the other hand, Pach and Tóth [27] proved that, for any graph $G$, CR$(G) \leq 2($PAIR-CR$(G))^2$. Probably this bound is very far from being optimal, we can not even rule out that CR$(G) =$ PAIR-CR$(G)$ for any graph $G$.

The determination of the crossing numbers is extremely difficult. Even the crossing numbers of the complete graphs are not known. Let

$$\gamma_{\text{PAIR-CR}} = \lim_{n\to\infty} \frac{\text{PAIR-CR}(K_n)}{\binom{n}{2}^2}, \quad \gamma_{\text{CR}} = \lim_{n\to\infty} \frac{\text{CR}(K_n)}{\binom{n}{2}^2}, \quad \gamma_{\text{LIN-CR}} = \lim_{n\to\infty} \frac{\text{LIN-CR}(K_n)}{\binom{n}{2}^2}.$$

These limits are known to exist [18] and the best known bounds are $1/30 \leq \gamma_{\text{PAIR-CR}} \leq 1/16$, $1/20 \leq \gamma_{\text{CR}} \leq 1/16$, $0.05191 \leq \gamma_{\text{LIN-CR}} \leq 0.0639$ [1, 7] (see also [10, 18]).

In this paper we investigate the crossing numbers of *random graphs.* Let $G = G(n, p)$ be a *random* graph with $n$ vertices, whose edges are chosen independently with probability $p$. Let $e$ denote the *expected number* of edges of $G$, i.e., $e = p\binom{n}{2}$. We shall always have $e \to \infty$ [indeed, $p = \Omega(n^{-1})$] so that $G$ almost surely has $e(1 + o(1))$ edges.

In [16] it was shown that if $e > 10n$, then almost surely we have CR$(G) \geq \frac{e^2}{4000}$. Consequently, almost surely we also have LIN-CR$(G) \geq \frac{e^2}{4000}$. As we always *can* draw a graph with straight lines, the crossing number (in any form) is never larger than the number of pairs of edges and the expected number of pairs of edges is $\sim \frac{e^2}{2}$. Our interest will be in those regions of $p$ for which the various crossing numbers are, asymptotically, a positive proportion of the number of pairs of edges.

Let

$$\kappa_{\text{LIN-CR}}(n, p) = \frac{E[\text{LIN-CR}(G)]}{e^2}, \quad \kappa_{\text{CR}}(n, p) = \frac{E[\text{CR}(G)]}{e^2}, \quad \kappa_{\text{PAIR-CR}}(n, p) = \frac{E[\text{PAIR-CR}(G)]}{e^2}.$$

We have $\kappa_{\text{PAIR-CR}}(n, p) \leq \kappa_{\text{CR}}(n, p) \leq \kappa_{\text{LIN-CR}}(n, p)$ for any $n$, $p$.

**Theorem 1.** *For any fixed $n$, $\kappa_{\text{LIN-CR}}(n, p)$, $\kappa_{\text{CR}}(n, p)$, $\kappa_{\text{PAIR-CR}}(n, p)$ are increasing, continuous functions of $p$.*

With Theorem 1 we may express (roughly) our two central concerns. At which $p = p(n)$ are $\kappa_{\text{LIN-CR}}(n, p)$, $\kappa_{\text{CR}}(n, p)$, $\kappa_{\text{PAIR-CR}}(n, p)$ bounded away from zero? At which $p = p(n)$ are $\kappa_{\text{LIN-CR}}(n, p)$, $\kappa_{\text{CR}}(n, p)$, $\kappa_{\text{PAIR-CR}}(n, p)$ close to the values $\gamma_{\text{LIN-CR}}$, $\gamma_{\text{CR}}$, $\gamma_{\text{PAIR-CR}}$, respectively? Our results for these three crossing numbers shall be quite different. We are uncertain whether or not that represents the reality of the situation. The following relatively simple result shows basically that, for $p = \frac{1}{n}$, all three crossing numbers are asymptotically negligible and that, for $p = \frac{c}{n}$ with $c > 1$ fixed, the three crossing numbers have not reached their limiting values.

**Theorem 2.**

1. $\limsup_{n\to\infty} \kappa_{\text{LIN-CR}}(n, c/n) = 0$ *for* $c \leq 1$

**2.** $\limsup_{n\to\infty} \kappa_{\mathrm{CR}}(n, c/n) = 0$ *for* $c \leq 1$
**3.** $\limsup_{n\to\infty} \kappa_{\mathrm{PAIR\text{-}CR}}(n, c/n) = 0$ *for* $c \leq 1$
**4.** $\lim_{c\to 1} \limsup_{n\to\infty} \kappa_{\mathrm{LIN\text{-}CR}}(n, c/n) = 0$
**5.** $\lim_{c\to 1} \limsup_{n\to\infty} \kappa_{\mathrm{CR}}(n, c/n) = 0$
**6.** $\lim_{c\to 1} \limsup_{n\to\infty} \kappa_{\mathrm{PAIR\text{-}CR}}(n, c/n) = 0$
**7.** $\limsup_{n\to\infty} \kappa_{\mathrm{LIN\text{-}CR}}(n, c/n) < \gamma_{\mathrm{LIN\text{-}CR}}$ *for all* $c$
**8.** $\limsup_{n\to\infty} \kappa_{\mathrm{CR}}(n, c/n) < \gamma_{\mathrm{CR}}$ *for all* $c$
**9.** $\limsup_{n\to\infty} \kappa_{\mathrm{PAIR\text{-}CR}}(n, c/n) < \gamma_{\mathrm{PAIR\text{-}CR}}$ *for all* $c$.

