4. The Lovász Local Lemma

In this chapter, we introduce one of the most powerful tools of the probabilistic method: The Lovász Local Lemma . We present the Local Lemma by reconsidering the problem of 2-colouring a hypergraph.

Recall that in Sect. 3.1 we showed that any hypergraph with fewer than 2^{k-1} hyperedges, each of size at least k, has a proper 2-colouring because the expected number of monochromatic edges in a uniformly random 2-colouring of the vertices is less than 1.

Now suppose that a k-uniform hypergraph has many more than 2^{k-1} hyperedges, say 2^{2^k} hyperedges. Obviously, the First Moment Method will fail in this case. In fact, at first glance it appears that any attempt to apply the probabilistic method by simply selecting a uniformly random 2-colouring is doomed since the chances of it being a proper 2-colouring are typically very remote indeed. Fortunately however, for the probabilistic method to succeed we don't require a high probability of success, just a positive probability of success.

To be more precise, we will choose a uniformly random 2-colouring of the vertices, and for each hyperedge e, we denote by A_e the event that e is monochromatic. Suppose, for example, that our k-uniform hypergraph consisted of m completely disjoint hyperedges. In this case, the events A_e are mutually independent, and so the probability that none of them hold is exactly $(1-2^{-(k-1)})^m$ which is positive no matter how large m is. Therefore, the hypergraph is 2-colourable.[1]

Of course for a general hypergraph, H, the events $\{A_e | e \in E(\mathcal{H})\}$ are not independent as many pairs of hyperedges intersect. The Lovász Local Lemma is a remarkably powerful tool which says that in such situations, as long as there is a sufficiently limited amount of dependency, we can still claim a positive probability of success.

Here, we state the Lovász Local Lemma in its simplest form. We omit the proof for now, as we will prove it in a more general form in Chap. 19.

The Lovász Local Lemma [44]: *Consider a set $\mathcal{E}$ of (typically bad) events such that for each $A \in \mathcal{E}$*

(a) $\mathbf{Pr}(A) \le p < 1$, *and*

[1] The astute reader may have found an alternate proof of this fact.

(b) A is mutually independent of a set of all but at most d of the other events.

If $4pd \leq 1$ *then with positive probability, none of the events in* $\mathcal{E}$ *occur.*

Remark 4.1 *The inequality* $4pd \leq 1$ *can be replaced by* $\mathrm{e}p(d+1) < 1$, *which typically yields a slightly sharper result. (Here* $\mathrm{e} = 2.71\ldots$*) Only rarely do we desire such precision so we usually use the first form. Shearer [143] proved that we cannot replace "e" by any smaller constant.*

Our first application of the Lovász Local Lemma is the following:

Theorem 4.2 *If* $\mathcal{H}$ *is a hypergraph such that each hyperedge has size at least* k *and intersects at most* 2^{k-3} *other hyperedges, then* $\mathcal{H}$ *is 2-colourable.*

Remark This application is the one used in virtually every introduction to the Lovász Local Lemma. The authors do not apologize for using it again here, because it is by far the best example. We refer the reader who for once would like to see a different first example to [125] where this application is disguised as a satisfiability problem.

Proof We will select a uniformly random 2-colouring of the vertices. For each hyperedge e, we define A_e to be the event that e is monochromatic. We also define N_e to be the set of edges which e intersects (i.e. its neighbourhood in the line graph of $\mathcal{H}$). Recall that $|N_e| < 2^{k-3}$ by assumption. We shall apply the Local Lemma to the set of events $\mathcal{E} = \{A_e | e \in E(\mathcal{H})\}$.

Claim: Each event A_e is mutually independent of the set of events $\{A_f : f \notin N_e\} \cup A_e$.

The proof follows easily from this claim and the Lovász Local Lemma, as $\mathbf{Pr}(A_e) \leq 2^{-(k-1)}$ and $4 \times 2^{-(k-1)} \times 2^{k-3} \leq 1$. The claim seems intuitively clear, but we should take care to prove it, as looks can often be deceiving in this field.

Suppose that the vertices are ordered $v_1, \ldots, v_n$ where $e = \{v_1, \ldots, v_t\}$. Consider any edges $f_1, \ldots, f_r, g_1, \ldots, g_s \notin N_e$. Let Υ be the set of 2-colourings for which the event $B = A_{f_1} \cap \ldots \cap A_{f_r} \cap \overline{A_{g_1}} \cap \ldots \cap \overline{A_{g_s}}$ holds.

For any 2-colouring ρ of $G - V(e)$, define T_ρ to be the set of the 2^t different 2-colourings of G which extend ρ. It is straightforward to verify that for each ρ, Υ contains either all of T_ρ or none of T_ρ. In other words, there is an ℓ such that Υ is the disjoint union $T_{\rho_1} \cup \ldots \cup T_{\rho_\ell}$ for some $\rho_1, \ldots, \rho_\ell$. Thus, $\mathbf{Pr}(B) = \frac{2^t \ell}{2^n}$.

Within each T_{ρ_i}, there are exactly two 2-colourings in which e is monochromatic, and so $\mathbf{Pr}(A_e \cap B) = \frac{2\ell}{2^n}$. Thus, $\mathbf{Pr}(A_e | B) = (\frac{2\ell}{2^n})/\mathbf{Pr}(B) = 2^{-(t-1)} = \mathbf{Pr}(A_e)$ as claimed. □

The claim in the preceding proof is a special case of a very useful principle concerning mutual independence. In fact, we appeal to the following fact nearly every time we wish to establish mutual independence in this book.

