Oriented Matroid

and pseudosphere arrangements

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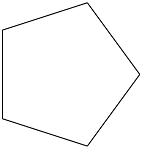
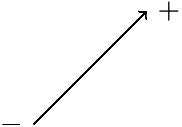
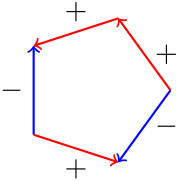
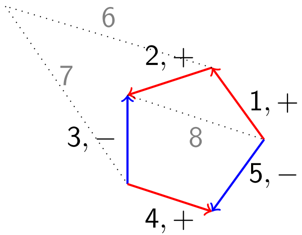
Matroid Circuits + **Orientation**



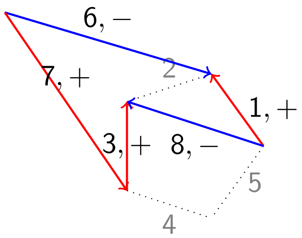




What is an Oriented Matroid?

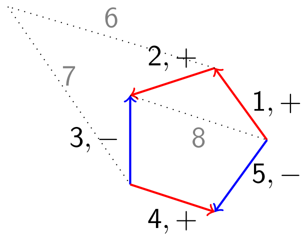


(a) $C = (+, +, -, +, -, 0, 0, 0)$

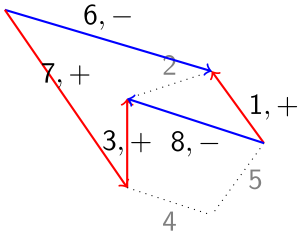


(b) $D = (+, 0, +, 0, 0, -, +, -)$

What is an Oriented Matroid?



(a) $C = (+,+,-,+,-,0,0,0)$



(b) $D = (+,0,+,0,0,-,+,-)$

Composition, Support, and Separation set of covectors

$$(C \circ D)_e := \begin{cases} C_e & \text{if } C_e \neq 0, \\ D_e & \text{otherwise} \end{cases} = (+,+,-,+,-,-,+,-)$$

$$\underline{C} := \{e \in E \mid C_e \neq 0\} = \{1,2,3,4,5\}$$

$$S(C,D) := \{e \in E \mid C_e = -D_e \neq 0\} = \{3\}$$

Definition of oriented matroid with covector axioms

An **oriented matroid** given in terms of its covectors is a pair $\mathcal{M} := (E, \mathcal{L})$, where $\mathcal{L} \subset \{-, 0, +\}^E$ satisfies

(CV0) $\mathbf{0} \in \mathcal{L}$

(CV1) $C \in \mathcal{L} \implies -C \in \mathcal{L}$

(CV2) $C, D \in \mathcal{L} \implies C \circ D \in \mathcal{L}$

(CV3) $C, D \in \mathcal{L},\ e \in S(C, D) \implies$ there is a $Z \in \mathcal{L}$ with $Z_e = 0$ and with $Z_f = (C \circ D)_f$ for $f \in E \setminus S(C, D)$.

Example: Hyperplane Arrangement

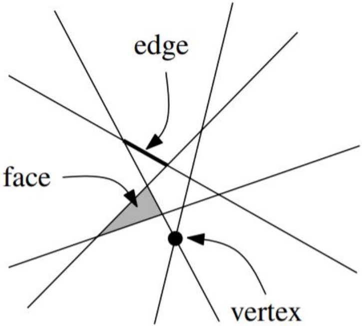


Figure: Line Arrangement[dCvO08]

Example: Hyperplane Arrangement

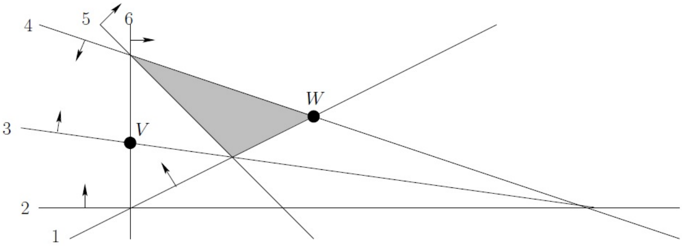


Figure: Oriented hyperplane arrangement[Hoc10]

$$H = \{x \in \mathbb{R}^2 \mid a^T x = c\}$$
$$H^+ = \{x \in \mathbb{R}^2 \mid a^T x > c\}$$
$$H^- = \{x \in \mathbb{R}^2 \mid a^T x < c\}$$