Theorem 2 gives only upper bounds for the various crossing numbers. The main results of this paper, given in Theorems 3, 4, 5, deal with lower bounds for the three crossing numbers. Our weakest result is for the pair-crossing number.

**Theorem 3.** *For any* $\varepsilon > 0$, $p = p(n) = n^{\varepsilon - 1}$,

$$\liminf_{n\to\infty} \kappa_{\mathrm{PAIR\text{-}CR}}(n, p) > 0.$$

For the crossing number we have a much stronger result.

**Theorem 4.** *For any* $c > 1$ *with* $p = p(n) = c/n$

$$\liminf_{n\to\infty} \kappa_{\mathrm{CR}}(n, p) > 0.$$

As $\textsc{lin-cr}(G) \geq \textsc{cr}(G)$, the lower bound of Theorem 4 applies also to the rectilinear crossing number. Our most surprising result is that with the rectilinear crossing number one reaches an asymptotically best limit in relatively short time.

**Theorem 5.** *If* $p = p(n) \gg \frac{\ln n}{n}$, *then*

$$\lim_{n\to\infty} \kappa_{\mathrm{LIN\text{-}CR}}(n, p) = \gamma_{\mathrm{LIN\text{-}CR}}(n, p).$$

## 2. UPPER BOUNDS

First we prove Theorem 1. Let $f$ be any real valued function on graphs. Then with $G \sim G(n, p)$

$$E[f(G)] = \sum_H f(H) p^{e(H)} (1-p)^{\binom{n}{2} - e(H)},$$

where $H$ runs over the labelled graphs on $n$ vertices and $e(H)$ is the number of edges of $H$. This is a polynomial and hence a continous function of $p$, giving the second part of Theorem 1. We argue that $\kappa_{\mathrm{CR}}(n, p)$ is an increasing function of $p$, the arguments in the other cases are identical. For $0 \leq p, q \leq 1$ we may view $G(n, p)$ as a two-step process, first creating $G(n, p)$ and then taking each edge from $G(n, p)$ with probability $q$. After the first stage consider a drawing with the minimal number of crossings $X$, so that $E[X] = \kappa_{\mathrm{CR}}(n, p)$. Now keep that drawing but take each edge with probability $q$. Each crossing is still in the new picture with probability $q^2$. This gives a drawing of $G(n, pq)$ with expected number of crossings $q^2 \kappa_{\mathrm{CR}}(n, p)$. We do not claim this drawing is optimal, but it does give the desired upper bound as $E[\mathrm{CR}(G(n, p))] \le q^2 E[\mathrm{CR}(G(n, p))]$, completing Theorem 1.

The first six parts of Theorem 2 will come as no surprise to those familiar with random graphs as in the classic papers of Erdős and Rényi it was shown that with $p = \frac{c}{n}$ the random graph $G(n, p)$ is almost surely planar when $c < 1$. Our argument is a bit technical, however, as we must bound the expected crossing number.

We prove only part 1, for $c < 1$. Parts 2 and 3 for $c < 1$ follow immediately, since they involve smaller crossing numbers. The statements for $c = 1$ follow from parts 4, 5, and 6, respectively. Fix $c < 1$, set $p = \frac{c}{n}$ and $X = \text{LIN-CR}(G)$ with $G \sim G(n, p)$. Let $Y$ be the number of cycles of $G$ and $Z$ the number of edges of $G$. Then we claim $X \le YZ$. Remove from $G$ one edge from each cycle. This leaves a forest which can be drawn with straight lines and no crossings. Now add back in those $Y$ edges as straight lines. At worst they could hit every edge, giving $\le YZ$ crossings. With $c < 1$ $E[Y] = \sum_{i=3}^{n} \frac{(n)_i}{2i} p^i < \sum_{i=3}^{\infty} c^i$ is bounded by a constant, say $A$. As $Z$ has Binomial Distribution standard bounds give, say $\Pr[Z > 10n] < \alpha^{-n}$ for some explicit $\alpha > 1$. As $X \le n^4$ always, $X \le 10nY + n^4 \chi(Z > 10n)$ where $\chi$ is the indicator random variable. Thus $E[X] \le 10An + n^4 \alpha^{-n} = o(n^2)$. This completes the proof of parts 1, 2, and 3 for the case $c < 1$.

