7. A First Glimpse of Total Colouring

In Part II, we introduced three probabilistic tools and saw an application of each of them. In the last chapter, we saw a more complicated application of one of them, the First Moment Method. In this chapter, we will illustrate the power of combining the other two, the Local Lemma and the Chernoff Bound, by discussing their application to total colouring.

Recall that a total colouring of a graph G consists of a colouring of the vertices and the edges so that:

(i) no two adjacent vertices receive the same colour,
(ii) no two incident edges receive the same colour,
(iii) no edge receives the same colour as one of its endpoints.

The total chromatic number of G, denoted $\chi_T(G)$, is the minimum k for which G has a total colouring using k colours. As mentioned in Chap. 1, Behzad and Vizing independently conjectured that every graph G has a total colouring using $\Delta(G)+2$ colours.

Now, finding a $\Delta+2$ vertex colouring presents no difficulty, as the greedy colouring procedure discussed in Sect. 1.7 will generate one for us. Colouring the edges with $\Delta+2$ colours is also straightforward, for we can apply Vizing's Theorem which ensures that an edge colouring using $\Delta+1$ colours exists. Complications arise when we try to put two such colourings together, as an edge may receive the same colour as one of its endpoints. The crux of the matter is to pair a $\Delta+2$ vertex colouring with a $\Delta+2$ edge colouring so that no such conflicts arise.

Actually, provided we can find a pairing which generates only a few conflicts then we can find a total colouring using not many more than $\Delta+2$ colours. For example, if only r conflicts arise then we can recolour the r edges involved in conflicts with r new colours to generate a $\Delta+r+2$ total colouring. Of course, we may be able to use fewer than r new colours. For example, if the edges involved in conflicts form a matching then we need only one new colour. More generally, if we let R be the graph formed by all those edges whose colour is rejected because they are involved in a conflict, then we can recolour the edges of R with $\chi_e(R) \le \Delta(R)+1$ new colours to obtain a $\Delta+\Delta(R)+3$ total colouring of G. This is the approach we take in this chapter.

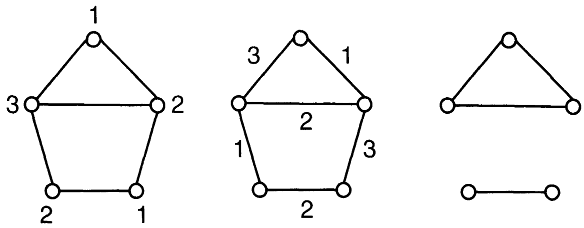

Fig. 7.1. An edge colouring, a vertex colouring and the resultant reject edges

To warm up, we present a result from [117] (the proof also appears in [74]) which uses the Chernoff Bound and the First Moment Method to show:

Theorem 7.1 *Every graph G satisfies:* $\chi_T(G) \leq \Delta + \lceil \log(|V(G)|) \rceil + 3$.

Proof We assume $|V(G)|$ is at least three, as otherwise the theorem is trivial. let $l = \lceil \log(|V(G)|) \rceil + 2$. Consider an arbitrary $\Delta + 1$ vertex colouring $C = \{S_1, \ldots, S_{\Delta+1}\}$ and an arbitrary $\Delta + 1$ edge colouring $D = \{M_1, \ldots, M_{\Delta+1}\}$ of G. Let $C_1, \ldots, C_{(\Delta+1)!}$ be the $(\Delta + 1)!$ vertex colourings which are obtained by permuting the colour class names of C. Note that if some of the colour classes are empty then some vertex colourings may appear more than once on this list. We show that for some i, combining C_i with D yields a reject graph R_i with $\Delta(R_i) \leq l - 1$ (to be precise, $R_i = \cup_{j=1}^{\Delta+1} \{xy | xy \in M_j,\ x \text{ or } y \text{ receives colour } j \text{ under } C_i\}$). Thus, we can edge colour R_i using l matchings, thereby completing the desired $\Delta + l + 1$ total colouring of G.

