Antimatroids and a curious relation to the k-SAT problem

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Closure Operator for Matroids

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$$\tau : \mathcal{P}(E) \to \mathcal{P}(E), A \mapsto \{x \in E |\, \mathrm{rk}(A \cup x) = \mathrm{rk}(A)\}$$

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  i) $A \subseteq \tau(A)$
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- Any such operator also defines a Matroid!

Introducing Convex Geometries

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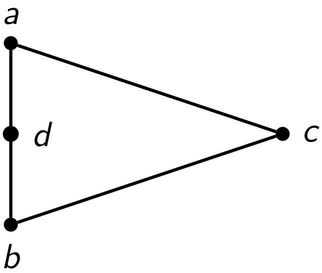
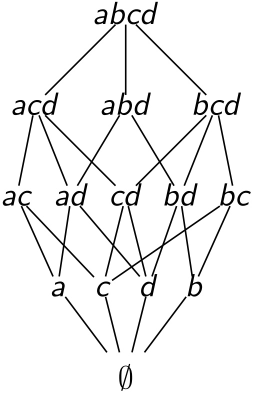
- Call A closed if $\tau(A) = A$
- Closed sets $\mathcal{N}$ of a convex geometry fulfill
  i) $E \in \mathcal{N}$
  ii) $A, B \in \mathcal{N} \implies A \cap B \in \mathcal{N}$
  iii) For $A \in \mathcal{N}$ with $A \neq E$ there exists $x \in E \setminus A$ s.t. $A \cup \{x\} \in \mathcal{N}$
- For a set system describing a convex geometry $\mathcal{N}$ define

$$\tau(D) = \bigcap_{C \supseteq D : C \in \mathcal{N}} C$$

Convex Geometries

Example





Antimatroids

Definition

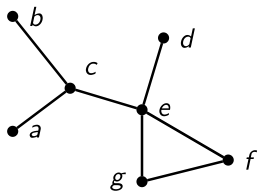
A set system $(\mathcal{L}, E)$ is an antimatroid if $\{E \setminus X | X \in \mathcal{L}\}$ is a convex geometry.

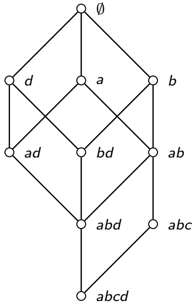
The set system $\mathcal{L}$ of an antimatroid has properties

i) $\emptyset \in \mathcal{L}$

ii) $A, B \in \mathcal{L} \implies A \cup B \in \mathcal{L}$

iii) For $S \in \mathcal{L}$ with $A \neq \emptyset$ exists $x \in S$ s.t. $S \setminus \{x\} \in \mathcal{L}$





Pruning Processes

Pruning Process:

- Remove Elements from a ground set one at a time
- Removable Elements always stay removable at every following step

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A new characterization for convex geometries

For a set system $\mathcal{N}$ for $A \in \mathcal{N}$ define

$$\mathrm{ex}(A) := \{x \in A | A - x \in \mathcal{N}\}$$

Theorem (Ardila, Maneva)

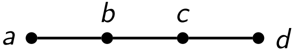
These three Statements are equivalent:

i) *$\mathcal{N}$ is the collection of closed sets of a convex geometry*
ii) *As A ranges over $\mathcal{N}$ the intervals $[\mathrm{ex}(A), A]$ partition the Boolean lattice 2^E*
iii) *For any p_i, q_i for $i \in E$ with $p_i + q_i = 1$, we have*

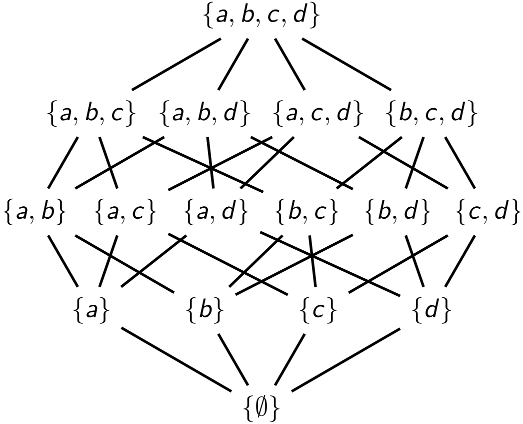
$$\sum_{A \in \mathcal{N}} \prod_{i \notin A} p_i \prod_{j \in \mathrm{ex}(A)} q_j = 1$$