Example: Linear Matroid

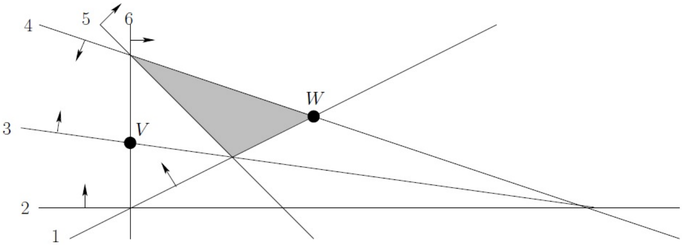
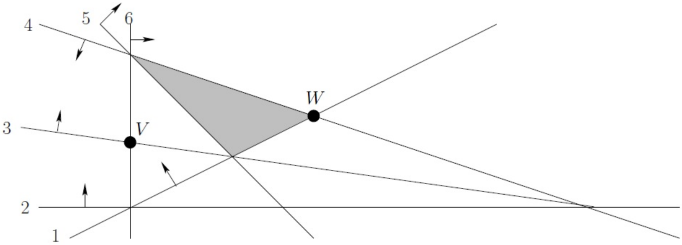


Figure: Oriented hyperplane arrangement[Hoc10]

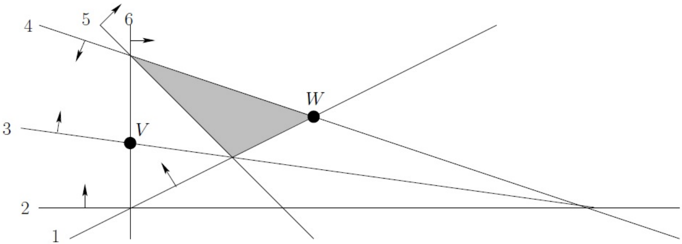
$$SignVector(V) = (+ + 0 + -0)$$
$$SignVector(W) = (0 + +0 + +)$$
$$SignVector(\text{Shaded Region}) = (+ + + + ++)$$

Example: Hyperplane Arrangement



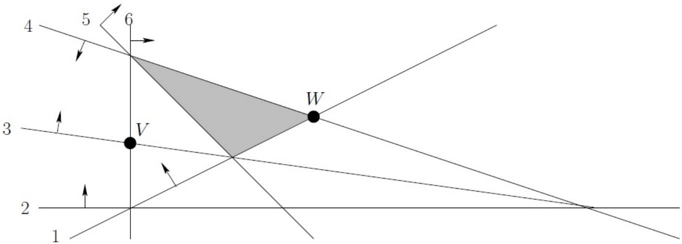
$$\mathcal{L} = \{\pm\text{SignVector}(x) \mid x \in \mathbb{R}^2\} \cup \{\mathbf{0}\}$$
$$= \{\pm\text{SignVector}(C) \mid C \text{ cell of the arrangement}\} \cup \{\mathbf{0}\}$$

Example: Hyperplane Arrangement



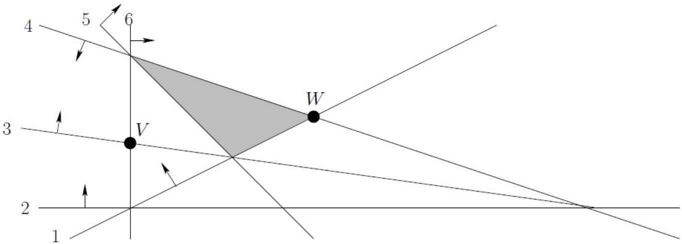
(CV0) $\mathbf{0} \in \mathcal{L}$

Example: Hyperplane Arrangement



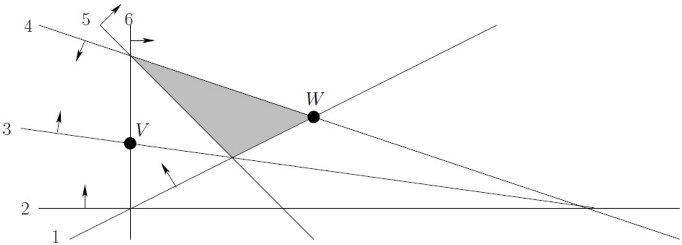
(CV1) $C \in \mathcal{L} \implies -C \in \mathcal{L}$

Example: Hyperplane Arrangement



$$\text{(CV2)} \quad C, D \in \mathcal{L} \implies C \circ D \in \mathcal{L}$$

Example: Hyperplane Arrangement



(CV3) $C, D \in \mathcal{L},\ e \in S(C, D) \implies$ there is a $Z \in \mathcal{L}$ with $Z_e = 0$ and with $Z_f = (C \circ D)_f$ for $f \in E \setminus S(C, D)$.