Now fix $c = 1 + \varepsilon$ with $\varepsilon$ positive and small. Set $p = \frac{1+\varepsilon}{n}$, $p' = \frac{1-\varepsilon}{n}$ and let $p^*$ satisfy $p' + p^* - p'p^* = p$ so that $p^* \sim \frac{2\varepsilon}{n}$. We may consider $G(n, p)$ as the union of independently chosen $G(n, p')$ and $G(n, p^*)$. Say the first has rectilinear crossing number $X$ and $Y$ edges and the second has $Z$ edges. Then their union has rectilinear crossing number at most $X + Z(Y + Z)$ as we can draw $G(n, p')$ optimally and assume all other pairs of edges do intersect. But $E[X] = o(n^2)$, and it is easy to show that $E(Z(Y + Z)) \sim E(Z)(E(Y + Z)) \sim \frac{1}{2} n^2 \varepsilon(1 + \varepsilon)$. Thus

$$E[\text{LIN-CR}(G)] \le (1 + o(1))\tfrac{1}{2}\, \varepsilon(1 + \varepsilon) n^2,$$

from which part 4 of Theorem 2 follows. Parts 5 and 6 then also follow as they involve smaller crossing numbers.

The final three parts of Theorem 2 are also natural to those familiar with random graphs. For $c > 1$ fixed $G(n, \frac{c}{n})$ has a "giant component" with $\Omega(n)$ vertices. Outside the giant component there are $\Omega(n)$ edges all lying in trees of unicyclic components. These edges may be drawn with no crossings, and that will involve a positive proportion of the potential edge crossings. Again, our argument will be a bit technical as we must deal with expectations. We state the argument only for rectilinear crossing number but it is the same in all three cases.

We first note a deterministic result: Let $G$ be *any* graph on $n$ vertices with $e$ edges. Then

$$\frac{\text{LIN-CR}(G)}{e^2} \le \frac{4 \cdot \text{LIN-CR}(K_n)}{(n)_4}.$$

Fix a drawing of $K_n$ with LIN-CR($K_n$) crossings. Define a random drawing of $G$ by randomly mapping its $n$ vertices bijectively to the $n$ vertices of the drawing. Let $e_1$, $e_2$ be two edges of $G$ with no common vertex, there being at most $e^2/2$ such unordered pairs. They may be mapped to a particular crossing of the drawing of $K_n$ in eight ways, so they have probability $8 \cdot \text{LIN-CR}(K_n)/(n)_4$ of being mapped to a crossing. Now the expected number of crossings of $G$ in this random drawing is at most, by Linearity of Expectation, $\frac{e^2}{2}\frac{8\cdot\text{LIN-CR}(K_n)}{(n)_4}$ and thus there exists a drawing of $G$ with at most that many crossings.

As the right-hand side approaches $\gamma_{\text{LIN-CR}}$, we have

$$\frac{\text{LIN-CR}(G)}{e^2} \leq \gamma_{\text{LIN-CR}} + o(1),$$

where the $o(1)$ term approaches zero in $n$, uniformly over all graphs $G$.

With $c > 0$ fixed (this argument is only needed for $c > 1$ but works for all positive $c$), $p = \frac{c}{n}$, and $G \sim G(n, p)$, let $X$ denote the number of edges and $Y$ denote the number of isolated edges. The savings comes from noting that isolated edges can always be added to a graph with no additional crossings. Thus

$$E[\text{LIN-CR}(G)] \leq E[(X - Y)^2](\gamma_{\text{LIN-CR}} + \text{o}(1)).$$

Here $E[X] \sim \frac{c}{2}n$ and $E[Y] = \binom{n}{2}p(1-p)^{2n-4} \sim \frac{c}{2}e^{-2c}n$ and elementary calculations give

$$E[(X-Y)^2] \sim E[X-Y]^2 \sim \left[\frac{c}{2}(1-e^{-2c})n\right]^2$$

With $e := p\binom{n}{2} \sim \frac{c}{2}n$ we have

$$\frac{E[\text{LIN-CR}(G)]}{e^2} \leq \gamma_{\text{LIN-CR}}(1-e^{-2c})^2(1+o(1))$$

*Comments and Open Questions.* We note that as $c$ approaches infinity the $(1 - e^{2c})^2$ term above approaches 1. The above bound may be improved somewhat by letting $Y$ denote the edges in isolated trees and unicyclic components and there are even further improvements possible. Still, all these improvements seem to approach 1 as $c$ approaches infinity. This leads to an intriguing conjecture: If $p(n) \gg \frac{1}{n}$ then $\kappa_{\text{LIN-CR}}(n, p) \to \gamma_{\text{LIN-CR}}$. One may make the same conjecture for all three variants of the crossing number. Indeed, this entire paper may be viewed as an attempt (thus far unsuccessful) of the authors to resolve these conjectures.

We conjecture that for any $c \geq 0$, the limits $\lim_{n\to\infty} \kappa_{\text{LIN-CR}}(n, c/n)$, $\lim_{n\to\infty}(n, c/n)$, and $\lim_{n\to\infty} \kappa_{\text{PAIR-CR}}(n, c/n)$ exist. This follows from Theorem 2, for $c \leq 1$. If this conjecture is true, it is not hard to see that the functions $f_{\text{LIN-CR}}(c) = \lim_{n\to\infty} \kappa_{\text{LIN-CR}}(n, c/n)$, $f_{\text{CR}}(c) = \lim_{n\to\infty} \kappa_{\text{CR}}(n, c/n)$, and $f_{\text{PAIR-CR}}(c) = \lim_{n\to\infty} \kappa_{\text{PAIR-CR}}(n, c/n)$ are continuous and increasing for all $c \geq 0$.