The Mutual Independence Principle *Suppose that $\mathcal{X} = X_1, \ldots, X_m$ is a sequence of independent random experiments. Suppose further that $A_1, \ldots, A_n$ is a set of events, where each A_i is determined by $F_i \subseteq \mathcal{X}$. If $F_i \cap (F_{i_1}, \ldots, F_{i_k}) = \emptyset$ then A_i is mutually independent of $\{A_{i_1}, \ldots, A_{i_k}\}$.*

The proof follows along the lines of that of the preceding claim, and we leave the details as an exercise.

We end this chapter with another application of the Local Lemma.

4.1 Constrained Colourings and the List Chromatic Number

As discussed in Sect. 3.3, Alon has shown that a graph has bounded list chromatic number if and only if it has bounded colouring number. Thus, if we impose no extra conditions on our lists, to approximately determine how big our lists must be to ensure that an acceptable colouring exists, we need only consider the colouring number. In this section, we show that we can ensure the existence of acceptable colourings for much shorter lists, if we impose a (natural) constraint on the ways in which the lists can intersect. The results discussed in this section first appeared in [133]. For further discussion, including some conjectures, the reader should consult that paper.

As we mentioned in Chap. 1, the greedy colouring procedure yields a bound of $\Delta + 1$ on $\chi_l(G)$. The following theorem suggests that a much stronger result, stated as a conjecture below, may be true. The theorem is quite powerful in its own right and will be used repeatedly throughout the book.

Theorem 4.3 *If there are at least ℓ acceptable colours for each vertex, and each colour is acceptable for at most $\frac{\ell}{8}$ of the neighbours of any one vertex, then there there is an acceptable colouring.*

Conjecture 4.4 *The $\frac{\ell}{8}$ in the above theorem can be replaced by $\ell - 1$.*

Remark The $\frac{\ell}{8}$ in the above theorem can be replaced by $\frac{\ell}{2e}$ by using the more precise version of the Local Lemma. Furthermore, using different techniques, Haxell [77] has proven that the result holds if the value is $\frac{\ell}{2}$, and by iteratively applying the Local Lemma, Reed and Sudakov [134] have shown that $\ell - o(\ell)$ is sufficient.

We now prove Theorem 4.3, which requires an application of the Local Lemma.

Proof of Theorem 4.3. Fix a graph G and an acceptable list of colours L_v for each vertex v, which satisfy the conditions of the theorem. For ease of exposition, we truncate each L_v so that it has exactly ℓ colours.

Now, we consider the random colour assignment in which each vertex is independently assigned a uniform element of L_v. For each edge $e = xy$ and colour $i \in L_x \cap L_y$, we let $A_{i,e}$ be the event that both x and y are coloured with i. We let $\mathcal{E}$ be the set of all such events. We use the Local Lemma to show that with positive probability none of the events in $\mathcal{E}$ occur, i.e. the colouring obtained is acceptable.

Consider first the probability of $A_{i,e}$, clearly this is $\left(\frac{1}{\ell}\right)^2$. Consider next the dependency between events. If e has endpoints x and y, then $A_{i,e}$ depends only on the colours assigned to x and y. Thus, letting $E_x = \{A_{j,f} | j \in L_x, x$ is an endpoint of $f\}$ and letting $E_y = \{A_{j,f} | j \in L_y, y$ is an endpoint of $f\}$, we see that $A_{i,e}$ is mutually independent of $\mathcal{E} - E_x - E_y$. Now, since L_x has exactly ℓ elements, and x has at most $\frac{\ell}{8}$ neighbours of colour i for each $i \in L_x$, we see that $|E_x| \leq \frac{\ell^2}{8}$. Similarly, $|E_y| \leq \frac{\ell^2}{8}$. Thus, setting $d = \frac{\ell^2}{4}$, we see that each $A_{e,i}$ is mutually independent of a set of all but at most d of the other events in $\mathcal{E}$. Since $\left(\frac{1}{\ell}\right)^2 \times \frac{\ell^2}{4} \leq \frac{1}{4}$, the Local Lemma implies that an acceptable colouring exists. This yields the desired result. $\square$

As is often the case with the Local Lemma, once we choose our bad events, the proof is straightforward. However, choosing the good bad events can sometimes be a bit tricky. For example, in Exercise 4.2, we see that two natural attempts at defining the bad events for this application do not lead to proofs.

Exercises

Exercise 4.1 Prove the Mutual Independence Principle.

Exercise 4.2 Show what would go wrong if you attempted to prove Theorem 4.3 by giving each vertex a uniformly random colour from its list and applying the Local Lemma to either of the following sets of bad events.

1. For each vertex v, A_v is the event that v receives the same colour as one of its neighbours.
2. For each edge e, A_e is the event that the endpoints of e both receive the same colour.

Exercise 4.3 Consider a graph G with maximum degree Δ where every vertex v of G has a list L_v of acceptable colours. Each colour $c \in L_v$ has a weight $w_v(c)$ such that $\sum_{c \in L_v} w_v(c) = 1$. Prove that if for every edge uv we have $\sum_{c \in L_u \cap L_v} w_u(c) w_v(c) \leq \frac{1}{8\Delta}$ then G has an acceptable colouring.