A new characterization of convex geometries

Example



$$\mathcal{N} = \{\{a,b,c,d\},\\ \{a,b,c\},\{b,c,d\},\\ \{a,b\},\{b,c\},\{c,d\},\\ \{a\},\{b\},\{c\},\{d\},\emptyset\}$$



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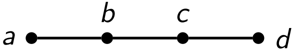

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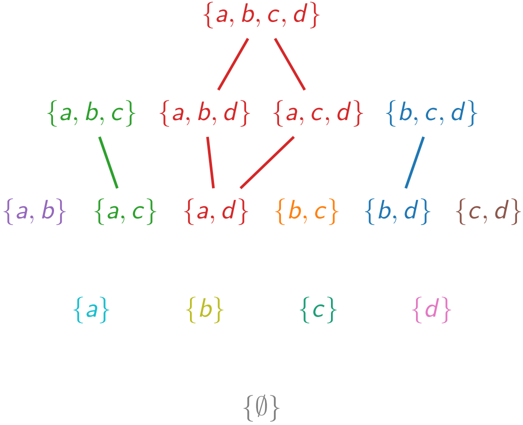

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Main theorem

i) $\implies$ *ii*)

Let $\mathcal{N}$ be subsets of closed geometry. Show that intervals $[\text{ex}(A), A]$ partition the boolean lattice.

1.Uniqueness: First recognize that $\tau(\text{ex}(A)) = A$.

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If $A \in \mathcal{N}$ with $\text{ex}(A) \subseteq D \subseteq A$ we have

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2.Existence: Clearly $D \subseteq \tau(D) =: A$. $\text{ex}(A)$ is minimal among sets $A = \tau(B)$. Therefore $D \supseteq \text{ex}(A)$

Main theorem

ii) $\implies$ *i*)

Let $\mathcal{N} \subseteq \mathcal{P}(E)$ such that for each $D \in 2^E$ there is exactly one $A \in \mathcal{N}$ with $D \in [\mathrm{ex}(A), A]$. Show three axioms for convex geometry

1. $E \in \mathcal{N}$
2. $A, B \in \mathcal{N} \implies A \cap B \in \mathcal{N}$
3. For $A \in \mathcal{N}$ with $A \neq E$ there exists $x \in E \setminus A$ s.t. $A \cup \{x\} \in \mathcal{N}$

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Proof:
E has to be covered by E.

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Proof:

- Define $\phi(D)$ as A s.t. $D \in [\text{ex}(A), A]$.
- Show for $A \in \mathcal{N}$, $D \subseteq A \implies \phi(D) \subseteq A$.

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$$
\begin{aligned}
\phi(A \cap B) \subseteq A \text{ and } \phi(A \cap B) \subseteq B &\implies \phi(A \cap B) \subseteq A \cap B \\
&\implies \phi(A \cap B) = A \cap B
\end{aligned}
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Proof:
Let $A, B \in \mathcal{N}$ with $A \subseteq B$. We show that there is $x \in B \setminus A$ with $x \in \text{ex}(B)$:

Main theorem

ii) $\implies$ iii)

Recall we want to prove

$$\sum_{A \in \mathcal{N}} \prod_{i \notin A} p_i \prod_{j \in \text{ex}(A)} q_j = 1$$

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Observe that for any E we have

$$\sum_{D \subseteq E} \prod_{i \notin D} p_i \prod_{j \in D} q_j = \prod_{h \in E} (p_h + q_h) = 1$$

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Therefore it is enough to show

$$\prod_{i \notin A} p_i \prod_{j \in \mathrm{ex}(A)} q_j = \sum_{D \in [\mathrm{ex}(A), A]} \prod_{i \notin D} p_i \prod_{j \in D} q_j$$

Main theorem

ii) $\implies$ ii)

$$
\begin{aligned}
&\sum_{D\in[\mathrm{ex}(A,A)]} \prod_{i\notin D} p_i \prod_{j\in D} q_j \\
&= \prod_{i\notin A} p_i \prod_{j\in \mathrm{ex}(A)} q_j \sum_{R\subseteq A\setminus \mathrm{ex}(A)} \left(\prod_{i\in A\setminus(\mathrm{ex}(A)\cup R)} p_i \prod_{j\in R} q_j \right) \\
&= \prod_{i\notin A} p_i \prod_{j\in \mathrm{ex}(A)} q_j
\end{aligned}
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\end{aligned}
$$