## 3. THE PAIR-CROSSING NUMBER

Here we prove Theorem 3. Fix $\varepsilon > 0$ and set $p = p(n) = n^{\varepsilon - 1}$. Our object is to show

$$\liminf_{n \to \infty} \kappa_{\text{PAIR-CR}}(n, p) > 0.$$

This is equivalent to showing that for $n$ sufficiently large

$$E[\text{PAIR-CR}(G(n, p))] > \delta n^4 p^2$$

for some $\delta$ dependent only on $\varepsilon$. For $L \geq 1$ we let $K_5(L)$ denote the following graph:

- There are five vertices $x_1, \ldots, x_5$.
- For each distinct pair $x_i, x_j$ there is a path between them of length $L$.

There are no other vertices nor edges so $K_5(L)$ has $5 + 10(L - 1)$ vertices and $10L$ edges. Note that $K_5(L)$ is a topological $K_5$. Hence in any drawing of $K_5(L)$ there must be at least one crossing. We shall fix $L$ such that $L\varepsilon > 1$. We shall show that $G$ contains many $K_5(L)$. Each $K_5(L)$ will force at least one crossing. With $L$ fixed this is a positive (albeit only $0.01L^{-2}$) proportion of the square of the number of edges involved. When this is carefully counted over all $K_5(L)$, we shall see that the total number of crossings is at least this constant times the square of the total number of edges.

We use three results about the almost sure behavior of $G(n, p)$. In the third $K$ is any fixed constant.

1. Every vertex has degree $\sim np$.
2. Between every pair of distinct vertices there are $\sim n^{L-1}p^L$ paths of length $L$.
3. For any distinct $x, y, z_1, \ldots, z_K$ there are $\sim n^{L-1}p^L$ paths of length $L$ between $x$ and $y$ that do not use any of the $z_j$.

The first result holds whenever $np \gg \ln n$ and follows from basic Large Deviation bounds on the degree of a vertex. Both the first and the second result are examples of a more general result [19] on counting extensions. For the third we note from [19] that the probability that the number of paths of length $L$ between fixed $x$ and $y$ is not in $[(1 - \epsilon)n^{L-1}p^L, (1 + \epsilon)n^{L-1}p^L]$ is exponentially small. Fix $x, y, z_1, \ldots, z_K$. Consider $L$-paths from $x$ to $y$ on $G$ with $z_1, \ldots, z_K$ deleted, which has distribution $G(n - K, p)$. The $K$ has negligible effect and so with exponentially small failure this number is as desired—hence almost surely it is as desired for all $O(n^{K+2})$ choices of $x, y, z_1, \ldots, z_K$.

Now we count the $K_5(L)$. There are $\binom{n}{5} \sim \frac{1}{5!} n^5$ choices for $x_1, \ldots, x_5$. We want to know how many ways we can add the ten paths, say $P_1, \ldots, P_{10}$, one between each pair of vertices. Suppose we have already selected $P_1, \ldots, P_i$. Then a constant number of vertices have been taken. Thus the number of choices for $P_{i+1}$ is $\sim n^{L-1}p^L$. So we are making ten choices, and each time we have $\sim n^{L-1}p^L$ choices so the total number of choices is $\sim [n^{L-1}p^L]^{10}$. This gives a total of $\sim \frac{1}{5!} n^{10L-5}p^{10L}$ copies of $K_5(L)$. For each one we count one crossing. Now consider a crossing between, say, edges $uv$ and $wz$. How many $K_5(L)$ do they lie on? Renumbering for convenience say the path from $x_1$ to $x_2$ has $u$ as its $i$th and $v$ as its $(i + 1)$st point and the path from $x_3$ to $x_4$ has $w$ as its $j$th and $z$

as its $(j + 1)$st point. There are $L^2$ choices for $i$, $j$. Now fix $u$, $v$, $w$, $z$ and $i$, $j$. From the first property there are $\sim (np)^i$ paths of length $i$ starting at $u$, $\sim (np)^{L-i-1}$ paths of length $L - i$ starting at $v$ and similarly for $w$, $z$. Further these numbers are not asymptotically effected when we require that they miss a fixed number of points. So we extend $u$, $v$, $w$, $z$ to some $x_1$, $x_2$, $x_3$, $x_4$ in $\sim (np)^{2(L-1)}$ ways. We have $n$ choices for $x_5$ and then $\sim (n^{L-1}p^L)^8$ ways to complete the remaining eight paths forming $K_5(L)$. Thus edges $uv$, $wz$ lie on $\sim L^2 n^{10L-9} p^{10L-2}$ different $K_5(L)$. So each crossing has been counted at most that many times and hence the number of crossings is at least asymptotically

$$\frac{\frac{1}{5!}\, n^{10L-5} p^{10L}}{L^2 n^{10L-9} p^{10L-2}} = \frac{1}{120L^2}\, n^4 p^2,$$

as desired.