Recall: Cocircuit Axiom

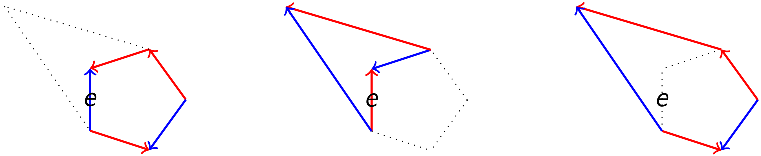
A collection $\mathcal{C}^* \subset \{-, 0, +\}^E$ is the set of cocircuits of an oriented matroid $\mathcal{M}$ if and only if it satisfies

(CC0) $\mathbf{0} \notin \mathcal{C}^*$

(CC1) $C \in \mathcal{C}^* \implies -C \in \mathcal{C}^*$

(CC2) for all $C, D \in \mathcal{C}^*$ we have $\underline{C} \subset \underline{D} \implies C = D$ or $C = -D$

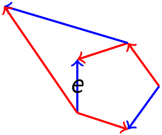
(CC3) $C, D \in \mathcal{C}^*$, $C \neq -D$, and $e \in S(C, D) \implies$ there is a $Z \in \mathcal{C}^*$ with $Z^+ \subset (C^+ \cup D^+) \setminus \{e\}$ and $Z^- \subset (C^- \cup D^-) \setminus \{e\}$

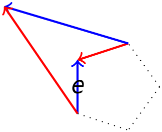


(a) C (b) D (c) Z

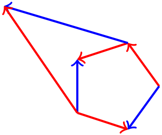
Cocircuit and Covector

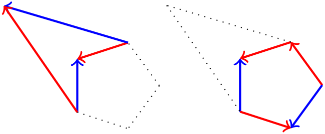
Covector C and $e \in C$ $\Longrightarrow$ smallest $M \in \mathcal{L}$ s.t. $e \in M \subset C$



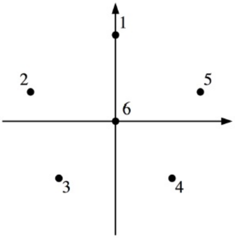


Covector C
$C = M_1 \circ \cdots \circ M_m$ $\Longleftarrow$ **Cocircuits** $M_1, \cdots, M_m$
$S(M_i, M_j) = \emptyset$, for $i \neq j$





Example: Linear Matroid



$$X := \begin{pmatrix} 0 & 3 & 1 \\ -3 & 1 & 1 \\ -2 & -2 & 1 \\ 2 & -2 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Figure: Homogenized coordinates of points [RZ17]

$$E = \{x_i \mid i = 1, \cdots, 6\}$$

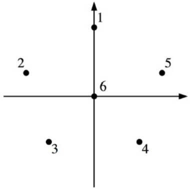
$$\mathcal{I} = \{\text{linearly independent } I \subset E\}$$

$$\mathcal{B} = \{\text{basis } B \subset E \text{ of } \mathbb{R}^d = \mathbb{R}^3\}$$

$$\mathcal{C} = \{\text{minimal linearly dependent } S \subset E\}$$

$$\mathcal{C}^* = \{\text{minimal } S \subset E \text{ that intersects each basis}\}$$

Example: Linear Matroid



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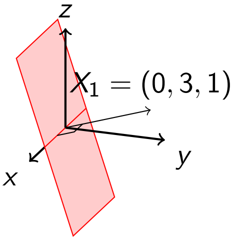

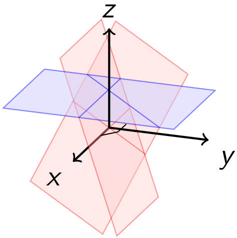

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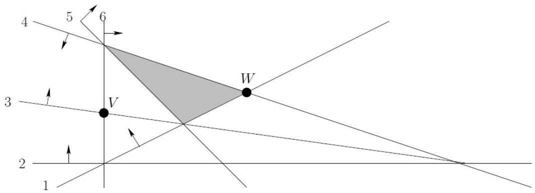
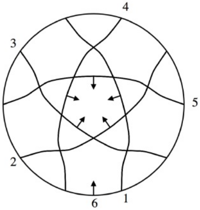


$$
\begin{aligned}
\mathcal{C}^* &= \{\text{minimal } S \subset E \text{ that intersects each basis}\} \\
&= \{\text{minimal } S \subset E \text{ such that } \dim(E \setminus S) < d = 3\} \\
&= \{\text{minimal } S \subset E \mid \dim(E \setminus S) = d - 1 = 2\} \\
&= \{\text{minimal } S \subset E \mid \dim\left(\mathrm{Ker}(E \setminus S)\right) = 1\} \\
&= \{S = E \setminus D \mid \text{maximal } D \subset E, dim(\mathrm{Ker}(D)) = 1\} \\
&= \{S = E \setminus D \mid D = \underline{SignVector(P)}^0, P \text{ vertex}\} \\
&= \{S = \underline{SignVector(P)} \mid P \text{ vertex}\}
\end{aligned}
$$

Example: Linear Matroid



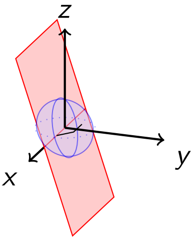


Figure: Hypersphere Arrangement[RZ17] from oriented hyperplanes

What is a Pseudosphere?