Overall leading to:

$$
\begin{aligned}
\sum_{A\in\mathcal{N}}\prod_{i\notin A}p_i\prod_{j\in\mathrm{ex}(A)}q_j &= \sum_{A\in\mathcal{N}}\sum_{D\in[\mathrm{ex}(A,A)]}\prod_{i\notin D}p_i\prod_{j\in D}q_j\\
&=\sum_{D\subseteq E}\prod_{i\notin D}p_i\prod_{j\in D}q_j=1
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iii) $\implies$ ii)

- For any assignment of p_i, q_i with $p_i + q_i = 1$ we have

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- Show that 2^E is partitioned into intervals $[\mathrm{ex}(A), A]$

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- Show that 2^E is partitioned into intervals $[\text{ex}(A), A]$
- Pick $D \subseteq E$, set $p_a = 0$ if $a \in D$ and $p_a = 1$ otherwise, then we have

$$1 = \sum_{A \in \mathcal{N}} \prod_{i \notin A} p_i \prod_{j \in \text{ex}(A)} q_j = \sum_{A \in \mathcal{N}} \sum_{D \in [\text{ex}(A), A]} \prod_{i \in D} p_i \prod_{j \in D} q_j$$

The k-SAT Problem

Goal: find a valid assignment for boolean formula

$$(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (x_2 \vee \bar{x}_3 \vee \bar{x}_4) \wedge (\bar{x}_1 \vee x_3 \vee x_4)$$

$$x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 1$$

$$\Big\downarrow$$

$$(\bar{0} \vee \bar{1} \vee 0) \wedge (1 \vee \bar{0} \vee \bar{1}) \wedge (\bar{1} \vee 0 \vee 1) = 1$$

Partial Assignments

Call an assignment of the variables $\mathbf{a} = (a_1, \ldots, a_n)$ with $a_1, \ldots a_n \in \{0, 1, *\}$ an invalid partial assignment if

- Applying $\mathbf{a}$ to the formula results in 0
- One clause is of the form $(0 \wedge \cdots \wedge 0 \wedge * \wedge 0 \wedge \cdots \wedge 0)$

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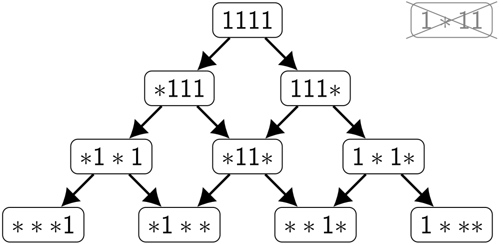
Poset on valid partial assignments



Poset on valid partial assignments for
$(\bar{x}_1 \vee \bar{x}_2 \vee x_3) \wedge (x_2 \vee \bar{x}_3 \vee \bar{x}_4)$ and $(1, 1, 1, 1)$

A curious property of valid assignments

For any partial assignment **a** let $S(\mathbf{a})$ be the set of variables which are stars and $U(\mathbf{a})$ the set of unconstrained variables.

Theorem (Maneva, Mossel, Wainwright)

For any valid assignment **a** *and* $p \in [0,1]$, $q = 1 - p$ *holds*

$$\sum_{\mathbf{b} \leq \mathbf{a}} p^{|S(\mathbf{b})|} q^{|U(\mathbf{b})|} = 1$$

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Theorem (Ardila, Maneva)

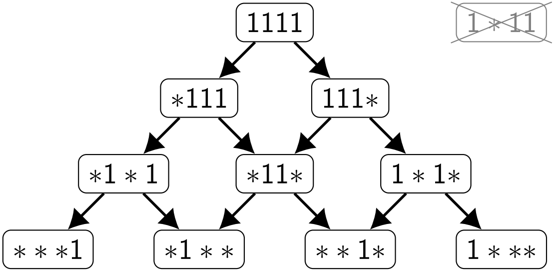
For the set of Variables V and any $p_i \in [0,1]$, $q_i = 1 - p_i$ for $i \in V$ we get

$$\sum_{\mathbf{b} \leq \mathbf{a}} \prod_{i \in S(\mathbf{b})} q_i \prod_{U(\mathbf{b})} p_j = 1$$

A curious property of valid assignments

Proof

Natural pruning process arising on the poset of valid assignments:



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