*Comments and Open Questions.* As we must take $L > \varepsilon^{-1}$ the constant $\frac{1}{120} L^{-2}$ in this result goes to zero as $\varepsilon \to 0$. This is in surprising contrast to the crossing number $\text{CR}(G)$ discussed in the next section. That crossing number becomes a positive proportion of the square of the number of edges already at $p = \frac{c}{n}$ when $c > 1$. Can the pair-crossing number and the crossing number have such different behavior? We doubt it. As mentioned in the Introduction, we cannot rule out the possibility that the pair-crossing number and the crossing number are always exactly the same. We can certainly make the weaker conjecture that the expectation of the pair-crossing number of $G(n, p)$ becomes $\Lambda(n^4 p^2)$ already at $p = \frac{1+\varepsilon}{n}$. We further note that we have no idea at which $p$ $\kappa_{\text{PAIR-CR}}(n, p)$ gets within $o(1)$ of its limit $\gamma_{\text{PAIR-CR}}$.

Pach and Tóth [17] introduced another variant of the crossing number. The *odd-crossing number,* $\text{ODD-CR}(G)$, of any graph $G$ is the minimum number of pairs of edges that cross an odd number of times, over all drawings of $G$. Clearly, for any graph, $\text{ODD-CR}(G) \leq \text{PAIR-CR}(G)$. With a little modification, the above argument works also for the odd-crossing number, therefore, the statement of Theorem 3 holds also for the odd-crossing number.

## 4. THE CROSSING NUMBER

Here we prove Theorem 4. Fix $c > 1$ and set $p = \frac{c}{n}$. Let $G \sim G(n, p)$. Our object is to show

$$\liminf_{n\to\infty} \frac{E[\text{CR}(G)]}{\left[\binom{n}{2}p\right]^2} > 0.$$

As $c$ is constant this is equivalent to showing that for $n$ sufficiently large

$$E[\text{CR}(G)] > \delta n^2$$

for some $\delta$ dependent only on $c$.

We begin by reviewing in outline form the argument of Pach and Tóth [17], which requires that $c$ be a sufficiently large constant. We will see why their argument does not work for $c = 1 + \epsilon$ with $\epsilon > 0$ small and then how a modification of their argument, combined with results on $G(n, p)$, does work.

Define the *bisection width* of $G$, denoted by $b(G)$, as the minimal number of edges running between $T$ (top) and $B$ (bottom) over all partitions of the vertex set into two disjoint parts $V = T \cup B$ such that $\frac{2}{3}|V| \geq |T|, |B| \geq \frac{1}{3}|V|$. (The specific constant $\frac{2}{3}$ is not essential here, we need only to assure that the sizes of $T$ and $B$ are within a constant factor.) Leighton observed that there is an intimate relationship between the bisection width and the crossing number of a graph [12], which is based on the Lipton-Tarjan separator theorem for planar graphs [13]. The following version of this relationship was obtained by Pach, Shahrokhi, and Szegedy [15]. Let $G$ be a graph on vertex set $V$ with $d_v$ denoting the degree of vertex $v$. Then

$$b(G) \leq 10\sqrt{\mathrm{CR}(G)} + 2\sqrt{\sum_{v \in V(G)} d_v^2}.$$

With $G \sim G(n, \frac{c}{n})$, $E[d_v^2] \sim c^2 = O(1)$ and almost surely $2\sqrt{\sum_{v \in V} d_v^2} = O(\sqrt{n})$, which proves to be negligible. For $c$ a *large* constant basic probabilistic methods give that almost surely *every* partition $V = T \cup B$ with $\frac{2}{3}|V| \geq |T|, |B| \geq \frac{1}{3}|V|$ has a constant proportion of the edges running between them. That is, almost surely $b(G) = \Omega(n)$. Hence almost surely $\mathrm{CR}(G) = \Omega(n^2)$.

Now suppose $c = 1 + \epsilon$ with $\epsilon > 0$ small. The difficulty is: almost surely $b(G)$ is zero! Why? From classic Erdős-Rényi results $G$ will have a "giant component" of size $\sim kn$ with $k = k(c)$ and all other components will have size $O(\ln n)$. The function $k = k(c)$ was given explicitly by Erdős and Rényi, but we need here only to note that $\lim_{c \to 1^+} k(c) = 0$. For $\epsilon$ a small (actually, not so small) but fixed constant and $c = 1 + \epsilon$, the giant component has size $kn$ with $k < \frac{2}{3}$. Place the giant component in the top $T$. Now take all other components sequentially. Add them to the top $T$ if $|T|$ remains below $\frac{2}{3}n$; otherwise place them in the bottom $B$. This gives a partition with $\frac{2}{3}|V| \geq |T|, |B| \geq \frac{1}{3}|V|$ and no edges running between $T$ and $B$.

Our approach shall be to show, effectively, that the giant component of $G(n, \frac{c}{n})$ has high bisection width for any $c > 1$. To do this, we employ an "enhancement" approach which we take, with only slight modification, from the work of Łuczak and McDiarmid [14].