To do so, we consider picking a C_i uniformly at random and let $R = R_i$ be the random reject graph thereby obtained. We show that the expected number of vertices of degree at least l in R is less than one and thereby prove that there exists an R_i with maximum degree less than l.

By the Linearity of Expectation, to show that the expected number of vertices of degree at least l is less than 1, it is enough to show that for each vertex v, $\mathbf{Pr}(d_R(v) \geq l) < \frac{1}{n}$.

Now, at most one edge incident to v is in R because it conflicts with v. So we consider the event that there are $l - 1$ edges incident to v which conflict with their other endpoint. We need only show that the probability of this event is less than $\frac{1}{n}$.

We actually show that for any vertex v, the expected number of sets of $l-1$ edges incident to v, all of which are in R because they conflict with their other endpoint is less than $\frac{1}{n}$. Applying Markov's Inequality, we obtain the desired result. To this end, we first compute the probability that a particular set $\{vu_1, .., vu_{l-1}\}$ of $l-1$ edges incident to v are all in R because they conflict with their other endpoint. We let α_i be the colour of vu_i. We let β_i be the colour that u_i is assigned under C. We are computing the probability that our random permutation takes β_i to α_i for $1 \leq i \leq l - 1$. This probability is zero if the β_i are not distinct. Otherwise, the probability that the permutation does indeed take each of the $l-1$ colours β_i to the corresponding α_i is: $\frac{(\Delta+1-(l-1))!}{(\Delta+1)!}$.

Now, there are at most $\binom{\Delta}{l-1}$ sets of $l-1$ edges incident to v in G. So the expected number of sets of $l-1$ edges incident with v which conflict with their other endpoint is at most:

$$\binom{\Delta}{l-1}\frac{(\Delta+1-(l-1))!}{(\Delta+1)!} < \frac{1}{(l-1)!}.$$

It is easy to see that $(\lceil \log n \rceil + 1)!$ is greater than n provided n is at least three, so the result holds. □

We now want to apply the same technique to obtain a bound on $\chi_T(G)$ which is independent of $|V(G)|$. To do so, we wish to apply the Local Lemma. However, the Local Lemma will only work if we are analyzing a random procedure for which the conflicts in distant parts of the graph occur independently. One way of ensuring that this is true is to assign each vertex a uniformly random colour without considering the colours assigned to the other vertices. Our bad events would each be determined only by the colours on a cluster of vertices which are all very close together, and so events corresponding to clusters in distant parts of the graph would occur independently.

The problem with this approach is that it is very unlikely to generate a proper vertex colouring. To overcome this problem, we will consider a two phase procedure, consisting of a random initial phase which retains the flavour of the random procedure proposed in the preceding paragraph, followed by a deterministic phase which ensures that we have a proper total colouring. We first randomly partition V into k sets $V_1, \ldots, V_k$ such that for each i, the graph H_i induced by V_i has maximum degree at most $l-1$ with l near $\frac{\Delta}{k}$. We then greedily colour the vertices of each H_i using the colours in $C_i = \{(i-1)l, \ldots, il-1\}$. This yields a kl colouring of $V(G)$.

We fix any $\Delta+1$ edge colouring $\{M_1, \ldots, M_{\Delta+1}\}$ before performing this process. We say that an edge xy *conflicts* with the endpoint x if xy is coloured with a colour in C_i and x is assigned to V_i. We note that if e does not conflict with x then in the second phase, the colour assigned to x will be different from that used on e. The advantage to widening our definition of conflict in this way is that now the conflicts depend only on the random phase of the procedure, and this allows us to apply the Local Lemma. Forthwith the details.