A **pseudosphere** is the image $s \subset \mathcal{S}^{d-1}$ of the equator $\{x \in \mathcal{S}^{d-1} \mid x_d = 0\}$ in the unit sphere under a self-homeomorphism $\phi : \mathcal{S}^{d-1} \to \mathcal{S}^{d-1}$. (Pseudospheres behave "nicely" in the sense that they divide $\mathcal{S}^{d-1}$ into two open sets, its **sides**, that are homeomorphic to open $(d-1)$-balls.)

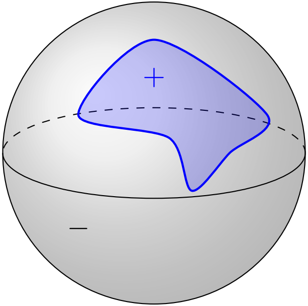


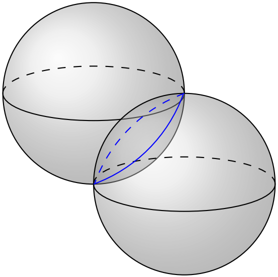
Figure: pseudosphere and its sides, for $d = 3$

What is an Arrangement of Pseudospheres?

A finite collection $\mathcal{P} = (s_1,\ s_2,\ \cdots,\ s_n)$ of pseudospheres in $\mathcal{S}^{d-1}$ is an **arrangement of pseudospheres** if the following conditions hold (we set $E := \{1, 2, \cdots, n\}$):

(PS1) For all $A \subset E$ the set $S_A = \cap_{e \in A} s_e$ is a topological sphere.

(PS2) If $S_A \not\subset s_e$, for $A \subset E$, $e \in E$, then $S_A \cap s_e$ is a pseudosphere in S_A with sides $S_A \cap s_e^+$ and $S_A \cap s_e^-$.



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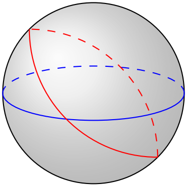

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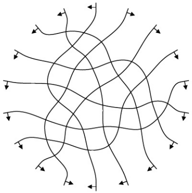


Figure: pseudoline arrangement [RZ17]

Oriented Matroids & Pseudosphere Arrangements

The Topological Representation Theorem

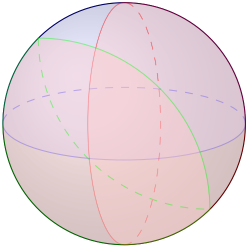
If $\mathcal{P}$ is an essential($S_E = \emptyset$) arrangement of pseudospheres on $\mathcal{S}^{d-1}$ then $\Gamma(\mathcal{P}) \cup \{\mathbf{0}\}$ forms the set of covectors of an oriented matroid of rank d. Conversely, for every oriented matroid $(E, \mathcal{L})$ of rank d (without loops) there exists an essential arrangement of pseudospheres $\mathcal{P}$ on $\mathcal{S}^{d-1}$ with $\Gamma(\mathcal{P}) = \mathcal{L} \setminus \{\mathbf{0}\}$.

$$\left\{\begin{array}{c}\textbf{Essential arrangement}\\ \textbf{of pseudospheres on } \mathcal{S}^{d-1}\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c}\textbf{Loopless oriented matroid}\\ \textbf{of rank } d\end{array}\right\}$$

Pseudospheres to Oriented Matroids

$$\left\{\begin{matrix}\text{Pseudosphere } s \\ s \subset \mathcal{S}^{d-1}\end{matrix}\right\} \Longrightarrow \left\{\begin{matrix}\text{Pseudohemispheres} \\ h_s = s^+ \text{ and } -h_s = s^-\end{matrix}\right\}$$

$$\mathcal{C}^* := \{\text{minimal set } C \neq \emptyset \text{ of pseudohemispheres such that}$$
$$C \cap -C = \emptyset \text{ and}$$
$$\cup_{h \in C}\, h = \mathcal{S}^{d-1}\}$$



Oriented Matroids to Pseudospheres

(Induction on $|E|$ for fixed rank d

- base case: $E = [d]$, $\mathcal{I} = 2^{[d]}$, $\mathcal{C}^* = \{\pm e_i \mid i \in [d]\}$)

$$P_d := Conv(\{\pm e_1, \cdots \pm e_d\})$$

$$e_i = (0, 0, \cdots, 0, 1, 0, \cdots, 0)$$

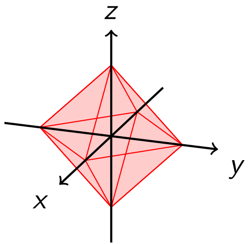


Figure: 3-dimensional Pyramid

$\implies$ Arrangement of ∂P_d

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