**Theorem 6.** *Let V be a set of m vertices. Let T be a tree on V. Let G be the random graph on V with edge probability* $p = \frac{a}{m}$. *For* $a > 0$ *fixed almost surely*

$$b(T \cup G) = \Omega(m).$$

*That is, there exists* $\eta > 0$ *dependent only on a such that* $\Pr[b(T \cup G) \leq m\eta]$ *approaches zero as m approaches infinity.*

Consider partitions $V = V_1 \cup V_2$ such that $T$ has at most $m\eta$ cut edges. For $i \leq m\eta$ we can choose $i$ cut edges in at most $\binom{m-1}{i} \leq \binom{m}{i}$ ways and orient them (selecting one endpoint for $V_1$ and the other for $V_2$) in at most $2^i$ ways. As $T$ is connected, these choices determine the partition. Hence the number of such partitions is at most

$$\sum_{i \le m\eta} \binom{m}{i} 2^i \le 2^{m\eta} \sum_{i \le m\eta} \binom{m}{i} \le 2^{m(\eta + H(\eta))},$$

where $H(\eta) := -\eta \log_2 \eta - (1 - \eta) \log_2(1 - \eta)$ is the standard entropy function.

Now fix a partition $V = V_1 \cup V_2$ with $\frac{m}{3} \le |V_i| \le \frac{2m}{3}$ for $i = 1, 2$. Let $b(V_1, V_2)$; $G$) denote the number of edges $\{x, y\}$ of $G$ with $x$, $y$ in different $V_i$. Then

$$b((V_1, V_2); G) \sim B\left[|V_1| \cdot |V_2|, \frac{a}{m}\right],$$

where $B$ is the binomial distribution. As $|V_1| \cdot |V_2| \ge \frac{2}{9} m^2$,

$$\Pr[b((V_1, V_2); G) \le m\eta] \le \Pr\left[B\left(\frac{2}{9} m^2, \frac{a}{m}\right) \le m\eta\right].$$

This last large deviation probability can be bounded in a number of ways. For our purposes let us assume $\eta < \frac{a}{9}$ and use that (see, e.g., the Appendix of [5]) $\Pr[B(n, p) \le \frac{1}{2} np] < e^{-np/8}$.

We select $\eta > 0$ such that $\eta < \frac{a}{9}$ and

$$(\ln 2)(\eta + H(\eta)) < \frac{a}{36}.$$

Each of the at most $2^{m(\eta + H(\eta))}$ partitions of $V$ which has fewer than $m\eta$ cut edges with respect to $T$, has probability less than $\exp[-\frac{1}{8} \frac{2a}{9} m]$ of having fewer than $m\eta$ cut edges with respect to $G$, and so the expected number of partitions with fewer than $m\eta$ cut edges with respect to both $T$ and $G$ is at most the product. Our selection of $\eta$ insures that the product approaches zero, completing the proof of Theorem 6.

Now we prove Theorem 4. Let $c > 1$ be fixed. Fix $c_1$, $a$ with $1 < c_1 < c$ and $c_1 + a = c$; for definiteness we may take $c_1 = \frac{1+c}{2}$ and $a = \frac{c-1}{2}$. Set $p = \frac{c}{n}$, $p_1 = \frac{c_1}{n}$ and $p_2$ such that $p = p_1 + p_2 - p_1 p_2$. Note $p_2 \sim \frac{a}{n}$. We may regard $G \sim G(n, p)$ as the union of independently chosen $G_1 \sim G(n, p_1)$ and $G_2 \sim G(n, p_2)$. On $G_1$ almost surely there exists a "giant component" $X$ with $|X| \sim d_1 n$, where $d_1$ is an explicit function of $c_1$ given in the classic Erdős-Rényi papers. Set $m = |X|$. Then $p_2 m \sim a d_1$. Then $G_{1|X}$ contains some tree $T$. From Theorem 6 there is a constant $\eta$ such that almost surely $(T \cup G_2)|_X$ has bisection width at least $m\eta \sim n(d_1 \eta)$. As adding edges can only increase the bisection width $b(G|_X) > n(d_1\eta)(1 - o(1))$. Applying the basic relationship between bisection width and crossing numbers gives $\text{CR}(G|_X) = \Omega(n^2)$. Hence $\text{CR}(G) \ge \text{CR}(G|_X) = \Omega(n^2)$.