Theorem 7.2 *For any graph G with maximum degree Δ sufficiently large, $\chi_T(G) \le \Delta + 2\Delta^{\frac{3}{4}}$*

Proof As usual, we can assume that G is Δ-regular by the construction in Sect. 1.5. Set $k = k_\Delta = \lceil \Delta^{\frac{1}{3}} \rceil$ and $l = l_\Delta = \lfloor \frac{\Delta + \Delta^{\frac{3}{4}}}{k} \rfloor$. We fix an arbitrary edge colouring of G using the colours $1, \ldots, \Delta+1$. We then specify a vertex colouring of G using the colours $0, \ldots, kl-1 \le \Delta + \Delta^{\frac{3}{4}} - 1$, as follows.

We first partition $V(G)$ into $V_1, \dots, V_k$ such that

(i) for each vertex v and part i, $|N_v \cap V_i| \le l - 1$,
(ii) For each vertex v, there are at most $\Delta^{\frac{3}{4}} - 3$ edges $e = (u,v)$ such that $u \in V_i$ and e has a colour in C_i.

Our next step will be to refine this partition into a proper colouring, colouring the vertices of V_i using the colours in C_i.

By (i), we can do so using the simple greedy procedure of Lemma 1.3, since the subgraph induced by V_i has maximum degree $l - 1$. By (ii), the reject graph formed has maximum degree at most $\Delta^{\frac{3}{4}} - 2$ (there is a 2 and not a 3 here because we may reject an edge incident to v because it has the same colour as v). Recolouring these edges with at most $\Delta^{\frac{3}{4}} - 1$ new colours yields the desired total colouring of G.

It only remains to show that we can actually partition the vertices so that (i) and (ii) hold. To do so, we simply assign each vertex to a uniformly random part (where of course, these choices are made independently). For each v, i we let $A_{v,i}$ be the event that (i) fails to hold for $\{v, i\}$ and B_v be the event that (ii) fails to hold for v. We will use the Local Lemma to prove that with positive probability none of these bad events occur. B_v and $A_{v,i}$ are determined by the colours of the vertices adjacent to v. Thus, by the Mutual Independence Principal, they are mutually independent of all events concerning vertices which are at distance more than 2 from v, and so every event is mutually independent of all but at most $(k+1)\Delta^2 < \Delta^3$ other events. We will show that the probability that any particular bad event holds is much less than $\frac{1}{4\Delta^3}$. Thus, by the Local Lemma, there exists a colouring satisfying (i) and (ii).

Consider first the event B_v. Let Rej_v be the set of edges $e = (u,v)$ with the property that e has a colour in C_i and $u \in V_i$. Since there are k parts, the probability that this occurs for a given e is exactly $\frac{1}{k}$. Furthermore, as the choices of the parts are independent, the size of Rej_v is just the sum of Δ independent 0-1 variables each of which is 1 with probability $p = \frac{1}{k}$. Applying the Chernoff Bound for $\mathrm{BIN}(\Delta, p)$ we obtain:

$$\mathbf{Pr}\left(\left| |\mathrm{Rej}_v| - \frac{\Delta}{k} \right| > \frac{\Delta}{k}\right) \le 2e^{-\frac{\Delta}{3k}},$$

Since $k = \lceil \Delta^{\frac{1}{3}} \rceil$ and $\frac{\Delta^{\frac{3}{4}}}{2} > \frac{\Delta}{k}$, it follows that for Δ sufficiently large,

$$\mathbf{Pr}(B_v) \le 2e^{-\Delta^{1/2}}.$$

The size of $N_v \cap V_i$ is just the sum of Δ independent 0-1 variables each of which is 1 with probability $\frac{1}{k}$, and so applying the Chernoff Bound as above we obtain that for large Δ,

$$\mathbf{Pr}(A_{v,i}) \leq \mathbf{Pr}\left(\left||N_v \cap V_i| - \frac{\Delta}{k}\right| > \frac{\Delta^{\frac{3}{4}}}{2}\right) \leq 2e^{-\Delta^{1/2}}$$

□

Remark Actually, we can obtain a $\Delta + O(\Delta^{\frac{2}{3}} \log \Delta)$ total colouring using exactly the same technique, but the computations are slightly more complicated.