*Comments and Open Questions.* From Theorem 1 we know $\kappa_{\text{CR}}(n, p) \to \gamma_{\text{CR}}$ as $p \to 1$, and we have just shown that $\kappa_{\text{CR}}(n, \frac{c}{n})$ is bounded from below. How large does $p = p(n)$ need to be so that $\kappa_{\text{CR}}(n, p(n)) \sim \gamma_{\text{CR}}$? We have already conjectured that for any

$p = p(n)$ with $np \to \infty$ we have $\kappa_{CR}(n, p) \to \gamma_{CR}$. But we cannot even show that $\kappa_{CR}(n, p) \to \gamma_{CR}$ when $p < 1$ is a constant. Suppose (which is surely true though we are unable to show it) that $\lim_n \kappa_{CR}(n, \frac{c}{n})$ exists and call it $f_{CR}(c)$. Then $f_{CR}(c)$ would be increasing so $\lim_{c\to\infty} f_{CR}(c)$ would exist but might be a value strictly less than $\gamma_{CR}$. Would there be a second (or even a third or more) region (something like $p = \Theta(n^{-1/2})$ or, more likely, $p = \Theta(1)$) where $\kappa_{CR}(n, p)$ increases (in some asymptotic sense) until it finally reaches $\gamma_{CR}$?

## 5. THE RECTILINEAR CROSSING NUMBER

Here we show Theorem 5. An *order type* of the points $x_1, x_2, \ldots, x_n$ in the plane (with no three colinear) is a list of orientations of all triples $x_i x_j x_k$, $i < j < k$ [14]. Elementary geometry gives that the order type of the four triples $x_i x_j x_k$, $x_i x_j x_l$, $x_i x_k x_l$, $x_j x_k x_l$ determines whether or not the straight line segments $x_i x_j$ and $x_k x_l$ intersect. Let $X$ be the set of all order types of the points $x_1, x_2, \ldots, x_n$ in the plane. We shall make critical use of a result of Goodman and Pollack [8, 9] that $|X| < n^{6n}$. We note that the Goodman-Pollack result is derived from the Milnor-Thom theorem, a now classical and very deep result concerning algebraic varieties.

First, however, we examine a fixed order type $\xi \in X$. For any graph $G$ with vertices $v_1, \ldots, v_n$ let $\text{LIN-CR}_\xi(G)$ denote the number of crossings in the straight line drawing of $G$, where $v_i$ is placed at $x_i$ in the plane and $x_1, \ldots, x_n$ have order type $\xi$.

**Theorem 7.** *Let $G(n, p)$ be a random graph with vertices $v_1, v_2, \ldots, v_n$, with edge probability $0 < p = p(n) < 1$, and let $e = p\binom{n}{2}$. Then*

$$\Pr\left[\left|\text{LIN-CR}_\xi(G) - \mathrm{E}[\text{LIN-CR}_\xi(G)]\right| > 3\alpha e^{3/2}\right] < 3\exp(-\alpha^2/4)$$

*holds for every $\alpha$ satisfying $(e/4)^3 \exp(-e/4) \le \alpha \le \sqrt{e}$.*

*Proof.* We follow the approach of Pach and Tóth [16]. (We note that general polynomial concentration results of Kim and Vu [11] could also be used.) Let $e_1, e_2, \ldots, e_{\binom{n}{2}}$ be the edges of the complete graph on $V(G)$. Define another random graph $G^*$ on the same vertex set, as follows. If $G$ has at most $2e$ edges, let $G^* = G$. Otherwise, there is an $i < \binom{n}{2}$ so that $|\{e_1, e_2, \ldots, e_i\} \cap \mathrm{E}(G)| = 2e$, and set $\mathrm{E}(G^*) = \{e_1, e_2, \ldots, e_i\} \cap \mathrm{E}(G)$. Finally, let $f(G) = \text{LIN-CR}_\xi(G^*)$.

The addition of any edge to $G$ can modify the value of $f$ by at most $2e$. Following the terminology of Alon–Kim–Spencer [3], we say that the *effect* of every edge is at most $2e$. The *variance* of any edge is defined as $p(1 - p)$ times the square of its effect. Therefore, the *total variance* cannot exceed

$$\sigma^2 = \binom{n}{2} p(2e)^2 = 4e^3.$$

Applying the Martingale Inequality of [3], which is a variant of Azuma's Inequality [5] (see also [4]), we obtain that for any positive $\alpha \le \sigma/e = 2\sqrt{e}$,

$$\Pr[|f(G) - \mathrm{E}[f(G)]| > \alpha\sigma = 2\alpha e^{3/2}] < 2\exp(-\alpha^2/4).$$

Our goal is to establish a similar bound for $\text{LIN-CR}_\xi(G)$ in place of $f(G)$. Obviously,

$$\Pr[f(G) \neq \text{LIN-CR}_\xi(G)] \leq \Pr[G \neq G^*] < \exp(-e/4).$$

Thus, we have

$$|\mathrm{E}[f(G)] - \mathrm{E}[\text{LIN-CR}_\xi(G)]| \leq \Pr[f(G) \neq \text{LIN-CR}_\xi(G)] \max \text{LIN-CR}_\xi(G) \leq \exp\left(-\frac{e}{4}\right)\frac{n^4}{8} \leq \alpha e^{3/2},$$

whenever $\alpha \geq (e/4)^3 \exp(-e/4)$ (say). Therefore,

$$\begin{aligned}
&\Pr[|\text{LIN-CR}_\xi(G) - \mathrm{E}[\text{LIN-CR}_\xi(G)]| > 3\alpha e^{3/2}] \\
&\quad \leq \Pr[\text{LIN-CR}_\xi(G) \neq f(G)] + \Pr[|f(G) - E[f(G)]| > 2\alpha e^{3/2}] \\
&\quad \leq \exp(-e/4) + 2\exp(-\alpha^2/4).
\end{aligned}$$

If $\alpha \leq \sqrt{e}$, the last sum is at most $3\exp(-\alpha^2/4)$, as required. This concludes the proof of Theorem 7.

Now we can prove Theorem 5. Fix $p = p(n)$ with $p(n) \gg \frac{\ln n}{n}$ and $G \sim G(n, p)$. Set $e = p\binom{n}{2}$. Let $C_n = \kappa_{\text{LIN-CR}}(n, 1)$. Since $\xi \in X$, $\mathrm{E}[\text{LIN-CR}_\xi(G)] = p^2 \text{LIN-CR}_\xi(K_n) \geq C_n e^2$. Let $\varepsilon > 0$ be arbitrarily small, but fixed. Then

$$\Pr[\text{LIN-CR}(G) < (C_n - \varepsilon)e^2] \leq \sum_{\xi \in X} \Pr[\text{LIN-CR}_\xi(G) < \mathrm{E}[\text{LIN-CR}_\xi(G)] - \varepsilon e^2].$$

We apply Theorem 7 with $3\alpha e^{3/2} = \varepsilon e^2$ so that $\alpha^2/4 = \frac{1}{36}\varepsilon^2 e$. The growth rate of $p(n)$ insures that this is $o(n^{-6n})$ for any fixed positive $\varepsilon$. The Goodman-Pollack result critically bounds $|X| \leq n^{6n}$. Hence the sum goes to zero, as desired.

*Comments and Open Quesetions.* We have not been able to determine if the condition $p \gg \frac{\ln n}{n}$ in Theorem 5 is necessary. We have already conjectured that for any $p = p(n)$ with $np \to \infty$ we already have $\lim_{n\to\infty} \kappa_{\text{LIN-CR}}(n, p) = \gamma_{\text{LIN-CR}}$. While the Goodman-Pollack theorem itself cannot be improved asymptotically [2], it might be the case that there are few (in some sense) near optimal drawings so that the $n^{-\Theta(n)}$ error probability used in the proof of Theorem 5 may not be fully necessary. This, however, remains highly speculative.

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## REFERENCES

[1] O. Aichholzer, F. Aurenhammer, and H. Krasser, Progress on rectilinear crossing numbers, manuscript.

[2] N. Alon, The number of polytopes, configurations, and real matroids, Mathematika 33 (1986), 62–71.

[3] N. Alon, J. H. Kim, and J. Spencer, Nearly perfect matchings in regular simple hypergraphs, Isr J Math 100 (1997), 171–187.

[4] N. Alon and J. Spencer, The probabilistic method, Wiley, New York, 1992.

[5] K. Azuma, Weighted sums of certain dependent random variables, Tohoku Math J 19(2) (1967), 357–367.

[6] D. Bienstock and N. Dean, Bounds for rectilinear crossing numbers, J Graph Theory 17 (1993), 333–348.

[7] A. Brodsky, S. Durocher, and E. Gethner, Toward the rectilinear crossing number of $K_n$: New drawings, upper bounds and applications, submitted.

[8] J. Goodman and R. Pollack, Upper bounds for configurations and polytopes in $R^d$, Discrete Comput Geom 1 (1986), 219–227.

[9] J. Goodman and R. Pollack, New bounds on higher dimensional configurations and polytopes, Ann NY Acad Sci 555 (1989), 205–212.

[10] R. K. Guy, Crossing numbers of graphs, Graph Theory Applications (Proc Conf Western Michigan Univ, Kalamazoo, 1972), Lecture Notes in Mathematics 303, Springer-Verlag, Berlin, 1972, pp. 111–124.

[11] J. H. Kim and V. H. Vu, Concentration of multivariate polynomials and its applications, Combinatorica 20 (2000), 417–434.

[12] F. T. Leighton, New lower bound techniques for VLSI, Math Syst Theory 17 (1984), 47–70.

[13] R. Lipton and R. Tarjan, A separator theorem for planar graphs, SIAM J Appl Math 36 (1979), 177–189.

[14] M. Łuczak and C. McDiarmid, Bisecting sparse random graphs, Random Struct Alg 18 (2001), 31–38.

[15] J. Pach, F. Shahrokhi, and M. Szegedy, Applications of the crossing number, Algorithmica 16 (1996), 111–117.

[16] J. Pach and G. Tóth, Thirteen problems on crossing numbers, Geombinatorics 9 (2000), 194–207.

[17] J. Pach and G. Tóth, Which crossing number is it, anyway?, Proc 39th Annu Symp Foundations of Computer Science, IEEE Press, New York, 1998, pp. 617–626; also J Combinat Theory Ser B 80 (2000), 225–246.

[18] R. B. Richter and C. Thomassen, Relations between crossing numbers of complete and complete bipartite graphs, Am Math Mon 104 (February 1997), 131–137.

[19] J. Spencer, Counting Extensions, J Combinat Theory Ser A 55 (1990), 247–255.